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On persistence of invariant tori and a theorem by Nekhoroshev

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Abstract

We give a proof of a theorem by N.N. Nekhoroshev concerning Hamiltonian systems with n degrees of freedom and s integrals of motion in involution, where $1 \leq s \leq n$. Such a theorem ensures persistence of s-dimensional invariant tori under suitable nondegeneracy conditions generalizing Poincaré's condition on the Floquet multipliers.

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1 Introduction.

A Hamiltonian system in n degrees of freedom having n independent integrals of motion in involution is integrable [1]. When the system has a number s of independent integrals of motion greater than one, but smaller than the number of degrees of freedom, i.e. $1 < s < n$, we say the system is "partially integrable".

In the early nineties Nekhoroshev stated, under suitable nondegeneracy conditions, an interesting result on the existence of s-parameter families of tori in partially integrable Hamiltonian systems with n degrees of freedom $(1 < s < n)$ [10]; this constitutes a bridge between the Poincaré-Lyapunov theorem (case of periodic trajectories, $s = 1$) and the Liouville-Arnold one (complete integrability, $s = n$).

Unfortunately Nekhoroshev never published a proof of this theorem. When we became interested in the problem, we decided to try to reconstruct the proof, following a line of thought which is naturally suggested by Nekhoroshev's formulation of the nondegeneracy condition. However, after doing this we realized that one can also reformulate the nondegeneracy condition of the theorem in terms of standard objects (Floquet multipliers), and obtain a simpler proof: this is given in the present paper.

We recall that, roughly speaking (the precise statement will be recalled in sect. 1), Nekhoroshev's theorem states that if one has s integrals of motion in involution and an invariant torus $\Lambda \simeq \mathbf{T}^s$, then (under suitable nondegeneracy conditions) there is a $2s$–dimensional symplectic submanifold N, with $\Lambda \subset N$, which is fibered by invariant tori $\Lambda_\beta \simeq \mathbf{T}^s$; moreover it is possible to build action angle coordinates on N. Finally the invariant tori Λ_β persist under small perturbations of the Hamiltonian and of the integrals of motion. The main problem lies in identifying the relevant nondegeneracy condition. We find that it can be expressed in terms of Floquet multipliers of periodic orbits of suitable Hamiltonian vector fields. In the case of reducible tori the condition can be reformulated in a purely algebraic form.

The **plan of the paper** is as follows. In section 1 we recall the precise statement of Nekhoroshev's theorem. Section 2 is devoted to a detailed proof of the main theorem, based on a lemma which provides "good" coordinates. In section 3 the theory is applied to the case of *reducible tori* (whose definition is recalled there); in such a case the nondegeneracy condition reduces to a condition on determinants of s-dimensional matrices, see eq.(4.18).

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2 Statement of results

Let (M,Ω) be a symplectic manifold (with symplectic form Ω) of dimension $\dim(M) = 2n$, differentiable of class C^r $(r \geq 2)$. Let $\mathbf{F}_\varepsilon := \{F_1^\varepsilon, ..., F_s^\varepsilon\}$ be s real functions on M depending in a C^r way on a small parameter $\varepsilon \in E$, with $E := (-\varepsilon_0, \varepsilon_0)$, and $\varepsilon_0 > 0$. We assume also that the functions F_i^ε are differentiable of class C^r, that they are independent and in involution, namely that $\left\{F_i^\varepsilon, F_j^\varepsilon\right\} = 0$, $i,j \in \{1,...,s\}$, $\forall \varepsilon \in E$.

We can then consider the Hamiltonian vector fields X_i^ε generated by the functions F_i^ε, namely defined by $X_i^\varepsilon \lrcorner \Omega = \mathrm{d}F_i^\varepsilon$.

We will *assume* that there is a compact and connected manifold $\Lambda \subset M$ of dimension s, differentiable of class C^r, which is invariant under all the unperturbed vector fields $X_i := X_i^0 \equiv X_i^\varepsilon|_{\varepsilon=0}$, and such that the X_i are linearly independent at all points $m \in \Lambda$. Obviously this Λ will be a submanifold of $\mathbf{F}_0^{-1}(\beta_0)$ for some $\beta_0 \in \mathbf{R}^s$.

Note that the condition of independence of the X_i (and of the F_i^ε) on Λ is equivalent to the requirement that

$$\eta^\varepsilon := \mathrm{d}F_1^\varepsilon \wedge ... \wedge \mathrm{d}F_s^\varepsilon \neq 0 \quad \text{on } \Lambda \quad \text{and for } \ \varepsilon = 0;$$

since the F_i^ε are smooth (i.e. C^1) functions, the relation $\eta^\varepsilon \neq 0$ holds in a neighbourhood of Λ and for ε small enough.

It is actually immediate to see that, being s-dimensional, connected, compact and invariant under the s commuting vector fields X_i, the C^r manifold Λ is necessarily a torus: $\Lambda = \mathbf{T}^s$; see e.g. the proof of the Liouville-Arnold theorem in [1]. Thus, under our assumptions the system has an invariant s-torus.

It is well known that for $s = 1$ (i.e. $\Lambda \approx S^1$ is a periodic orbit), if the Floquet multiplier 1 has multiplicity two, then Λ is part of a one-parameter local family of such orbits which persists under perturbation (Poincaré-Lyapunov theorem). We want to show that a similar property holds for arbitrary s, $1 \leq s \leq n$; the main problem lies in identifying the appropriate nondegeneracy condition.

In order to state the theorem we need to fix some notation.

First of all, using the same notation as in [1], we will denote by $g_{\varepsilon,i}^t(.)$ the flow of the vector field X_i^ε, and, for $\tau \equiv (\tau_1, \tau_2, ..., \tau_s) \in \mathcal{T} \subseteq \mathbf{R}^s$, we define

$$g_\varepsilon^\tau(p) = g_{\varepsilon,1}^{\tau_1} \circ g_{\varepsilon,2}^{\tau_2} \circ ... \circ g_{\varepsilon,s}^{\tau_s}(p) \ . \tag{2.1}$$

We will also denote

$$g^\tau := g_0^\tau \ .$$

Consider the first homotopy group $\pi_1(\Lambda) = \mathbf{Z}^s$ of Λ, and fix $\alpha = (\alpha_1, ..., \alpha_s) \in \pi_1(\Lambda)$; then there exist real constants $(c_1, ..., c_s) \equiv \mathbf{c}$ such that the flow of the vector field

$$X_\alpha \ := \ \sum_{i=1}^s \ c_i \, X_i \tag{2.2}$$

with initial data on Λ is periodic of period one (if $\alpha \not\equiv 0$) and has closed trajectories in the homotopy class α. Such periodic orbits are given by $g^{t\mathbf{c}}(m)$, $t \in [0,1]$, $m \in \Lambda$. We will denote by X^ε_α the vector field $X^\varepsilon_\alpha := \sum_{i=1}^s c_i X^\varepsilon_i$.

We recall that for a fixed $m \in \Lambda$, the Floquet multipliers of the periodic orbit $g^{t\mathbf{c}}(m)$ are the eigenvalues of the Jacobian matrix $[dg^{\mathbf{c}}](m)$ (where the differentiation is with respect to the variable m). It turns out (as discussed in lemma 3.8 below) that the Floquet multipliers of the orbit $g^{t\mathbf{c}}(m)$ do not depend on the point $m \in \Lambda$, but only on the vector field X_α. Moreover, due to the symmetries of the problem (see the proof of lemma 3.9 below), 1 is a Floquet multiplier of such a periodic orbit with multiplicity at least $2s$.

Theorem 2.3 *(***Nekhoroshev***) Let (M,Ω) be a $2n$-dimensional symplectic manifold. Let $\mathbf{F}_\varepsilon := \{F^\varepsilon_1, ..., F^\varepsilon_s\}$ be s real functions on $M \times E$, differentiable of class C^r jointly in the M variables and in ε; and denote by X^ε_i the corresponding Hamiltonian vector fields.*

Assume that:

(i) *there exists a C^r compact and connected s-dimensional manifold $\Lambda \subset M$ invariant under all the $X_i \equiv X^\varepsilon_i|_{\varepsilon=0}$;*

(ii) *the functions F^ε_i are independent and in involution – namely $\eta^\varepsilon \neq 0$ and $\left\{F^\varepsilon_i, F^\varepsilon_j\right\} = 0$ $\forall i,j \in \{1,...,s\}$ – in a neighbourhood W of Λ, for all $\varepsilon \in E$;*

(iii) *there exists $\alpha \in \pi_1(\Lambda)$ such that the Floquet multiplier 1 of the periodic orbits of X^0_α on Λ has multiplicity $2s$.*

Then there exists $\varepsilon_* > 0$, such that, for all $\varepsilon \in E_0$, $E_0 := (-\varepsilon_*, \varepsilon_*)$, the following holds true:

(1) *In a neighbourhood $U \subseteq W$ of Λ in M, there is a family of symplectic submanifolds $N_\varepsilon \subset U$, of dimension $2s$, which are fibered over a domain $B \subset \mathbf{R}^s$ having as fibers C^r-differentiable tori $\Lambda^\varepsilon_\beta = N_\varepsilon \cap \mathbf{F}^{-1}_\varepsilon(\beta) \simeq \mathbf{T}^s$ (with $\beta \in B$). For each $\varepsilon \in E_0$, the tori $\Lambda^\varepsilon_\beta$ are C^r-diffeomorphic to Λ and invariant under the X^ε_i; the tori $\Lambda^\varepsilon_\beta$ and the manifolds N_ε depend in a C^r way jointly on $(\varepsilon, \beta) \in E_0 \times B$.*

(2) *There exist symplectic action angle coordinates $(I^\varepsilon_1, ..., I^\varepsilon_s; \varphi^\varepsilon_1, ..., \varphi^\varepsilon_s)$ in N, such that $F^\varepsilon_i|_{N_\varepsilon} = F^\varepsilon_i|_{N_\varepsilon}(I^\varepsilon_1, ..., I^\varepsilon_s)$, with $i = 1, ..., s$. The coordinates $(I^\varepsilon; \varphi^\varepsilon)$ depend in a C^r way on $\varepsilon \in E$, and so do the functions $F^\varepsilon_i|_{N_\varepsilon}$.*

Remark 1. The theorem applies in particular to the case of a hamiltonian system with a compact symmetry group acting properly and freely, and which admits an Ad^* equivariant momentum map. To fix ideas suppose that F_1 is the Hamiltonian of the system and $F_2, ..., F_s$ are the integrals in involution related to the momentum map. In this case the invariant torus Λ is just a relative periodic orbit (namely a periodic orbit of the system obtained by the Marsden–Weinstein reduction procedure). Thus in this case the statement (1) is very close to the Poincaré continuation theorem for relative periodic orbits. This situation has been studied extensively (see [9, 13] and references therein). $\odot$

Remark 2. In the above situation, if the symmetry group acts linearly, the solutions of the equations of motion of the Hamiltonian vector fields of $F_2, ..., F_s$ are automatically periodic also for initial data outside Λ. We stress that this is not assumed here. $\odot$

Remark 3. A related result, based on topological arguments, has been recently proved by J.P.Ortega [11], who extended the Weinstein–Moser theorem in order to find a lower bound to the number of relative periodic orbits (i.e. invariant tori of the original system) close to relative equilibria. $\odot$

Remark 4. In the same situation one could try to apply KAM theory for lower dimensional tori [4, 5, 6, 12] (note that this would require a stronger nondegeneracy condition), and this would ensure persistence of a Cantor family of invariant tori; on the other hand we ensure here existence of a *continuous* family of invariant tori. In particular we ensure also persistence of the *resonant* tori. Obviously this is due to the fact that the systems admit some integrals of motions independent of the Hamiltonian, and thus our situation is exceptional. $\odot$

Remark 5. When studying infinite dimensional systems, one meets cases where a continuous spectrum arises, and this makes KAM theory non applicable at all. On the other hand theorem 2.3 extends immediately to some infinite dimensional situations of this kind; in particular it is used in [3] to construct quasiperiodic breathers in infinite lattices. $\odot$

Our formulation differs slightly from the original one [10] as it is focused to the perturbative frame. In this case the statement (2) is particularly useful in that it allows to characterize the dynamics on the invariant tori of the perturbed system. In particular one has the following

Corollary 2.4 *In the same hypotheses of theorem 2.3, assume also that for some $\kappa \in \{1, ..., s\}$, and for some $m \in \Lambda$ one has*

$$\det\left(\frac{\partial^2 F^0_\kappa|_\Lambda}{\partial I_j \partial I_k}\right)(m) \neq 0 \ , \tag{2.5}$$

and fix ε small enough; then there exists a neighbourhood B_0 of β_0 in $\mathbf{R}^s$ such that for almost all $\beta \in B_0$ the flow of X^ε_κ on $\Lambda^\varepsilon_\beta$ is quasiperiodic with s frequencies independent over the rationals.

Proof. By smooth dependence of action angle variables and of F^ε_κ on ε, eq.2.5 holds also for ε small but different from zero. It follows that the map from the actions to the frequencies is a local isomorphism. $\triangle$

3 Proofs

In this section we will always *assume that all the hypotheses of theorem 2.3 hold*, without stating this explicitly in each lemma.

In order to prove the theorem stated in the previous section, the main point is to introduce suitable coordinates in a neighbourhood U of an arbitrary point of Λ. In order to do that we introduce, for any m in Λ a manifold Σ_m of codimension s passing through m. By the tubular neighbourhood theorem [8], the manifolds Σ_m can be chosen in such a way to define a foliation in U.

Lemma 3.6 *Let m be an arbitrary point in Λ. There exists a neighbourhood $\mathcal{V}_m \subset W \subset M$ of m a positive ε_0 and, for any ε with $|\varepsilon| < \varepsilon_0$ a coordinate map*

$$\mathcal{V}_m \ni p \mapsto (\beta, \tau, y) \in \mathbf{R}^s \times \mathbf{R}^s \times \mathbf{R}^{2(n-s)}$$

with the following properties (where we identify a point with its coordinates):
(i) $F_i^\varepsilon(\beta, \tau, y) = \beta_i$,
(ii) when $\varepsilon = 0$ the coordinates of m are $(\beta_0, 0, 0)$
(iii) $(\partial/\partial\tau_i) = X_i^\varepsilon$, $\forall i = 1, ..., s$
(iv) $p \in \Sigma_m \iff p = (\beta, 0, y)$
(v) the coordinate map depends smoothly on m and ε.

Proof. First one has that $\mathbf{F}_\varepsilon$ restricted to Σ_m is a submersion and therefore one can introduce coordinates (β, y) in Σ_m defined in an open set of $\mathbf{R}^s \times \mathbf{R}^{2(n-s)}$ with $\mathbf{F}_\varepsilon(\beta, y) = \beta$ and such that, when $\varepsilon = 0$, the coordinates of $m \in \Sigma_m$ are $(\beta_0, 0)$.

For (τ, z) in some subset of $\mathbf{R}^s \times \Sigma_m$, consider the map $(\tau, z) \mapsto g_\varepsilon^\tau(z)$. Using the coordinates (β, y) in Σ_m this can be expressed as a map $(\tau, \beta, y) \mapsto g_\varepsilon^\tau(\beta, y)$. It is easily verified that, for $\varepsilon = 0$, its differential at m is an isomorphism and therefore, by the implicit function theorem, the (β, τ, y) are a local coordinate system at m for any ε small enough. Moreover by construction they have the properties stated in the lemma. $\triangle$

The coordinates constructed in this lemma will be called *adapted coordinates based at the point m*

Let us now fix a nontrivial homotopy class $\alpha \in \pi_1(\Lambda)$ and, as in the previous section, consider a vector field $X_\alpha = \sum_i c_i X_i$ such that its trajectories are periodic of period 1 on Λ and belong to the homotopy class α. For ease of writing, we denote by Φ_ε the time 1 flow of X_α^ε, i.e. $\Phi_\varepsilon(p) = g_\varepsilon^{\mathbf{c}}(p)$ where $\mathbf{c} := (c_1, ..., c_s)$.

Consider now a point $m \in \Lambda$ and introduce adapted coordinates based at m; we write $(\widehat{\beta}_\varepsilon, \widehat{\tau}_\varepsilon, \widehat{y}_\varepsilon) := \Phi_\varepsilon(\beta, \tau, y)$. Remark that since $\Phi_0(\beta_0, 0, 0) = (\beta_0, 0, 0) \equiv m$, by smooth dependence of solutions on initial data and parameters there exists a neighbourhood $\widetilde{\mathcal{V}}_m$ of m which is mapped under Φ_ε in the domain $\mathcal{V}_m$ of definition of the above coordinates. We restrict Φ_ε to such a neighbourhood.

Lemma 3.7 *In the adapted coordinates, the map $\Phi_\varepsilon : (\beta,\tau,y) \to (\widehat{\beta}_\varepsilon, \widehat{\tau}_\varepsilon, \widehat{y}_\varepsilon)$ is described by*

$$\widehat{\beta}_\varepsilon(\beta,\tau,y) = \beta \quad ; \quad \widehat{\tau}_\varepsilon(\beta,\tau,y) = \tau + \bar{\tau}_\varepsilon(\beta,y) ,$$

with $\bar{\tau}_\varepsilon$ a suitable function. Moreover,

$$\partial \widehat{y}_\varepsilon / \partial \tau = 0 .$$

Proof. The first equality is a trivial consequence of the way the coordinates are defined. To prove the other two equalities fix $\mu \in \mathbf{R}^s$ small and consider

$$\widehat{\beta}_\varepsilon(\beta,\tau+\mu,y) \quad ; \quad \widehat{\tau}_\varepsilon(\beta,\tau+\mu,y) \quad ; \quad \widehat{y}_\varepsilon(\beta,\tau+\mu,y)$$

by definition these are the coordinates of the point

$$\Phi_\varepsilon(\beta,\tau+\mu,y) = \Phi_\varepsilon\left(g^\mu_\varepsilon(\beta,\tau,y)\right) = g^\mu_\varepsilon\left(\Phi_\varepsilon\left(\beta,\tau,y\right)\right) = \left(\beta, \hat{\tau}_\varepsilon(\beta,\tau,y)+\mu, \widehat{y}_\varepsilon(\beta,\tau,y)\right) .$$

From this

$$\widehat{\tau}_\varepsilon(\beta,\tau+\mu,y) = \widehat{\tau}_\varepsilon(\beta,\tau,y)+\mu , \quad \widehat{y}_\varepsilon(\beta,\tau+\mu,y) = \widehat{y}_\varepsilon(\beta,\tau,y)$$

which shows that $\widehat{y}_\varepsilon$ is independent of τ. To conclude the proof just put $\bar{\tau}_\varepsilon(\beta,y) := \hat{\tau}_\varepsilon(\beta,0,y)$. $\triangle$

It is useful to remark that the Floquet multipliers of the periodic orbits of X_α do not depend on the initial point of the orbit in Λ.

Lemma 3.8 *Let m and m_1 be two points of Λ; then the Floquet multipliers of $g^{t\mathbf{c}}(m)$ and of $g^{t\mathbf{c}}(m_1)$ coincide.*

Proof. Remark that there exists τ_1 such that $m_1 = g^{\tau_1}(m)$. Then, by the commutation of flows we have $g^{\mathbf{c}} = g^{-\tau_1} \circ g^{\mathbf{c}} \circ g^{\tau_1}$. Taking the differential (with respect to the space variable) of this equality at the point m we get

$$dg^{\mathbf{c}}(m) = dg^{-\tau_1}(m_1) dg^{\mathbf{c}}(m_1) dg^{\tau_1}(m) ,$$

which gives the relation between $dg^{\mathbf{c}}(m)$ and $dg^{\mathbf{c}}(m_1)$; the eigenvalues of the former are the Floquet multipliers at m, and the eigenvalues of the latter are the Floquet multipliers at m_1. This shows that $dg^{\mathbf{c}}(m)$ and $dg^{\mathbf{c}}(m_1)$ are conjugated and therefore have the same eigenvalues. $\triangle$

Lemma 3.9 *Fix $m \in \Lambda$ and introduce adapted coordinates based at m, consider the map $\Phi_\varepsilon : (\beta,\tau,y) \to (\widehat{\beta}_\varepsilon, \widehat{\tau}_\varepsilon, \widehat{y}_\varepsilon)$; then there exists β_* and ε_* independent of m such that*

$$\widehat{y}_\varepsilon(\beta, \rho^\varepsilon_m(\beta)) = \rho^\varepsilon_m(\beta) .$$

defines a unique smooth map $\rho^\varepsilon_m(\beta)$ for all β, ε with $|\beta-\beta_0| < \beta_*$ and $|\varepsilon| < \varepsilon_*$.

Proof. In order to be guaranteed that we can solve the equation $\widehat{y}_\varepsilon(\beta, y) = y$ by means of the implicit function theorem, we show that our assumption (iii) on the Floquet multipliers (see theorem 2.3) implies that 1 is not an eigenvalue of the Jacobian of the map $y \mapsto \hat{y}_0$ at $(\beta_0, 0, 0)$.

To prove this fact remark that by lemma 3.7 the Jacobian matrix J of Φ_0 at m takes the block form

$$J = \begin{pmatrix} \frac{\partial\hat{\beta}}{\partial\beta} & \frac{\partial\hat{\tau}}{\partial\beta} & \frac{\partial\hat{y}}{\partial\beta} \\ \frac{\partial\hat{\beta}}{\partial\tau} & \frac{\partial\hat{\tau}}{\partial\tau} & \frac{\partial\hat{y}}{\partial\tau} \\ \frac{\partial\hat{\beta}}{\partial y} & \frac{\partial\hat{\tau}}{\partial y} & \frac{\partial\hat{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\partial\bar{\tau}}{\partial\beta} & \frac{\partial\hat{y}}{\partial\beta} \\ 0 & 1 & 0 \\ 0 & \frac{\partial\bar{\tau}}{\partial y} & \frac{\partial\hat{y}}{\partial y} \end{pmatrix} . \tag{3.10}$$

where we dropped the index ε which is here equal to zero. It follows that the secular equation for J takes the form (with I the identity matrix)

$$\det(\lambda I - J) \;=\; (\lambda - 1)^{2s} \,\det\left(\lambda I - \frac{\partial\hat{y}}{\partial y}\right) \;=\; 0$$

which gives the relation between the Floquet multipliers λ and the eigenvalues of the Jacobian matrix of the map $y \mapsto \widehat{y}$. As J has, by assumption, *exactly* $2s$ Floquet multipliers equal to 1, it follows that 1 is not an eigenvalue of $(\partial\widehat{y}/\partial y)$. Note that this also shows that we have always a multiplicity at least $2s$ for the eigenvalue 1

We can thus apply the implicit function theorem, which ensures existence uniqueness and smoothness of the map ρ^ε_m. We can choose β_* and ε_* independent of m: indeed by compactness of Λ the C^r norm of $\widehat{y}$ can be bounded uniformly with respect to m and the eigenvalues of $(\partial\widehat{y}/\partial y)$ are uniformly bounded away from 1, see lemma 2.9. $\triangle$

Define now the map $\sigma^\varepsilon_\beta : \Lambda \to U$ by $\sigma^\varepsilon_\beta(m) \equiv (\beta, 0, \rho^\varepsilon_m(\beta))$ where we used adapted coordinates based at m. Remark that since $\sigma^\varepsilon_\beta(m) \in \Sigma_m$, one has

$$\sigma^\varepsilon_\beta(m) \neq \sigma^\varepsilon_\beta(m') \quad \text{if} \quad m \neq m' .$$

Define

$$\Lambda^\varepsilon_\beta := \sigma^\varepsilon_\beta(\Lambda) \ ,$$

and remark that this set is in one to one correspondence with Λ. We are going to prove that actually $\Lambda^\varepsilon_\beta$ is a smooth manifold, and that the above correspondence is a diffeomorphism. To this end, having fixed m, β and ε with $|\beta - \beta_0| < \beta_*$ $|\varepsilon| < \varepsilon_*$, define

$$\mathcal{M}^{\beta,\varepsilon}_m := \left\{ p \in \widetilde{\mathcal{V}}_m \ : \ p = (\beta, \tau, y) \ , \ y = \rho^\varepsilon_m(\beta) \right\}$$

where we use adapted coordinates based at m. Then $\sigma^\varepsilon_\beta(m) \in \mathcal{M}^{\beta,\varepsilon}_m$, and moreover $\mathcal{M}^{\beta,\varepsilon}_m$ is an s–dimensional smooth submanifold of M.

Lemma 3.11 *One has* $\mathcal{M}_m^{\beta,\varepsilon} \subset \Lambda_\beta^\varepsilon$.

Proof. In the proof we drop the index ε which is assumed to be fixed and different from zero. Let $p \in \mathcal{M}_m^\beta$, $p \neq \sigma_\beta(m)$, then there exists a unique m_1 such that $p \in \Sigma_{m_1}$. Consider $\widehat{p} := \Phi(p)$. Using adapted coordinates based at m, if $p = (\beta, \tau, y)$ then, by definition of $\mathcal{M}_m^\beta$, the coordinates of $\hat{p}$ are $(\beta, \widehat{\tau}, y)$. From this it follows $g^{-(\widehat{\tau}-\tau)}(\widehat{p}) = p$. Introduce adapted coordinates (β, τ_1, y_1) based at the point m_1; one has $p = (\beta, 0, y_1)$, and denote

$$\widehat{p} = (\beta, \widehat{\tau}_1, \widehat{y}_1) \ .$$

Applying $g^{-(\widehat{\tau}-\tau)}$ we obtain

$$(\beta, \widehat{\tau}_1 - (\widehat{\tau} - \tau), \widehat{y}_1) = g^{-(\widehat{\tau}-\tau)}(\widehat{p}) = p = (\beta, 0, y_1)$$

from which in particular one has $y_1 = \widehat{y}_1(\beta, y_1)$. By uniqueness of the solution to $y_1 = \widehat{y}(\beta, y_1)$, it follows $y_1 = \rho_{m_1}(\beta)$ and therefore $p = \sigma_\beta(m_1)$. $\triangle$

Remark 6. The sets $\mathcal{M}_m^{\beta,\varepsilon}$ are a covering of $\Lambda_\beta^\varepsilon$: indeed $\mathcal{M}_m^{\beta,\varepsilon}$ contains at least $\sigma_\beta^\varepsilon(m)$.

Lemma 3.12 $\Lambda_\beta^\varepsilon$ *is a smooth compact manifold,* C^r*-diffeomorphic to* Λ.

Proof. Also in this proof we drop the index ε. First remark that the map

$$\psi_m^\beta : \mathcal{U}_m \to \mathcal{M}_m^\beta \quad , \quad \psi_m^\beta(\beta_0, \tau, 0) = (\beta, \tau, \rho_m(\beta))$$

(where we used adapted coordinates based at m) is smooth from a neighbourhood $\mathcal{U}_m$ of m in Λ to a neighbourhood of $\sigma_\beta(m)$ in Λ_β. Use this map to introduce the τ as local coordinates in Λ_β. Then the maps ψ_m^β yield a C^r atlas of Λ_β; the transition functions between the coordinate system τ on $\mathcal{M}_m^\beta$ and the coordinate system τ_1 on $\mathcal{M}_{m_1}^\beta$ are given simply by the functions $\tau_1(\beta, \tau, \rho_m(\beta))$, which are C^r-smooth. It follows that Λ_β is a C^r manifold; it is also C^r diffeomorphic (by the map σ_β) to Λ, and therefore is compact. $\triangle$

Remark 7. By construction the fields X_i^ε are tangent to each of the $\mathcal{M}_m^{\beta,\varepsilon}$, and therefore the manifold $\Lambda_\beta^\varepsilon$ is invariant under the flows of all the X_i^ε.

Lemma 3.13 *The* C^r *manifold* $\widetilde{N}_\varepsilon$ *obtained as the union of* $\Lambda_\beta^\varepsilon$ *for* $|\beta - \beta_0| < \beta_*$ *and* ε *fixed is symplectic and fibered in isotropic tori* $\mathbf{T}^s$.

Proof. It is clear that $\widetilde{N}_\varepsilon$ is C^r and that it is fibered in tori $\mathbf{T}^s \equiv \Lambda_\beta^\varepsilon$. The tori $\Lambda_\beta^\varepsilon$ are integral manifolds of the Hamiltonian vector fields generated by the functions F_i^ε, and lie in common level manifolds of the F_i^ε; thus we can choose a basis of variables τ_i along $\Lambda_\beta^\varepsilon$ such that $\Omega(\partial/\partial F_i^\varepsilon, \partial/\partial\tau_j) = \delta_{ij}$. This shows at once that the $\Lambda_\beta^\varepsilon$ are isotropic and that the restriction $\Omega_{\widetilde{N}_\varepsilon}$ of the symplectic form Ω defined on M to the submanifold $\widetilde{N}_\varepsilon \subset M$ is non-degenerate. As $\mathrm{d}\Omega = 0$ implies $\mathrm{d}\Omega_{\widetilde{N}_\varepsilon} = 0$, the proof is complete. $\triangle$

Corollary 3.14 *There exist a $2s$ dimensional symplectic invariant submanifold N_ε, $\Lambda \subset N_\varepsilon \subset \widetilde{N}_\varepsilon$, and action angle coordinates on N_ε.*

Proof. The restriction of the system to $\widetilde{N}_\varepsilon$ is integrable in the Arnold Liouville sense. Thus the standard construction of action angle coordinates for integrable systems applies [1] and allows to construct action angle coordinates in a neighbourhood N_ε of $\Lambda^\varepsilon_{\beta_0}$ in $\widetilde{N}_\varepsilon$. $\triangle$

This concludes the proof of theorem 2.3.

4 Reducible tori

The nondegeneracy condition of theorem 2.3 takes a particularly useful form in the case of reducible tori that we are going to discuss.

Under the assumptions (i,ii) of theorem 2.3 assume also that there exists a system of canonical coordinates (I, φ, p, q), with $\varphi \in \mathbf{T}^s$ and (I, p, q) defined in a subset of $\mathbf{R}^s \times \mathbf{R}^{n-s} \times \mathbf{R}^{n-s}$, in which the functions F_i^0 take the form

$$F_i^0 \ = \ \sum_{j=1}^{s} \omega_j^{(i)} I_j \ + \ \sum_{j=1}^{r} \nu_j^{(i)} \left(\frac{p_j^2 + q_j^2}{2} \right) + \texttt{h.o.t.}$$

where $\texttt{h.o.t.}$ denotes higher order terms, i.e. terms which are at least quadratic in (I, p, q) if they depend on I, and terms which are independent of I and at least cubic in p, q, and Λ is the manifold $p = q = I = 0$, and $r := n - s$. *If such coordinates exist the torus Λ is said to be reducible.*

Concerning reducible tori we recall the following

Proposition 4.15 *(**Kuksin**[7]) Under the assumptions (i,ii) of theorem 2.3, let $\alpha^{(j)}$, $j = 1, ..., s$ be a basis of $\pi_1(\Lambda)$; assume that for all j all the Floquet multipliers of the periodic orbits of the fields $X_{\alpha^{(j)}}$ have modulus one, then Λ is reducible.*

We define, for ease of notation, matrices A and B with elements given by

$$A_{ij} \ := \ \omega_j^{(i)} \ \ ; \ \ B_{ij} = \nu_j^{(i)} \ .$$

Hence A is a $(s \times s)$ matrix built with the frequencies of motion in the invariant torus Λ, and B is a $(s \times r)$ matrix built with the frequencies of small oscillations in the transversal directions to the invariant torus.

We will denote by $\Omega(k; j)$ the matrix obtained from A by substituting its k-th column with the j-th column of B. For ease of notation, we also write $P := (A^T)^{-1}$. We will also denote by $|M|$ the determinant of a matrix M.

Theorem 4.16 *Let X_α be the vector field $X_\alpha = \sum_i c_i X_i$ having periodic orbits γ_α on Λ with period one and in the homotopy class $\alpha \in \mathbf{Z}^s$. Then:*

(1) *The condition that the multiplicity of the Floquet multiplier 1 of γ_α does not exceed $2s$ is equivalent to $Q_j(\alpha) \notin \mathbf{Z}$, where*

$$Q_j(\alpha) \ = \ \sum_{i,k=1}^{s} B^T_{ji} P_{ik} \alpha_k \ . \tag{4.17}$$

(2) *Condition (1) is equivalent to*

$$\sum_{k=1}^{s} \alpha_k \, |\Omega(k;j)| \ \neq \ m \ |A| \qquad \forall m \in \mathbf{Z} \ , \quad \forall j = 1, ..., r \ . \tag{4.18}$$

We stress that both the conditions given in the theorem can be checked once we know the matrices A and B; however condition 4.17 requires to consider only one matrix $B^T (A^T)^{-1}$ but also to perform the inversion of the $(s \times s)$ matrix A, while condition 4.18 requires to consider $s \cdot (n-s)$ matrices $\Omega(k,j)$, but does not require to consider any inversion of matrices. Thus it can be more convenient to use one or the other of them depending on the problem at hand; however condition 4.18 can always be explicitly checked and requires only simple linear algebra. As far as we know, condition 4.18 has not been considered before.

Remark 8. In the case $s=1$ condition 4.18 reduces to

$$\exists \alpha \in \mathbf{Z} \ : \ \alpha \nu_k \neq m \omega_1 \ , \ \forall m \in \mathbf{Z} \ , \ \forall k = 2, ..., n \ , \tag{4.19}$$

which is easily seen to be equivalent to the standard condition under which the Lyapunov center theorem holds, namely that $\nu_k / \omega_1 \notin \mathbf{Z}$, $\forall k = 2, ..., n$. Indeed, if $\nu_k/\omega_1 \in \mathbf{Z}$ then, for any choice of $\alpha \in \mathbf{Z}$ one has that $\alpha \nu_k / \omega_1 \in \mathbf{Z}$, and therefore 4.19 is also violated. Conversely, if $\nu_k/\omega_1 \notin \mathbf{Z}$ then simply choose $\alpha = 1$ and 4.19 also holds. $\odot$

Proof of theorem 4.16. In this section we will drop from F^0_i the higher order terms which do not change the linearized dynamics at the torus, we will drop also the index ε since here we are only interested in the case $\varepsilon = 0$.

Fix $\alpha \in \pi_1(\Lambda)$; first of all we determine the vector field X_α, to this end we remark that the projection Z_c of a general vector field $X_c = \sum_{i=1}^s c_i X_i$ to invariant tori is just

$$Z_c \ = \ \sum_{i=1}^{s} \sum_{k=1}^{s} c_i \, A_{ik} \partial_k \ \ ,$$

where $\partial_k := \frac{\partial}{\partial \varphi_k}$. When we require that this has closed orbits with period 1 and winding number α_i around the cycles of Λ, we are requiring

$$Z_c \ = \ \sum_{k=1}^{s} (2\pi) \, \alpha_k \, \partial_k \ .$$

Thus we get (recall $P := (A^T)^{-1}$)

$$c_i \;=\; \sum_{k=1}^{s} (2\pi)\, P_{ij}\, \alpha_j ,$$

which explicitly defines $X_\alpha := \sum_{j=1}^s c_i X_i$.

In order to compute the transversal Floquet multipliers we have to study the dynamics of X_α in the transversal direction. In particular the flow generated by X_α in the planes p_j, q_j $(j = 1, ..., r)$ takes the form

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix} \mapsto \begin{pmatrix} p_j \cos(2\pi\, Q_j(\alpha)\, t) - q_j \sin(2\pi\, Q_j(\alpha)\, t) \\ q_j \cos(2\pi\, Q_j(\alpha)\, t) + q_j \sin(2\pi\, Q_j(\alpha)\, t) \end{pmatrix} ,$$

This shows that the transversal Floquet exponents are indeed the $Q_j(\alpha)$ and therefore condition (iii) of theorem 2.3 is equivalent to $Q_j(\alpha) \notin \mathbf{Z}$.

It remains to prove (2). Let us consider the matrix

$$M \;:=\; B^T\, P \;\equiv\; B^T (A^T)^{-1}$$

and the matrices $\Omega(k,j)$ introduced above. We have to show that the elements m_{kj} of M can be written as a ratio of determinants, $m_{kj} = |\Omega(k,j)|/|A|$.

Let us introduce a useful notation: given a matrix M of elements m_{ij}, we denote by $\widetilde{M}$ the matrix of its algebraic complements. We recall that $\sum_j m_{ij} (\widetilde{m})_{kj} = \delta_{ik} |M|$, that $(\widetilde{M^T}) = \left(\widetilde{M}\right)^T$, and that by Cramer's theorem M^{-1} is obtained as $(M^{-1})_{ij} = |M|^{-1} \left(\widetilde{M}\right)^T$.

For ease of writing, we will think of k and j as fixed and write simply R for $[\Omega(k,j)]^T$. One has $|R| = \sum_\ell R_{i\ell} (\widetilde{R})_{i\ell}$, for any $i = 1, ..., s$; we consider $i = k$ and compute the determinant in this way. By definition, however, $R_{k\ell} = (B^T)_{j\ell}$. As for $(\widetilde{R})_{k\ell}$, we note that R differs from A^T only on the row k; thus the k-th rows of $\widetilde{R}$ and of $\widetilde{A^T}$ are identical, i.e. $\widetilde{R}_{k\ell} = \widetilde{A^T}_{k\ell} = \widetilde{A}_{\ell k}$.

In this way we obtain that

$$|R| \;=\; \sum_\ell R_{k\ell}\, (\widetilde{R})_{k\ell} \;=\; \sum_\ell (B^T)_{j\ell}\, \widetilde{A}_{\ell k} \;=\; \sum_\ell (B^T)_{j\ell}\, [(A^T)^{-1}]_{\ell k}\, (|A|) \; .$$

Using $|R^T| = |R|$ and writing again in full $R^T \equiv \Omega(k,j)$, we have shown that

$$M_{jk} \;:=\; (B^T P)_{jk} \;\equiv\; (B^T (A^T)^{-1})_{jk} \;=\; \frac{|\Omega(k,j)|}{|A|} \; ,$$

hence that $Q_j(\alpha) = \sum_k \alpha_k (|\Omega(k,j)|/|A|)$; this completes the proof of (2) and thus of the theorem. $\triangle$

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