M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 5, 1999

Paper 2
Received: Oct 19, 1998, Accepted: Apr 8, 1999
Editor: P. Collet

Eigenvalue Asymptotics for the Dirac Operator in Strong Constant Magnetic Fields

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Abstract. We consider the three-dimensional Dirac operator H with constant magnetic field and electric potential which decays at infinity. We study the asymptotic behaviour of the discrete spectrum of H as the norm of the magnetic field grows unboundedly.

1 Introduction

Let $H_0(b)$ be the three-dimensional Dirac operator in constant magnetic field $B = (0, 0, b)$, $b > 0$. Choosing an appropriate gauge, system of units, and coordinates, we can write

$$H_0(b) = \sum_{j=1,2,3} \alpha_j \Pi_j(b) + \beta$$

where

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \; j = 1, 2, 3, \; \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

σ_j, $j = 1, 2, 3$, are the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

I_2 is the unit 2×2 matrix, Π_j, $j = 1, 2, 3$, are the components of the extended momentum

$$\Pi_1 = \Pi_1(b) := -i\frac{\partial}{\partial x} + \frac{by}{2}, \Pi_2 = \Pi_2(b) := -i\frac{\partial}{\partial y} - \frac{bx}{2}, \Pi_3 := -i\frac{\partial}{\partial z},$$

and $X = (x, y, z) \in \mathbf{R}^3$. It is well-known that for each $b \geq 0$ we have

$$\sigma(H_0(b)) = \sigma_{\rm ess}(H_0(b)) = (-\infty, -1] \cup [1, +\infty). \tag{1.1}$$

Further, let $V : \mathbf{R}^3 \to \mathbf{R}$ be the electric (scalar) potential. We shall say that V is in the class $\mathcal{L}$ if and only if for each $\varepsilon > 0$ it can be written as $V = V_1 + V_2$ with $V_1 \in L^3(\mathbf{R}^3)$, and $\sup_{X \in \mathbf{R}^3} |V_2(X)| \leq \varepsilon$. Throughout the paper we assume $V \in \mathcal{L}$, unless more restrictive assumptions are imposed.
In particular, $V \in \mathcal{L}$ entails the compactness of the operator $VH_0(b)^{-1}$. Set

$$H(b) := H_0(b) + VI_4 = H_0(b) + V$$

where I_4 is the unit 4×4 matrix. Since the operator $VH_0(b)^{-1}$ is compact, we have $\sigma_{\rm ess}(H(b)) = \sigma_{\rm ess}(H_0(b))$, and hence (1.1) implies

$$\sigma_{\rm ess}(H(b)) = (-\infty, -1] \cup [1, +\infty).$$

However, the discrete spectrum of the operator $H(b)$ might be non-empty. The aim of the present paper is to investigate the asymptotic distribution as $b \to \infty$ of the eigenvalues of $H(b)$ lying in the gap $(-1, 1)$ of its essential spectrum.

2 Statement of the main result

Let $T = T^*$ be a selfadjoint operator in a Hilbert space. Denote by $P_{\mathcal{I}}(T)$ its spectral projection corresponding to the interval $\mathcal{I} \subset \mathbf{R}$. Set

$$\mathcal{N}(\lambda_1, \lambda_2; T) = \mathrm{rank}\, P_{(\lambda_1,\lambda_2)}(T), \ \lambda_1, \lambda_2 \in \mathbf{R}, \ \lambda_1 < \lambda_2,$$

$$N(\lambda; T) := \mathrm{rank}\, P_{(-\infty,\lambda)}(T), \ \lambda \in \mathbf{R},$$

$$n_{\pm}(s; T) := \mathrm{rank}\, P_{(s,+\infty)}(\pm T), s > 0.$$

If T is a linear compact operator which is not necessarily selfadjoint, put

$$n_*(s; T) := \mathrm{rank}\, P_{(s^2,+\infty)}(T^*T), s > 0.$$

In what follows if $X = (x, y, z) \in \mathbf{R}^3$ we shall write occasionally $X = (X_\perp, z)$ where $X_\perp = (x, y)$ are the variables on the plane perpendicular to the magnetic field $B = (0, 0, b)$, while z is the variable along B. Fix $X_\perp \in \mathbf{R}^2$ and set

$$\chi(X_\perp) := \chi_0 + V(X_\perp, .)\, I_2$$

where

$$\chi_0 := \begin{pmatrix} 1 & -i\frac{d}{dz} \\ -i\frac{d}{dz} & -1 \end{pmatrix}.$$

Proposition 2.1 *Let $V \in \mathcal{L}$. Then for almost every $X_\perp \in \mathbf{R}^2$ the operator $\chi(X_\perp)$ is defined as an operator sum selfadjoint in $L^2(\mathbf{R}; \mathbf{C}^2)$. Moreover, for almost every $X_\perp \in \mathbf{R}^2$ the operator $V(X_\perp, .)\chi_0^{-1}$ is compact and, therefore,*

$$\sigma_{\mathrm{ess}}(\chi(X_\perp)) = \sigma_{\mathrm{ess}}(\chi_0) = (-\infty, -1] \cup [1, +\infty).$$

The proof of the proposition is contained in Section 7.
Let λ_1 and λ_2 be real numbers such that $-1 < \lambda_1 < \lambda_2 < 1$. Introduce the magnetic integrated density of states

$$\mathcal{D}(\lambda_1, \lambda_2) = \mathcal{D}_V(\lambda_1, \lambda_2) := \int_{\mathbf{R}^2} \mathcal{N}(\lambda_1, \lambda_2; \chi(X_\perp))\, dX_\perp.$$

Proposition 2.2 *Let $V \in \mathcal{L}$, $\lambda_1, \lambda_2 \in \mathbf{R}$, $-1 < \lambda_1 < \lambda_2 < 1$. Then $\mathcal{D}_V(\lambda_1, \lambda_2) < \infty$.*

The proof of this proposition can also be found in Section 7.
We shall say that a point $\lambda \in (-1, 1)$ is regular if and only if

$$\mathrm{vol}\ \left\{X_\perp \in \mathbf{R}^2 | \, \dim \mathrm{Ker}\, (\chi(X_\perp) - \lambda) \geq 1\right\} = 0.$$

Note that λ_1 (respectively, λ_2) is a regular point if and only if $\lim_{\varepsilon \to 0} \mathcal{D}(\lambda_1 + \varepsilon, \lambda_2) = \mathcal{D}(\lambda_1, \lambda_2)$ (respectively, $\lim_{\varepsilon \to 0} \mathcal{D}(\lambda_1, \lambda_2 + \varepsilon) = \mathcal{D}(\lambda_1, \lambda_2)$).

Theorem 2.1 *Let $V \in \mathcal{L}$, $\lambda_1, \lambda_2 \in \mathbf{R}$, $-1 < \lambda_1 < \lambda_2 < 1$. Assume that the points λ_1 and λ_2 are regular. Then we have*

$$\lim_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1, \lambda_2; H(b)) = \frac{1}{2\pi}\mathcal{D}(\lambda_1, \lambda_2). \tag{2.1}$$

The present paper could be regarded as a supplement to [7] where strong-magnetic-field spectral asymptotics for the Schrödinger and Pauli operators have been considered. The methods applied here are close to the ones used in [7]. However, the Dirac operator $H(b)$ studied in this paper as well as the auxiliary operator $\chi(X_\perp)$ are not semibounded in contrast to the Schrödinger and Pauli operators. This additional difficulty is overcome by the application of a simple but yet non-trivial generalization of the well-known Birman-Schwinger principle.
Various types of spectral properties and, in particular, eigenvalue asymptotics for the Dirac operator with or without magnetic field have been studied in [8], [3], [4], [5], [6]. However, the asymptotic behaviour as $b \to \infty$ of $\mathcal{N}(\lambda_1, \lambda_2; H(b))$ has never been investigated before.
The paper is organized as follows. The next four brief sections contain auxiliary results. A formulation of the Kac-Murdock-Szegö theorem borrowed from [7], can be founded in Section 3. Section 4 is devoted to the generalization of the Birman-Schwinger principle concerning the number of the eigenvalues situated in a gap of the essential spectrum of a selfadjoint operator. In Section 5 we describe certain spectral properties of the unperturbed operator $H_0(b)$. In Section 6 we perform some preliminary estimates. Finally, Propositions 2.1-2.2 are proved in Section 7, and Theorem 2.1 – in Section 8.

3 The Kac-Murdock-Szegö theorem

In this section we follow closely the exposition of [7, Subsection 3.1]. For the reader's convenience, we reproduce a suitable version of the Kac-Murdock-Szegö theorem whose proof can be found in [7, Subsection 3.1].
In the sequel we shall denote by S_∞ the space of linear compact operators acting in a given Hilbert space, and by S_p, $p \in [1, \infty)$, – the Schatten–von Neumann spaces of operators $T \in S_\infty$ for which the norm $\|T\|_p := (\mathrm{Tr}\ |T|^p)^{1/p}$ is finite.
Moreover, we shall say that the function ν defined on $\mathbf{R} \setminus \{0\}$ is in the class $\mathcal{C}$ if it is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$, non-negative on $(-\infty, 0)$, and non-positive on $(0, \infty)$.

Lemma 3.1 *Let $\{T(b)\}_{b>0}$ be a family of selfadjoint compact operators satisfying the estimate $\|T(b)\| \le t_0$ with $t_0 > 0$ independent of b. Let $\nu \in \mathcal{C}$. Assume that $\nu(t) = 0$ for $|t| > t_0$. Suppose that there exists a real $p \ge 1$ such that the following three conditions are fulfilled:*
(i) *$T(b) \in S_p$ for each $b > 0$;*

(ii) *the quantity* $\int_{\mathbf{R}\setminus\{0\}} |t|^p \, d\nu(t)$ *is finite;*
(iii) *the limiting relations*

$$\lim_{b\to\infty} b^{-1} \operatorname{Tr} T(b)^l = \int_{\mathbf{R}\setminus\{0\}} t^l \, d\nu(t)$$

hold for each integer $l \geq p$.
Let $t \neq 0$ *be a continuity point of* ν. *Then we have*

$$\lim_{b\to\infty} b^{-1} \, n_-(-t; T(b)) = \nu(t) \quad \text{if} \quad t < 0,$$

$$\lim_{b\to\infty} b^{-1} \, n_+(t; T(b)) = -\nu(t) \quad \text{if} \quad t > 0.$$

Remark. We shall use Lemma 3.1 only with $t < 0$.

4 The generalized Birman-Schwinger principle

One of the versions of the classical Birman-Schwinger principle (cf. [2, Lemma 1.1]) says that if $\mathcal{H}_0 = \mathcal{H}_0^* \geq 0$, $\mathcal{V} = \mathcal{V}^*$, and $|\mathcal{V}|^{1/2}(\mathcal{H}_0 + 1)^{-1/2} \in S_\infty$, then for each $\lambda > 0$ we have

$$N(-\lambda; \mathcal{H}_0 + \mathcal{V}) = n_-(1; (\mathcal{H}_0 + \lambda)^{-1/2}\mathcal{V}(\mathcal{H}_0 + \lambda)^{-1/2}) \tag{4.1}$$

where the sum $\mathcal{H}_0 + \mathcal{V}$ should be understood in the quadratic-forms sense.
Lemma 4.1 below contains a generalization of (4.1) to the case where $\mathcal{H}_0$ is not necessarily semibounded.
Related arguments in the special case where $\mathcal{H}_0$ coincides with the free Dirac operator have already appeared in [2, Section 5]. Much later arguments of this type have been employed in [4] and [5].
Let $\mathcal{H}_0$ be a linear operator selfadjoint in the Hilbert space H. Assume $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 < \lambda_2$, $[\lambda_1, \lambda_2] \subset \rho(\mathcal{H}_0)$.
Set $\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0) := ((\mathcal{H}_0 - \lambda_1)(\mathcal{H}_0 - \lambda_2))^{-1/2}$. Since $[\lambda_1, \lambda_2]$ is in the resolvent set of $\mathcal{H}_0$ the operator $(\mathcal{H}_0 - \lambda_1)(\mathcal{H}_0 - \lambda_2)$ is positive-definite, and hence the operator $\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)$ is well-defined and bounded. Set

$$\mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0) := \left(\mathcal{H}_0 - \frac{1}{2}(\lambda_1 + \lambda_2)\right) \mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0).$$

Evidently, $\mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0)$ is bounded. Further, let $\mathcal{V}$ be a symmetric operator on $D(\mathcal{H}_0)$ such that $\mathcal{V}(\mathcal{H}_0 + i)^{-1} \in S_\infty$, which is equivalent to $\mathcal{V}\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0) \in S_\infty$. Set

$$\mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}) :=$$

$$\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)\mathcal{V}^2\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0) + 2\mathrm{Re}\, \mathcal{G}(\lambda_1, \lambda_2; \mathcal{H}_0)\mathcal{V}\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0).$$

Obviously, $\mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}) \in S_\infty$.

Lemma 4.1 *Let $\mathcal{H}_0$ be a linear operator selfadjoint in the Hilbert space* H, $\lambda_1, \lambda_2 \in \mathbf{R}$, $\lambda_1 < \lambda_2$, *and* $[\lambda_1, \lambda_2] \subset \rho(\mathcal{H}_0)$. *Let $\mathcal{V}$ be a symmetric operator on $D(\mathcal{H}_0)$ such that $\mathcal{V}(\mathcal{H}_0 + i)^{-1} \in S_\infty$. Then we have*

$$\mathcal{N}(\lambda_1, \lambda_2; \mathcal{H}_0 + \mathcal{V}) = n_-(1; \mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V})) \tag{4.2}$$

where the sum $\mathcal{H}_0 + \mathcal{V}$ should be understood in the operator sense.

Proof. Obviously,

$$\mathcal{N}(\lambda_1, \lambda_2; \mathcal{H}_0 + \mathcal{V}) = N\left(\frac{1}{4}(\lambda_1 - \lambda_2)^2; (\mathcal{H}_0 + \mathcal{V} - \frac{1}{2}(\lambda_1 + \lambda_2))^2\right). \tag{4.3}$$

The minimax principle implies that the quantity at the right-hand side of (4.3) is equal to the maximal dimension of the linear subsets of $D(\mathcal{H}_0)$ whose non-zero elements u satisfy the inequality

$$\left\|\mathcal{H}_0 u + \mathcal{V}u - \frac{1}{2}(\lambda_1 + \lambda_2)u\right\|^2 < \frac{1}{4}(\lambda_1 - \lambda_2)^2\|u\|^2$$

where $\|.\|$ denotes the norm in H. This last inequality can be re-written as

$$\|\sqrt{(\mathcal{H}_0 - \lambda_1)(\mathcal{H}_0 - \lambda_2)}u\|^2 < -\|\mathcal{V}u\|^2 - 2\mathrm{Re}\left\langle\left(\mathcal{H}_0 - \frac{1}{2}(\lambda_1 + \lambda_2)\right)u, \mathcal{V}u\right\rangle \tag{4.4}$$

where $\langle ., . \rangle$ denotes the inner product in H. Note that the operator $\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)$ maps bijectively H on $D(\mathcal{H}_0)$. Set $u = \mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)w$, $w \in$ H, in (4.4). Hence, (4.4) is equivalent to

$$\begin{aligned}\|w\|^2 < -\|\mathcal{V}\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)w\|^2 - \\ 2\mathrm{Re}\left\langle\left(\mathcal{H}_0 - \frac{1}{2}(\lambda_1 + \lambda_2)\right)\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)w, \mathcal{V}\mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)w\right\rangle = \\ -\langle \mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V})w, w\rangle.\end{aligned} \tag{4.5}$$

By (4.3), the quantity $\mathcal{N}(\lambda_1, \lambda_2; \mathcal{H}_0 + \mathcal{V})$ coincides with the maximal dimension of the subspaces of H whose non-zero elements w satisfy (4.5). By the minimax principle this maximal dimension equals $n_-(1; \mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}))$. Hence, (4.2) is valid. □

In this paper we shall apply Lemma 4.1 in the case $\mathcal{H}_0 = H_0(b)$, $\mathcal{V} = V$, and $[\lambda_1, \lambda_2] \subset (-1, 1)$. Set

$$R(\lambda_1, \lambda_2) \equiv R_b(\lambda_1, \lambda_2) := \mathcal{R}(\lambda_1, \lambda_2; H_0(b)), \tag{4.6}$$

$$G(\lambda_1, \lambda_2) \equiv G_b(\lambda_1, \lambda_2) := \mathcal{G}(\lambda_1, \lambda_2; H_0(b)), \tag{4.7}$$

$$K(\lambda_1, \lambda_2) \equiv K_b(\lambda_1, \lambda_2) := \mathcal{K}(\lambda_1, \lambda_2; H_0(b), V). \tag{4.8}$$

By analogy with (4.6) and (4.7) introduce the operators $\varrho(\lambda_1, \lambda_2)$ and $\gamma(\lambda_1, \lambda_2)$ replacing $H_0(b)$ by χ_0. Similarly, fix $X_\perp \in \mathbf{R}^2$ such that the operator $\chi(X_\perp)$ is well-defined,

and $V(X_\perp, .)\chi_0^{-1} \in S_\infty$, and define the operator $\kappa(X_\perp) \equiv \kappa(\lambda_1, \lambda_2; X_\perp)$ substituting in (4.8) the operator $H_0(b)$ for χ_0, and V for $V(X_\perp, .)$.
Applying (4.2), we obtain

$$\mathcal{N}(\lambda_1, \lambda_2; H(b)) = n_-(1; K_b(\lambda_1, \lambda_2)), \tag{4.9}$$

$$\mathcal{D}(\lambda_1, \lambda_2) = \int_{\mathbf{R}^2} n_-(1; \kappa(\lambda_1, \lambda_2; X_\perp))\, dX_\perp. \tag{4.10}$$

For further references we formulate here a lemma which is closely related to the generalized Birman–Schwinger principle.

Lemma 4.2 *Let the operators $\mathcal{H}_0$ and $\mathcal{V}$ and the numbers $\lambda_1,\ \lambda_2 \in \mathbf{R}$ satisfy the hypotheses of Lemma 4.1. Then the spectrum of $\mathcal{H}_0 + \mathcal{V}$ contains at least one of the points λ_1 and λ_2 if and only if the operator $\mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V})$ has an eigenvalue equal to -1. Moreover,*

$$\sum_{j=1,2} \dim \operatorname{Ker}(\mathcal{H}_0 + \mathcal{V} - \lambda_j) = \dim \operatorname{Ker}(\mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V}) + 1).$$

Proof. It suffices to note that the equations

$$(\mathcal{H}_0 + \mathcal{V} - \lambda_1)(\mathcal{H}_0 + \mathcal{V} - \lambda_2)u = 0,\ u \in D(\mathcal{H}_0),$$

and

$$\mathcal{K}(\lambda_1, \lambda_2; \mathcal{H}_0, \mathcal{V})w + w = 0,\ w \in \mathrm{H},$$

are equivalent for $u = \mathcal{R}(\lambda_1, \lambda_2; \mathcal{H}_0)w,\ w \in \mathrm{H}$. $\square$

Corollary 4.1 *Let $\lambda_1, \lambda_2 \in \mathbf{R}$, $-1 < \lambda_1 < \lambda_2 < 1$. Then λ_1 and λ_2 are simultaneously regular points if and only if*

$$\operatorname{vol}\left\{X_\perp \in \mathbf{R}^2 | \dim \operatorname{Ker}(\kappa(\lambda_1, \lambda_2; X_\perp) + 1) \geq 1\right\} = 0.$$

5 The ground-levels projection

The unperturbed Hamiltonian can be written as

$$H_0(b) = \begin{pmatrix} I_2 & F(b) \\ F(b) & -I_2 \end{pmatrix}$$

where

$$F(b) := \sum_{j=1}^{3} \sigma_j \Pi_j(b) = \begin{pmatrix} \Pi_3 & a(b) \\ a(b)^* & -\Pi_3 \end{pmatrix},$$

$$a(b) := \Pi_1(b) - i\Pi_2(b),\ a(b)^* := \Pi_1(b) + i\Pi_2(b).$$

The commutation relation $[\Pi_1(b), \Pi_2(b)] = ib$ implies

$$a(b)a(b)^* = \Pi_1(b)^2 + \Pi_2(b)^2 - b, \; a(b)^*a(b) = \Pi_1(b)^2 + \Pi_2(b)^2 + b.$$

Therefore $F(b)^2$ coincides with the Pauli operator

$$F(b)^2 = \left(\sum_{j=1}^{3} \sigma_j \Pi_j(b) \right)^2 = \begin{pmatrix} \Pi(b)^2 - b & 0 \\ 0 & \Pi(b)^2 + b \end{pmatrix} \tag{5.1}$$

where

$$\Pi(b)^2 := \sum_{j=1,2,3} \Pi_j^2.$$

Moreover,

$$H_0(b)^2 = \begin{pmatrix} F(b)^2 + I_2 & 0 \\ 0 & F(b)^2 + I_2 \end{pmatrix}. \tag{5.2}$$

Define the orthogonal projection p_b by

$$(p_b u)(x, y, z) = \int_{\mathbf{R}^2} \mathcal{P}_b(x, y; x', y') u(x', y', z) \, dx' dy', \; u \in L^2(\mathbf{R}^3), \tag{5.3}$$

where

$$\mathcal{P}_b(x, y; x', y') := \frac{b}{2\pi} \exp\left\{ -\frac{b}{4} \left[(x - x')^2 + (y - y')^2 + 2i(xy' - yx') \right] \right\}. \tag{5.4}$$

It is essential that $\mathcal{P}_b$ is the integral kernel of the orthogonal projection on $\operatorname{Ker} a(b)^*$ $= \operatorname{Ker} a(b)a(b)^* \subset L^2(\mathbf{R}^2)$. Evidently, p_b commutes with Π_3.
On $L^2(\mathbf{R}^3, \mathbf{C}^4)$ introduce the orthogonal projection

$$P_b := \begin{pmatrix} p_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

Obviously, P_b commutes with Π_3 and H_0. Moreover, if $u = (u_1, u_2, u_3, u_4) \in D(H_0)$, we have

$$H_0 P_b u = \begin{pmatrix} 1 & 0 & \Pi_3 & 0 \\ 0 & 0 & 0 & 0 \\ \Pi_3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_b u_1 \\ 0 \\ p_b u_3 \\ 0 \end{pmatrix}. \tag{5.6}$$

Put

$$Q_b := \mathrm{Id} - P_b. \tag{5.7}$$

On $\{D(\Pi_3)\}^2 \subset L^2(\mathbf{R}^3;\mathbf{C}^2)$ introduce the operator

$$h_0 := \begin{pmatrix} 1 & \Pi_3 \\ \Pi_3 & -1 \end{pmatrix}. \tag{5.8}$$

Evidently, $\sigma(h_0) = \sigma_{\rm ess}(h_0) = (-\infty,-1]\cup[1,+\infty)$. Note that if we replace Π_3 by $-i\frac{d}{dz}$ in (5.8), we shall obtain the operator χ_0.
Define the operators r and g substituting $H_0(b)$ for h_0 respectively in (4.6) and (4.7). Taking into account (5.6), we find that the spectral theorem for selfadjoint operators entails the following lemma.

Lemma 5.1 *The restrictions of the operators $H_0(b)$ (respectively, R_b and G_b) on $P_bD(H_0(b))$ (respectively, $P_bL^2(\mathbf{R}^3;\mathbf{C}^4)$) are unitarily equivalent to the restrictions of h_0 (respectively, r and g) on $P_bD(h_0)$ (respectively, $p_bL^2(\mathbf{R}^3;\mathbf{C}^2)$).*

6 Preliminary estimates

Lemma 6.1 *Let $\lambda_1,\ \lambda_2\in\mathbf{R}$, $-1<\lambda_1<\lambda_2<1$. Then the estimates*

$$c_1\||H_0(b)|^{-1}u\|^2 \le \|R_b(\lambda_1,\lambda_2)u\|^2 \le c_2\||H_0(b)|^{-1}u\|^2, \quad \forall u\in L^2(\mathbf{R}^3;\mathbf{C}^4), \tag{6.1}$$

$$c_1\||h_0|^{-1}v\|^2 \le \|r(\lambda_1,\lambda_2)v\|^2 \le c_2\||h_0|^{-1}v\|^2, \quad \forall v\in L^2(\mathbf{R}^3;\mathbf{C}^2), \tag{6.2}$$

$$c_1\||\chi_0|^{-1}w\|^2 \le \|\varrho(\lambda_1,\lambda_2)w\|^2 \le c_2\||\chi_0|^{-1}w\|^2, \quad \forall w\in L^2(\mathbf{R};\mathbf{C}^2), \tag{6.3}$$

hold for some $c_j(\lambda_1,\lambda_2)>0$, $j=1,2$.

Proof. In order to deduce (6.1), it suffices to note that $\|R_b(\lambda_1,\lambda_2)u\|^2 = \|((H_0(b)-\lambda_1)(H_0(b)-\lambda_2))^{-1/2}u\|^2$, and the quantity $|\lambda^2(\lambda-\lambda_1)^{-1}(\lambda-\lambda_2)^{-1}|$ is bounded and strictly positive if $\lambda\in\sigma(H_0(b)) = (-\infty,-1]\cup[1,+\infty)$. Estimates (6.2) and (6.3) are completely analogous. $\Box$

Let the matrix $M(X):\mathbf{C}^4\to\mathbf{C}^4$ be defined for $X\in\mathbf{R}^3$. Denote by $|M(X)|$ the norm of $M(X)$, $X\in\mathbf{R}^3$.

Lemma 6.2 *Let $|M|\in L^p(\mathbf{R}^3)$, $p\ge 2$, $\lambda_1,\ \lambda_2\in\mathbf{R}$, $-1<\lambda_1<\lambda_2<1$. Then the estimate*

$$\|MR_bP_b\|_p^p \le bc_3\int_{\mathbf{R}^3}|M(X)|^p\,dX, \quad b>0, \tag{6.4}$$

holds with c_3 which depends on λ_1 and λ_2, but is independent of b and M.

Proof. Evidently,

$$\begin{gathered}\|MR_bP_b\| \le \|MR_b\| \le c_2\|M|H_0(b)|^{-1}\| \le \\ c_2\|\,|H_0(b)|^{-1}\|\|\,|M|\,\|_{L^\infty(\mathbf{R}^3)} = c_2\|\,|M|\,\|_{L^\infty(\mathbf{R}^3)}.\end{gathered} \tag{6.5}$$

On the other hand,

$$\|MR_bP_b\|_2^2 = \|Mp_br\|_2^2 \le c_2^2 \|\,|M|\,p_b|h_0|^{-1}\|_2^2 = 2c_2^2\|\,|M|\,p_b(\Pi_3^2+1)^{-1/2}\|_2^2. \tag{6.6}$$

Taking into account (5.3)–(5.4) and

$$((\Pi_3^2+1)^{-1/2}u)(x,y,z) = \frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{e^{i(z-z')\zeta}}{(\zeta^2+1)^{1/2}} u(x,y,z')\,d\zeta dz',$$

we get

$$\|\,|M|\,p_b(\Pi_3^2+1)^{-1/2}\|_2^2 =$$

$$\frac{b^2}{(2\pi)^3}\int_{\mathbf{R}^3}|M(x,y,z)|^2dxdydz\int_{\mathbf{R}^2} e^{-\frac{b}{2}\left((x-x')^2+(y-y')^2\right)}\,dx'dy'\int_{\mathbf{R}}\frac{d\zeta}{\zeta^2+1} =$$

$$= \frac{b}{4\pi}\int_{\mathbf{R}^3}|M(X)|^2\,dX. \tag{6.7}$$

Combining (6.6) with (6.7), we obtain

$$\|MR_bP_b\|_2^2 \le b\frac{c_2^2}{2\pi}\int_{\mathbf{R}^3}|M(X)|^2\,dX. \tag{6.8}$$

Interpolating between (6.5) and (6.8), we find that (6.4) holds with $c_3 = c_2^p/2\pi$. $\square$.

Recall that if $T \in S_p$, $p \ge 1$, then $n_*(\varepsilon;T) \le \varepsilon^{-p}\|T\|_p^p$, $\varepsilon > 0$.

Corollary 6.1 *Under the assumptions of Lemma* 6.2 *the estimate*

$$n_*(\varepsilon; MR_bP_b) \le b\,c_3\varepsilon^{-p}\int_{\mathbf{R}^3}|M(X)|^p\,dX \tag{6.9}$$

holds for every $\varepsilon > 0$ and $p \ge 2$.

Lemma 6.3 *Let $|M| \in L^3(\mathbf{R}^3)$.*
(i) *There exists a constant c_4 such that for every $\varepsilon > 0$ we have*

$$n_*(\varepsilon; MR_bQ_b) \le c_4\varepsilon^{-3}\int_{\mathbf{R}^3}|M(X)|^3\,dX. \tag{6.10}$$

(ii) *Moreover, for every $\varepsilon > 0$, λ_1, $\lambda_2 \in \mathbf{R}$, $-1 < \lambda_1 < \lambda_2 < 1$, and $|M| \in L^3(\mathbf{R}^3)$, there exists a number b_0 such that $b \ge b_0$ entails*

$$n_*(\varepsilon; MR_bQ_b) = 0. \tag{6.11}$$

Proof. By Lemma 6.1 we have

$$n_*(\varepsilon; MR_bQ_b) \leq n_*(\varepsilon c_2^{-1}; |M||H_0(b)|^{-1}Q_b), \quad \varepsilon > 0. \tag{6.12}$$

Further, (5.1)-(5.2) entail

$$|H_0(b)|^{-1}Q_b \leq (\Pi^2 + b)^{-1/2}Q_b \leq (\Pi^2 + b)^{-1/2}I_4.$$

Moreover, the operators $|H_0(b)|^{-1}$, Q_b and $(\Pi^2 + b)^{-1/2}I_4$ are pairwise commuting. Therefore, we have

$$n_*(\varepsilon c_2^{-1}; |M||H_0(b)|^{-1}Q_b) \leq 4n_*(\varepsilon c_2^{-1}; |M|(\Pi^2 + b)^{-1/2}). \tag{6.13}$$

The classical Birman-Schwinger principle (see (4.1)) entails

$$n_*(\varepsilon c_2^{-1}; |M|(\Pi^2 + b)^{-1/2}) = N(-b; \Pi^2 - c_2^2\varepsilon^{-2}|M|^2) \leq N(0; \Pi^2 - c_2^2\varepsilon^{-2}|M|^2), \tag{6.14}$$

while the magnetic version of the Cwickel-Lieb-Rozenblioum estimate implies

$$N(0; \Pi^2 - c_2^2\varepsilon^{-2}|M|^2) \leq c_5c_2^3\varepsilon^{-3}\int_{\mathbf{R}^3} |M(X)|^3\, dX \tag{6.15}$$

where c_5 is independent of M and ε (see [1, Theorem 2.15]).
Now, the combination of (6.12)–(6.15) immediately yields (6.10) with $c_4 = c_5c_2^3$.
On the other hand, by the Kato–Simon inequality we have

$$\| \, |M| \, (\Pi^2 + b)^{-1/2}\| \leq \| \, |M| \, (-\Delta + b)^{-1/2}\|. \tag{6.16}$$

Since $|M| \in L^3(\mathbf{R}^3)$, the multiplier by $|M|^2$ is -Δ-form-compact. Therefore

$$\lim_{b\to\infty} \| \, |M| \, (-\Delta + b)^{-1/2}\| = 0. \tag{6.17}$$

Fix $\varepsilon > 0$, and taking into account (6.16)–(6.17), choose b_0 so that $b \geq b_0$ entails

$$\| \, |M| \, (\Pi^2 + b)^{-1/2}\| < \varepsilon c_2^{-1}. \tag{6.18}$$

Now, (6.12) and (6.13) combined with (6.18) imply (6.11). □

Corollary 6.2 *Let $M \in L^3(\mathbf{R}^3)$. Then for every $\varepsilon > 0$ and $b > 0$ we have*

$$n_*(\varepsilon; MR_b) \leq (c_3b + c_4)\left(\frac{\varepsilon}{2}\right)^{-3}\int_{\mathbf{R}^3} |M(X)|^3\, dX. \tag{6.19}$$

Proof. Since $Q_b + P_b = \mathrm{Id}$, we have

$$n_*(\varepsilon; MR_b) = n_*(\varepsilon; MR_bP_b + MR_bQ_b) \leq n_*(\varepsilon/2; MR_bP_b) + n_*(\varepsilon/2; MR_bQ_b).$$

Applying (6.9) with $p = 3$ and (6.10), we get (6.19). □

Remark. In most cases we shall apply Lemmas 6.2 - 6.3 and Corollaries 6.1 - 6.2 with $M = V\, I_4$.

7 Proof of Propositions 2.1 – 2.2

Let $V : \mathbf{R}^3 \to \mathbf{R}$ be a measurable function. Fix $\varepsilon > 0$ and set

$$V_1(X) = V_{1,\varepsilon}(X) = \begin{cases} V(X) \text{ if } |V(X)| > \varepsilon, \\ 0 \text{ otherwise}, \end{cases} \tag{7.1}$$

$$V_2(X) = V_{2,\varepsilon}(X) = V(X) - V_{1,\varepsilon}(X). \tag{7.2}$$

It is easy to check that $V \in \mathcal{L}$ is equivalent to $V_{1,\varepsilon} \in L^3(\mathbf{R}^3)$ for all $\varepsilon > 0$. In what follows if $V \in \mathcal{L}$ and $\varepsilon > 0$ we shall choose the decomposition $V = V_1 + V_2$ with $V_1 \in L^3(\mathbf{R}^3)$ and $\sup_{X \in \mathbf{R}^3} |V_2(X)| \leq \varepsilon$, as in (7.1)–(7.2), and shall call it briefly the *ε-decomposition* of V.

Lemma 7.1 *Let $V \in \mathcal{L}$. Then for almost every $X_\perp \in \mathbf{R}^2$ the operator $V(X_\perp, .)\chi_0^{-1}$ is compact in $L^2(\mathbf{R})$.*

Proof. Fix ε and write the ε-decomposition of V. Choose $X_\perp \in \mathbf{R}^2$ so that

$$\int_{\mathbf{R}} |V_1(X_\perp, z)|^3 \, dz < \infty. \tag{7.3}$$

Evidently, the complement of the set of $X_\perp$ satisfying (7.3), is a null-set. Moreover, (7.3) implies that the operator $V_1(X_\perp, .)\chi_0^{-1}$ is compact.
Now, pick a sequence ε_n such that $\varepsilon_n > 0$, and $\lim_{n\to\infty} \varepsilon_n = 0$. Write the ε_n-decomposition of $V = V_1^{(n)} + V_2^{(n)}$ with $V_j^{(n)} := V_{j,\varepsilon_n}$, $j = 1, 2$. Fix $X_\perp \in \mathbf{R}^2$ such that $\int_{\mathbf{R}} |V_1^{(n)}(X_\perp, z)|^3 \, dz < \infty$ for all n. The complement of such $X_\perp$ is again a null-set being a countable union of null-sets. The operators $V_1^{(n)}(X_\perp, .)\chi_0^{-1}$ are compact, and we have

$$\|V_1(X_\perp, .)\chi_0^{-1} - V_1^{(n)}(X_\perp, .)\chi_0^{-1}\| = \|V_2^{(n)}(X_\perp, .)\chi_0^{-1}\| \leq \varepsilon_n.$$

Since the operator $V(X_\perp, .)\chi_0^{-1}$ can be approximated in norm by compact operators, it is a compact operator itself. □

Remark. Lemma 7.1 entails immediately Proposition 2.1.

Lemma 7.2 *Let $v \in L^p(\mathbf{R})$, $p \geq 2$. Then we have*

$$\|v\varrho\|_p^p \leq c_6 \int_{\mathbf{R}} |v(z)|^p \, dz \tag{7.4}$$

where $c_6 = c_6(p)$ is independent of v.

Proof. Applying Lemma 6.1, we get

$$\|v\varrho\|_p^p \leq c_2^p \|v|\chi_0|^{-1}\|_p^p. \tag{7.5}$$

Evidently,

$$\|v|\chi_0|^{-1}\|_p^p = 4\left\|v\left(-\frac{d^2}{dz^2}+1\right)^{-1/2}\right\|_p^p. \tag{7.6}$$

If $v \in L^\infty(\mathbf{R})$, we have

$$\left\|v\left(-\frac{d^2}{dz^2}+1\right)^{-1/2}\right\| \leq \sup_{\zeta\in\mathbf{R}}(\zeta^2+1)^{-1/2}\ \|v\|_{L^\infty(\mathbf{R})} = \|v\|_{L^\infty(\mathbf{R})}. \tag{7.7}$$

If $v \in L^2(\mathbf{R})$, we have

$$\left\|v\left(-\frac{d^2}{dz^2}+1\right)^{-1/2}\right\|_2^2 = \frac{1}{2\pi}\int_{\mathbf{R}}\frac{d\zeta}{\zeta^2+1}\ \|v\|_{L^2(\mathbf{R})}^2 = \frac{1}{2}\|v\|_{L^2(\mathbf{R})}^2. \tag{7.8}$$

Interpolating between (7.7) and (7.8), and bearing in mind (7.5) and (7.6), we find that (7.4) holds with $c_6 = 2c_2^p$. $\square$

Corollary 7.1 *Let $V \in L^3(\mathbf{R}^3)$. Then the estimate*

$$\int_{\mathbf{R}^2} n_*(\varepsilon; V(X_\perp,.)\varrho)\, dX_\perp \leq c_6\varepsilon^{-3}\int_{\mathbf{R}^3}|V(X)|^3\, dX \tag{7.9}$$

holds for each $\varepsilon > 0$ with $c_6 = c_6(3)$.

Proof. Fix $X_\perp \in \mathbf{R}^2$ for which $\int_{\mathbf{R}^2}|V(X_\perp, z)|^3\, dz < \infty$; the complement of the set of such $X_\perp$ is a null set. Applying (7.4), we get

$$n_*(\varepsilon; V(X_\perp,.)\varrho) \leq \varepsilon^{-3}\|V(X_\perp,.)\varrho\|_3^3 \leq c_6\varepsilon^{-3}\int_{\mathbf{R}}|V(X_\perp,z)|^3\, dz.$$

Integrating with respect to $X_\perp \in \mathbf{R}^2$, we get (7.9). $\square$

Corollary 7.2 *Let $V \in L^3(\mathbf{R}^3)$, $-1 < \lambda_1 < \lambda_2 < 1$. Then we have*

$$\mathcal{D}(\lambda_1,\lambda_2) \leq c_7\varepsilon^{-3}\int_{\mathbf{R}^3}|V(X)|^3\, dX \tag{7.10}$$

with $c_7 = c_7(\lambda_1,\lambda_2) = 2^4(c_2(\lambda_1,\lambda_2)\|\gamma(\lambda_1,\lambda_2)\|)^3$.

Proof. First, by (4.10) and $\varrho V(X_\perp,.)^2\varrho \geq 0$, we have

$$\mathcal{D}(\lambda_1,\lambda_2) \leq \int_{\mathbf{R}^2} n_-(1; 2\mathrm{Re}\gamma V(X_\perp,.)\varrho)dX_\perp \leq \int_{\mathbf{R}^2} n_*(1; 2\|\gamma\|V(X_\perp.,)\varrho)dX_\perp.$$

Further, Corollary 7.1 implies

$$\int_{\mathbf{R}^2} n_*(1; 2\|\gamma\|V(X_\perp,.)\varrho)dX_\perp \leq c_6 2^3\|\gamma\|^3\int_{\mathbf{R}^3}|V(X)|^3\, dX. \tag{7.11}$$

Inserting the value of c_6 into (7.11), we obtain (7.10). $\square$

Remark. Proposition 2.2 is implied almost immediately by Corollary 7.2. In order to see that, we assume that $-1 < \lambda_1 < \lambda_2 < 1$, fix $\varepsilon > 0$ such that $\lambda_1 - \varepsilon > -1$ and $\lambda_2 + \varepsilon < 1$, and write the ε-decomposition of V. Then we have

$$\mathcal{D}_V(\lambda_1, \lambda_2) \leq \mathcal{D}_{V_1}(\lambda_1 - \varepsilon, \lambda_2 + \varepsilon),$$

and by (7.10)

$$\mathcal{D}_{V_1}(\lambda_1 - \varepsilon, \lambda_2 + \varepsilon) \leq c_7(\lambda_1 - \varepsilon, \lambda_2 + \varepsilon) \int_{\mathbf{R}^3} |V_1(X)|^3 \, dX.$$

Therefore $\mathcal{D}_V(\lambda_1, \lambda_2) < \infty$.

8 Proof of Theorem 2.1

Let $-1 < \lambda_1 < \lambda_2 < 1$. Introduce the operator

$$k_b \equiv k_b(\lambda_1, \lambda_2) := r(\lambda_1, \lambda_2) p_b V^2 p_b r(\lambda_1, \lambda_2) + 2\mathrm{Re}\, g(\lambda_1, \lambda_2) p_b V p_b r(\lambda_1, \lambda_2)$$

where the operator p_b is defined by (5.3), while the operators $r(\lambda_1, \lambda_2)$ and $g(\lambda_1, \lambda_2)$ are introduced at the end of Section 5.
It is easy to check that if $V \in L^p(\mathbf{R}^3)$ then $V p_b r \in S_p$, $p \geq 2$; hence, $V p_b r$ itself as well as $r p_b V^2 p_b r$, $g p_b V p_b r$ and $r p_b V p_b g$ are Hilbert-Schmidt operators.

Proposition 8.1 *Let $V \in C_0^\infty(\mathbf{R}^3)$, $-1 < \lambda_1 < \lambda_2 < 1$. Then the asymptotic relations*

$$\lim_{b \to \infty} b^{-1} \mathrm{Tr}\, k_b(\lambda_1, \lambda_2)^l = \frac{1}{2\pi} \int_{\mathbf{R}^2} \kappa(\lambda_1, \lambda_2; X_\perp)^l \, dX_\perp$$

are valid for every integer $l \geq 2$.

Proof. Throughout the proof the parameters λ_1 and λ_2 are fixed, and we omit them in the notations.
For $l \geq 1$ write

$$k_b^l = \sum_{j=1}^{3^l} k_{j,l}(b), \ \kappa(X_\perp)^l = \sum_{j=1}^{3^l} \kappa_{j,l}(X_\perp),$$

where the terms $k_{j,l}$ and $\kappa_{j,l}$, $j = 1, \ldots, 3^l$, are defined recurrently:

$$k_{1,1}(b) := r p_b V^2 p_b r, \ k_{2,1}(b) := g p_b V p_b r, \ k_{3,1}(b) := r p_b V p_b g,$$

$$\kappa_{1,1}(X_\perp) := \varrho V(X_\perp, .)^2 \varrho, \ \kappa_{2,1}(X_\perp) := \gamma V(X_\perp, .) \varrho, \ \kappa_{3,1}(X_\perp) := \varrho V(X_\perp, .) \gamma,$$

$$k_{j,l}(b) = \begin{cases} k_{1,1}(b)\ k_{j,l-1}(b), \ j=1,\ldots,3^{l-1}, \\ k_{2,1}(b)\ k_{j,l-1}(b), \ j=3^{l-1}+1,\ldots,2.3^{l-1}, \ \ l \geq 2, \\ k_{3,1}(b)\ k_{j,l-1}(b), \ j=2.3^{l-1}+1,\ldots,3^{l}, \end{cases}$$

$$\kappa_{j,l}(X_\perp) = \begin{cases} \kappa_{1,1}(X_\perp)\ \kappa_{j,l-1}(X_\perp), \ j=1,\ldots,3^{l-1}, \\ \kappa_{2,1}(X_\perp)\ \kappa_{j,l-1}(X_\perp), \ j=3^{l-1}+1,\ldots,2.3^{l-1}, \ \ l \geq 2. \\ \kappa_{3,1}(X_\perp)\ \kappa_{j,l-1}(X_\perp), \ j=2.3^{l-1}+1,\ldots,3^{l}, \end{cases}$$

The operators $k_{j,l}(b)$, $j=1,\ldots,l$, $l \geq 2$, can be written in the form

$$k_{j,l} = E^-_{j,l} W_{1,j,l} T_{1,j,l} \ldots W_{l-1,j,l} T_{l-1,j,l} W_{l,j,l} E^+_{j,l}$$

where the operators $E^-_{j,l}$ and $E^+_{j,l}$ coincide either with r or with g, the operators $W_{s,j,l}$, $s=1,\ldots,l$, coincide either with $p_b V^2 p_b$ or with $p_b V p_b$, and the operators $T_{s,j,l}$, $s = 1,\ldots,l-1$, coincide either with r^2, or with gr, or with g^2. Note that among the operators $T_{s,j,l}$, $s=1,\ldots,l-1$, and $E^+_{j,l} \times E^-_{j,l}$, there are either at least one operator r^2, or at least two operators gr.
Analogously,

$$\kappa_{j,l}(X_\perp) = \epsilon^-_{j,l}\omega_{1,j,l}(X_\perp)\tau_{1,j,l}\ldots\omega_{l-1,j,l}(X_\perp)\tau_{l-1,j,l}\omega_{l,j,l}(X_\perp)\epsilon^+_{j,l}$$

where $\epsilon^\pm_{j,l} = \varrho$ if $E^\pm_{j,l} = r$ and $\epsilon^\pm_{j,l} = \gamma$ if $E^\pm_{j,l} = g$, $\omega_{s,j,l}(X_\perp) = V(X_\perp,.)^2$ if $W_{s,j,l} = p_b V^2 p_b$, and $\omega_{s,j,l}(X_\perp) = V(X_\perp,.)$ if $W_{s,j,l} = p_b V p_b$, $s=1,\ldots,l$, $\tau_{s,j,l} = \varrho^2$ if $T_{s,j,l} = r^2$, $\tau_{s,j,l} = \varrho\gamma$ if $T_{s,j,l} = rg$, and $\tau_{s,j,l} = \gamma^2$ if $T_{s,j,l} = g^2$, $s=1,\ldots,l-1$.
Obviously,

$$\mathrm{Tr}\ k_b^l = \sum_{j=1}^{3^l} \mathrm{Tr}\ k_{j,l}(b),$$

$$\int_{\mathbf{R}^2} \mathrm{Tr}\ \kappa(X_\perp)^l dX_\perp = \sum_{j=1}^{3^l} \int_{\mathbf{R}^2} \mathrm{Tr}\ \kappa_{j,l}(X_\perp)\ dX_\perp, \ l \geq 2.$$

Hence, it suffices to prove that

$$\lim_{b\to\infty} b^{-1}\mathrm{Tr}\ k_{j,l}(b) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \mathrm{Tr}\ \kappa_{j,l}(X_\perp)\ dX_\perp, \ j=1,\ldots,3^l, \ l\geq 2. \tag{8.1}$$

It is not difficult to show that

$$\mathrm{Tr}\ k_{j,l} = \int_{\mathbf{R}^{3l}} {\Pi'}^{l}_{s=1} w_{s,j,l}(x_{s+1}, y_{s+1}, \zeta_{s+1}-\zeta_s)\mathcal{P}_b(x_{s+1}, y_{s+1}; x_s, y_s)\times$$

$$\mathrm{Tr}\ \Pi^l_{s=1} t_{s,j,l}(\zeta_s)\ \Pi^l_{s=1} dx_s dy_s d\zeta_s \tag{8.2}$$

where

$$w_{s,j,l}(x,y,\zeta) = \begin{cases} \frac{1}{2\pi}\int_{\mathbf{R}} e^{-iz\zeta} V^2(x,y,z)dz \quad \text{if} \quad W_{s,j,l} = p_b V^2 p_b, \\ \frac{1}{2\pi}\int_{\mathbf{R}} e^{-iz\zeta} V(x,y,z)dz \quad \text{if} \quad W_{s,j,l} = p_b V p_b, \end{cases}$$

$\mathcal{P}_b$ is introduced in (5.4), $t_{s,j,l}(\zeta)$, $s=1,\dots,l-1$, coincides with the matrix-valued symbol of the operator $T_{s,j,l}$, and $t_{l,j,l}(\zeta)$ is the matrix-valued symbol of the operator $E^+_{j,l}\times E^-_{j,l}$. Moreover, the notation ${\Pi'}^l_{s=1}$ means that in the product of l factors the variables x_{l+1}, y_{l+1}, and ζ_{l+1}, should be set equal respectively to x_1, y_1, and ζ_1.
Analogously, we have

$$\mathrm{Tr}\ \kappa_{j,l}(X_\perp)\equiv \mathrm{Tr}\ \kappa_{j,l}(x,y)=$$
$$\int_{\mathbf{R}^l}{\Pi'}^l_{s=1}w_{s,j,l}(x,y,\zeta_{s+1}-\zeta_s)\mathrm{Tr}\ \Pi^l_{s=1}t_{s,j,l}(\zeta_s)\Pi^l_{s=1}d\zeta_s,\ X_\perp\equiv(x,y)\in\mathbf{R}^2. \tag{8.3}$$

In order to prove (8.1), we insert (5.4) into (8.2), and obtain

$$\mathrm{Tr}\ k_{j,l}=\frac{b^l}{(2\pi)^l}\int_{\mathbf{R}^{3l}}{\Pi'}^l_{s=1}w_{s,j,l}(x_{s+1},y_{s+1},\zeta_{s+1}-\zeta_s)\times$$
$$\exp\left\{-\frac{b}{4}\left[(x_{s+1}-x_s)^2+(y_{s+1}-y_s)^2+2i(x_{s+1}y_s-y_{s+1}x_s)\right]\right\}$$
$$\mathrm{Tr}\ \Pi^l_{s=1}t_{s,j,l}(\zeta_s)\Pi^l_{s=1}dx_sdy_sd\zeta_s.$$

Change the variables

$$x_1=x_1',\ y_1=y_1',$$
$$x_s=b^{-1/2}x_s'+x_1',\ y_s=b^{-1/2}y_s'+y_1',\ s=2,\dots,l.$$

Note that the corresponding Jacobian is equal to b^{1-l}. Thus we get

$$\mathrm{Tr}\ k_{j,l}=\frac{b}{(2\pi)^l}\int_{\mathbf{R}^{3l}}w_{l,j,l}(x_1',y_1',\zeta_1-\zeta_l)e^{\Phi(x_2',\dots,x_l',y_2',\dots,y_l')}$$
$$\Pi^{l-1}_{s=1}w_{s,j,l}(x_1'+b^{-1/2}x_{s+1}',y_1'+b^{-1/2}y_{s+1}',\zeta_{s+1}-\zeta_s)\ \mathrm{Tr}\ \Pi^l_{s=1}t_{s,j,l}(\zeta_s)\ \Pi^l_{s=1}dx_s'dy_s'd\zeta_s,$$

where

$$\Phi(x_2,\dots,x_l,y_2,\dots,y_l):=$$
$$-\frac{1}{4}\left\{{x_2}^2+{y_2}^2+{x_l}^2+{y_l}^2+\sum_{s=2}^{l-1}\Big((x_{s+1}-x_s)^2+(y_{s+1}-y_s)^2+2i(x_{s+1}y_s-y_{s+1}x_s)\Big)\right\}.$$

Therefore,

$$\lim_{b\to\infty}b^{-1}\mathrm{Tr}\ k_{j,l}(b)=$$
$$\frac{1}{(2\pi)^l}\int_{\mathbf{R}^{3l}}e^{\Phi(x_2',\dots,x_l',y_2',\dots,y_l')}{\Pi'}^l_{s=1}w_{s,j,l}(x_1',y_1',\zeta_{s+1}-\zeta_s)\mathrm{Tr}\ \Pi^l_{s=1}t_{s,j,l}(\zeta_s)\Pi^l_{s=1}dx_s'dy_s'd\zeta_s. \tag{8.4}$$

Changing the variables in the integral at the right hand side of (8.4)

$$x_1'=x_1,\ y_1'=y_1,$$
$$x_s'=x_s-x_1,\ y_s'=y_s-y_1,\ s=2,\dots,l,$$

we get

$$\lim_{b\to\infty} b^{-1}\mathrm{Tr}\ k_{j,l}(b) = \int_{\mathbf{R}^{3l}} \Pi'^{l}_{s=1} w_{s,j,l}(x_1, y_1, \zeta_{s+1}-\zeta_s)$$

$$\mathcal{P}_1(x_{s+1}, y_{s+1}; x_s, y_s)\mathrm{Tr}\ \Pi^l_{s=1} t_{s,j,l}(\zeta_s)\Pi^l_{s=1} dx_s dy_s d\zeta_s. \tag{8.5}$$

Since

$$\int_{\mathbf{R}^2} \mathcal{P}_1(x,y;x'',y'')\mathcal{P}_1(x'',y'';x',y')dx''dy'' = \mathcal{P}_1(x,y;x',y'),\ x,y,x',y' \in \mathbf{R},$$

$$\mathcal{P}_1(x,y;x,y) = (2\pi)^{-1},\ x,y \in \mathbf{R},$$

we find that (8.5) is equivalent to

$$\lim_{b\to\infty} b^{-1}\mathrm{Tr}\ k_{j,l}(b) =$$

$$\frac{1}{2\pi}\int_{\mathbf{R}^{l+2}} \Pi'^{l-1}_{s=1} w_{s,j,l}(x_1, y_1, \zeta_{s+1}-\zeta_s)\mathrm{Tr}\ \Pi^l_{s=1} t_{s,j,l}(\zeta_s)\,\Pi^l_{s=1} d\zeta_s\, dx_1 dy_1 =$$

$$\frac{1}{2\pi}\int_{\mathbf{R}^2} \mathrm{Tr}\ \kappa_{j,l}(X_\perp)\, dX_\perp,\ j = 1,\dots,l,\ l \geq 2,$$

(see (8.3)), which is identical to (8.1). $\square$

Set

$$\nu(s) := \begin{cases} \frac{1}{2\pi}\int_{\mathbf{R}^2} n_-(-s;\kappa(X_\perp))\, dX_\perp,\ s < 0, \\ -\frac{1}{2\pi}\int_{\mathbf{R}^2} n_+(s;\kappa(X_\perp))\, dX_\perp,\ s > 0, \end{cases} \tag{8.6}$$

Note that $s \neq 0$ is a continuity point of ν if and only if

$$\mathrm{vol}\ \left\{X_\perp \in \mathbf{R}^2 | \dim\mathrm{Ker}\ (\kappa(X_\perp) - s) \geq 1\right\} = 0.$$

Corollary 8.1 *Let $t < 0$ be a continuity point of ν. Then we have*

$$\lim_{b\to\infty} b^{-1} n_-(-t; k_b) = \nu(t).$$

The corollary follows immediately from Proposition 8.1 and Lemma 3.1 with $T(b) = k_b$, ν defined as in (8.6), and $t_0 = \|r\|^2\|V\|^2_{L^\infty(\mathbf{R})} + 2\|g\|\|r\|\|V\|_{L^\infty(\mathbf{R})}$.

Proposition 8.2 *Let $V \in C_0^\infty(\mathbf{R}^3)$. Assume that $t < 0$ is a continuity point of ν. Then we have*

$$\lim_{b\to\infty} b^{-1} n_-(-t; K_b) = \nu(t). \tag{8.7}$$

Proof. Evidently

$$n_-(-t; K_b) \geq n_-(-t; P_b K_b P_b) = n_-(-t; k_b).$$

Applying Corollary 8.1, we get

$$\liminf_{b\to\infty} b^{-1}n_-(-t; K_b) \geq \liminf_{b\to\infty} b^{-1}n_-(-t; k_b) = \lim_{b\to\infty} b^{-1}n_-(-t; k_b) = \nu(t). \tag{8.8}$$

On the other hand we have

$$K_b = P_bK_bP_b + Q_bK_bQ_b + 2\mathrm{Re}\ P_bK_bQ_b =$$

$$P_bK_bP_b + Q_bK_bQ_b + 2\mathrm{Re}\ P_bR_bV^2R_bQ_b + 2\mathrm{Re}\ P_bG_bVR_bQ_b + 2\mathrm{Re}\ P_bR_bVG_bQ_b.$$

Note that

$$R_bVG_b = G_bVR_b + R_bJR_b$$

where

$$J := [V, H_0(b) - \frac{1}{2}(\lambda_1 + \lambda_2)] = [V, H_0(b)] = i\left(\frac{\partial V}{\partial x}\alpha_1 + \frac{\partial V}{\partial y}\alpha_2 + \frac{\partial V}{\partial z}\alpha_3\right).$$

It is essential that J is independent of b.
Apply the estimates

$$K_b = P_bK_bP_b + Q_bK_bQ_b + 2\mathrm{Re}\ P_bR_bV^2R_bQ_b + 4\mathrm{Re}\ P_bG_bVR_bQ_b + 2\mathrm{Re}\ P_bR_bJR_bQ_b \geq$$

$$P_bK_bP_b + Q_bK_bQ_b - \varepsilon P_bR_bV^2R_bP_b - \varepsilon^{-1}Q_bR_bV^2R_bQ_b -$$

$$2\varepsilon P_bG_b^2P_b - 2\varepsilon^{-1}Q_bR_bV^2R_bQ_b - \varepsilon P_bR_b^2P_b - \varepsilon^{-1}Q_bR_bJ^*JR_bQ_b \geq$$

$$P_b(K_b - \varepsilon(R_bV^2R_b + 2G_b^2 + R_b^2))P_b - Q_b(\varepsilon G_b^2 + \varepsilon^{-1}R_b(4V^2 + J^*J)R_b)Q_b,\ \varepsilon > 0. \tag{8.9}$$

Now fix $\mu \in (0, -t)$, and choose ε so small that we have $\varepsilon(3\|G_b\|^2 + \|R_b\|^2) \leq \mu/3$; hence $\varepsilon\|2P_bG_b^2P_b + P_bR_b^2P_b + Q_bG_b^2Q_b\| \leq \mu/3$. Then (8.9) entails

$$n_-(-t; K_b) \leq n_-(-t-\mu; P_bK_bP_b) +$$

$$n_+(\mu/3; \varepsilon P_bR_bV^2R_bP_b) + n_+(\mu\varepsilon/3; Q_bR_b(4V^2 + J^*J)R_bQ_b). \tag{8.10}$$

Lemma 6.3(ii) combined with the estimate

$$n_+(2\delta^2; Q_bR_b(4V^2 + J^*J)R_bQ_b) \leq n_*(\delta/2; VR_bQ_b) + n_*(\delta; JR_bQ_b), \quad \delta > 0,$$

implies that the quantity $n_+(\mu\varepsilon/3; Q_bR_b(4V^2 + J^*J)R_bQ_b)$ vanishes for b large enough. Hence, (8.10) entails

$$\limsup_{b\to\infty} b^{-1}n_-(-t; K_b) \leq \limsup_{b\to\infty} b^{-1}n_-(-t-\mu; P_bK_bP_b) +$$

$$\limsup_{b\to\infty} b^{-1}n_+(\mu/3; \varepsilon P_bR_bV^2R_bP_b),\ \mu \in (0, -t),\ \varepsilon > 0. \tag{8.11}$$

Corollary 6.1 combined with the estimate $n_+(\delta^2; P_bR_bV^2R_bP_b) \leq n_*(\delta; VR_bP_b)$, $\delta > 0$, implies

$$\limsup_{b\to\infty} b^{-1}n_+(\mu/3; \varepsilon P_bR_bV^2R_bP_b) \leq c_3\left(\frac{3\varepsilon}{\mu}\right)^{3/2}\int_{\mathbf{R}^3}|V|^3\,dX.$$

Letting $\varepsilon \downarrow 0$, we find that (8.11) entails

$$\limsup_{b\to\infty} b^{-1}n_-(-t; K_b) \leq \limsup_{b\to\infty} b^{-1}n_-(-t-\mu; P_bK_bP_b),\ \mu \in (0,-t). \tag{8.12}$$

Now choose a sequence $\{\mu_l\}_{l\geq 1}$ such that $\mu_l \in (0,-t)$, $\lim_{l\to\infty}\mu_l = 0$, and all the points $-t-\mu_l$ are continuity points of ν. Then Corollary 8.1 implies

$$\limsup_{b\to\infty} b^{-1}n_-(-t-\mu_l; P_bK_bP_b) = \lim_{b\to\infty} b^{-1}n_-(-t-\mu_l; k_b) = \nu(t+\mu_l). \tag{8.13}$$

Letting $l \to \infty$, we find that (8.11)–(8.13) entail

$$\limsup_{b\to\infty} b^{-1}n_-(-t; K_b) \leq \nu(t). \tag{8.14}$$

The combination of (8.8) and (8.14) immediately yields (8.7). $\square$

Proposition 8.3 *Let $V \in L^3(\mathbf{R}^3)$. Assume that $t < 0$ is a continuity point of ν. Then* (8.7) *remains valid.*

Proof. Pick a sequence $\{\delta_l\}_{l\geq 1}$, $\lim_{l\to\infty}\delta_l = 0$, and write $V = V_0 + V_1$, where $V_0 = V_{0,l} \in C_0^\infty(\mathbf{R}^3)$, $V_1 = V_{1,l} \in L^3(\mathbf{R}^3)$, and $\|V_{1,l}\|_{L^3(\mathbf{R}^3)} \leq \delta_l$.
Introduce the operators $K_{b,0}$, $k_{b,0}$ and $\kappa_0(X_\perp)$, replacing V by $V_{0,l}$. Analogously, define the function $\nu_l \equiv \nu_{0,l}$ substituting κ for κ_0. Choose the sequence $\{\varepsilon_r\}$ such that $0 < \varepsilon_r < \min\{1, -t/2\}$, $\lim_{r\to\infty}\varepsilon_r = 0$, and the points $-t \pm \varepsilon_r$ are continuity points of all functions $\nu_{0,l}$. Evidently,

$$K_b \geq (1-\varepsilon_r^2)K_{b,0} + (1-\varepsilon_r^{-2})R_bV_1^2R_b + 2\mathrm{Re}G_bV_1R_b + 2\varepsilon_r^2\mathrm{Re}G_bV_0R_b,$$

$$K_b \leq (1+\varepsilon_r^2)K_{b,0} + (1+\varepsilon_r^{-2})R_bV_1^2R_b + 2\mathrm{Re}G_bV_1R_b - 2\varepsilon_r^2\mathrm{Re}G_bV_0R_b,$$

and, hence,

$$n_-(-t; K_b) \leq n_-(-t-\varepsilon_r; K_{b,0}) + n_-(\varepsilon_r/3; (1-\varepsilon_r^{-2})R_bV_1^2R_b) +$$
$$n_-(\varepsilon_r/3; 2\mathrm{Re}\ G_bV_0R_b) + n_-(\varepsilon_r/3; 2\varepsilon_r^2\mathrm{Re}\ G_bV_1R_b),$$
$$n_-(-t; K_b) \geq n_-(-t+\varepsilon_r; K_{b,0}) - n_+(\varepsilon_r/3; (1+\varepsilon_r^{-2})R_bV_1^2R_b) -$$
$$n_+(\varepsilon_r/3; 2\mathrm{Re}G_bV_1R_b) - n_-(\varepsilon_r/3; 2\varepsilon_r^2\mathrm{Re}G_bV_0R_b).$$

Utilizing Corollary 6.2 and Proposition 8.2, we get

$$\limsup_{b\to\infty} b^{-1}n_+(-t; K_b) \leq \nu_{0,l}(t+\varepsilon_r) + c_8\delta_l^3 + c_9\varepsilon_r^3\int_{\mathbf{R}^3}|V_0|^3\,dX, \tag{8.15}$$

$$\liminf_{b\to\infty} b^{-1}n_+(-t;K_b) \geq \nu_{0,l}(t-\varepsilon_r) - c_8\delta_l^3 - c_9\varepsilon_r^3 \int_{\mathbf{R}^3} |V_0|^3\,dX, \tag{8.16}$$

where c_8 depends on ε_r but is independent on δ_l, while c_9 is independent of both ε_r and δ_l.

Similarly, using Corollaries 7.1 and 7.2, we obtain the estimates

$$\nu_{0,l}(t+\varepsilon_r) \leq \nu(t+2\varepsilon_r) + c_8'\delta_l^3 + c_9'\varepsilon_r^3 \int_{\mathbf{R}^3} |V_0|^3\,dX, \tag{8.17}$$

$$\nu_{0,l}(t-\varepsilon_r) \geq \nu(t-2\varepsilon_r) - c_8'\delta_l^3 - c_9'\varepsilon_r^3 \int_{\mathbf{R}^3} |V_0|^3\,dX, \tag{8.18}$$

where c_8' depends on ε_r but is independent on δ_l, while c_9' is independent of both ε_r and δ_l.

Letting at first $l \to \infty$ (hence, $\delta_l \downarrow 0$), and then $r \to \infty$ (hence, $\varepsilon_r \downarrow 0$), in (8.15) - (8.18), and taking into account that t is a continuity point of ν, we obtain (8.7). □

Using Lemma 4.1, Corollary 4.1, and Proposition 8.3 with $t = -1$, we deduce the following corollary.

Corollary 8.2 *Let the hypotheses of Theorem* 2.1 *hold. Assume in addition* $V \in L^3(\mathbf{R}^3)$. *Then* (2.1) *holds.*

In order to complete the proof of Theorem 2.1 it remains to show that we can approximate $V \in \mathcal{L}$ by $V \in L^3(\mathbf{R}^3)$.

Let $\lambda_1,\ \lambda_2 \in \mathbf{R}^3$, $-1 < \lambda_1 < \lambda_2 < 1$, be regular points. Fix $\varepsilon > 0$ such that $\lambda_1 - 3\varepsilon > -1$, $\lambda_2 + 3\varepsilon < 1$, $\lambda_1 + 3\varepsilon < \lambda_2 - 3\varepsilon$, and write the ε-decomposition of V. Evidently,

$$\mathcal{N}(\lambda_1+\varepsilon,\lambda_2-\varepsilon;H_0(b)+V_1) \leq \mathcal{N}(\lambda_1,\lambda_2;H(b)) \leq \mathcal{N}(\lambda_1-\varepsilon,\lambda_2+\varepsilon;H_0(b)+V_1). \tag{8.19}$$

Now chose the numbers $\varepsilon_j \in [0,\varepsilon)$ such that

$$\mathrm{vol}\left\{X_\perp \in \mathbf{R}^2 | \dim \mathrm{Ker}\ (\chi_0 + V_1(X_\perp,.) - (\lambda_j \pm \varepsilon \pm \varepsilon_j)) \geq 1\right\} = 0, \quad j=1,2.$$

Applying Corollary 8.2, we deduce from (8.19) the following estimates

$$\begin{gathered}
\limsup_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1,\lambda_2;H(b)) \leq \limsup_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1-\varepsilon,\lambda_2+\varepsilon;H_0(b)+V_1) \leq \\
\limsup_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1-\varepsilon-\varepsilon_1,\lambda_2+\varepsilon+\varepsilon_2;H_0(b)+V_1) = \\
\mathcal{D}_{V_1}(\lambda_1-\varepsilon-\varepsilon_1,\lambda_2+\varepsilon+\varepsilon_2) \leq \mathcal{D}_{V_1}(\lambda_1-2\varepsilon,\lambda_2+2\varepsilon), \qquad (8.20) \\
\liminf_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1,\lambda_2;H(b)) \geq \liminf_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1+\varepsilon,\lambda_2-\varepsilon;H_0(b)+V_1) \geq \\
\liminf_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1+\varepsilon+\varepsilon_1,\lambda_2-\varepsilon-\varepsilon_2;H_0(b)+V_1) =
\end{gathered}$$

$$\mathcal{D}_{V_1}(\lambda_1+\varepsilon+\varepsilon_1, \lambda_2-\varepsilon-\varepsilon_2) \leq \mathcal{D}_{V_1}(\lambda_1+2\varepsilon, \lambda_2-2\varepsilon). \tag{8.21}$$

Finally, note the obvious inequalities

$$\mathcal{D}_{V_1}(\lambda_1-2\varepsilon, \lambda_2+2\varepsilon) \leq \mathcal{D}_V(\lambda_1-3\varepsilon, \lambda_2+3\varepsilon),$$

$$\mathcal{D}_{V_1}(\lambda_1+2\varepsilon, \lambda_2-2\varepsilon) \geq \mathcal{D}_V(\lambda_1+3\varepsilon, \lambda_2-3\varepsilon). \tag{8.22}$$

Putting together (8.19)–(8.22), we get

$$\limsup_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1, \lambda_2; H(b)) \leq \mathcal{D}_V(\lambda_1-3\varepsilon, \lambda_2+3\varepsilon), \tag{8.23}$$

$$\liminf_{b\to\infty} b^{-1}\mathcal{N}(\lambda_1, \lambda_2; H(b)) \geq \mathcal{D}_V(\lambda_1+3\varepsilon, \lambda_2-3\varepsilon). \tag{8.24}$$

Letting $\varepsilon \downarrow 0$ in (8.23)–(8.24), and bearing in mind that λ_1 and λ_2 are regular points, we come to (2.1).

Acknowledgements

This work has been done during author's visit to the University of Regensburg in the summer of 1998 as a DAAD Research Fellow. The financial support of the DAAD is gratefully acknowledged.
The author thanks Prof.H.Siedentop for his warm hospitality, and for several illuminating discussions on the spectral theory of the Dirac operator.
The author was partially supported by the Bulgarian Science Foundation under Grant MM 612/96.

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