M P E J

MATHEMATICAL PHYSICS ELECTRONIC JOURNAL

ISSN 1086-6655
Volume 5, 1999

Paper 1
Received: June 19, 1998, Accepted: Jan 6, 1999
Editor: J. Avron

Lower bounds on wave packet propagation by packing dimensions of spectral measures

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Abstract

We prove that, for any quantum evolution in $\ell^2(\mathbf{Z}^D)$, there exist arbitrarily long time scales on which the qth moment of the position operator increases at least as fast as a power of time given by q/D times the packing dimension of the spectral measure. Packing dimensions of measures and their connections to scaling exponents and box-counting dimensions are also discussed.

1 Introduction.

Let H be a selfadjoint operator in a separable Hilbert space $\mathcal{H}$. Let $|\psi\rangle$ be any vector in $\mathcal{H}$, μ its spectral measure relative to H and $\mathcal{B} \equiv \{|n\rangle\}_{n\in\mathbf{N}}$ a Hilbert basis in $\mathcal{H}$. As *time* t increases, the *wave packet* $e^{-\imath tH}|\psi\rangle$ spreads out over the basis $\mathcal{B}$. In particular, it is known [G1, G2, La, Co] that, for $q > 0$, the qth moment of the probability distribution associated with the Fourier expansion of $e^{-\imath tH}|\psi\rangle$ over the basis $\mathcal{B}$ increases, in time average, at least as fast as a power of time given by q times the Hausdorff dimension of μ. This qualitatively means that propagation is faster the more continuous the spectral measure is, the degree of continuity being measured by the lower pointwise dimension of the spectral measure.

The present work is summarized by the qualitative remark that, at any given time t, the wave packet can only probe continuity of the spectral measure on the level of spectral resolution achieved at time t, hence on a spectral scale of the order of $1/t$. This remark is relevant to the case of non-exactly scaling spectral measures. For such measures, the upper and lower pointwise dimensions do not coincide, meaning that there are arbitrarily small spectral scales at which the measure is somewhere in the spectrum scaling faster than expressed by the Hausdorff dimension. We accordingly obtain that on arbitrarily long time scales a larger lower bound is valid, given by the packing dimension of the spectral measure.

There is at least one abstract example of a zero-Hausdorff dimensional spectral measure with packing dimension equal to 1, which leads to ballistic transport [G3]. Spectral measures of a similar nature are likely to occur also for concrete Schrödinger operators. Such may be the case with the quasi-ballistic dynamics exhibited by the Harper model with Liouville incommensuration [La].

The Hausdorff dimension of a measure is related to its lower pointwise dimensions by the theory of Rogers and Taylor [Ro], whose relevance in the present context was advocated by Last [La]. Our present result calls *upper* pointwise dimensions into play. Apart from [G3], these have not yet found their way in the theory of quantum transport. In the appendix, we therefore elaborate on results by Cutler [Cu] and develop a treatment, in a sense dual to Rogers' and Taylor's, which connects such dimensions to the packing dimension of a measure. Furthermore, we show that these packing dimensions can also be calculated by a box-counting procedure.

In the next two sections we establish preliminaries and notations. In Section 4 we state and prove our main result Theorem 1. The main element of its proof is Proposition 2 which is basically a restatement of existing results. However, we give here an alternative *ab initio* derivation which makes no use of Strichartz theorem, but is based on Proposition 1 combined with Last's argument [La]. A modified version of Proposition 1 allowed to prove also *upper* bounds for a special class of Hamiltonians [GS]. Finally, in Section 5, we transpose the main result to Hamiltonians on a D-dimensional lattice.

A useful discussion with Y. Last is acknowledged.

2 Growth exponents.

For given time T and $\epsilon \in (0,1)$, we define the *minimal carrier* $\overline{n}(\epsilon,T)$ of the wave packet on the basis $\mathcal{B}$ as follows:

$$\overline{n}(\epsilon,T) \;=\; \min\left\{ \overline{n}\in \mathbf{N} \;\middle|\; \sum_{n\geq \overline{n}} p_n(T) \leq \epsilon \right\} \;, \tag{1}$$

where the $p_n(T)$ is the average probability up to time T in the basis state $|n\rangle \in \mathcal{B}$, given by:

$$p_n(T) \;=\; \int_0^T \frac{dt}{T}\, |\langle n|e^{-\imath Ht}|\psi\rangle|^2 \;.$$

Upper and lower growth exponents of the minimal carriers are defined as:

$$\beta_0^+(\epsilon) \;=\; \limsup_{T\to\infty} \frac{\log(\overline{n}(\epsilon,T))}{\log(T)} \;, \qquad \beta_0^-(\epsilon) \;=\; \liminf_{T\to\infty} \frac{\log(\overline{n}(\epsilon,T))}{\log(T)} \;, \tag{2}$$

We also define $\beta_0^\pm = \lim_{\epsilon\to 0}\beta_0^\pm(\epsilon)$. Let us further introduce the qth moment $M_q(T)$ of the distribution $p_n(T)$ by

$$M_q(T) \;=\; \sum_{n\geq 0} n^q\, p_n(T) \;,$$

which can be interpreted as the time-average up to time T of the expectation value of the qth moment ($q \neq 0$) of the *position operator* associated with the basis $\mathcal{B}$. The corresponding growth exponents are

$$\beta_q^+(\epsilon) \;=\; \limsup_{T\to\infty} \frac{\log(M_q(T))}{q\,\log(T)} \;, \qquad \beta_q^-(\epsilon) \;=\; \liminf_{T\to\infty} \frac{\log(M_q(T))}{q\,\log(T)} \;, \tag{3}$$

We have $\beta_q^+ \leq \beta_0^-$ whenever $q<0$ and $\beta_q^- \geq \beta_0^+$ whenever $q>0$. Lower bounds on $\beta_0^\pm$ convey stronger information than lower bounds on $\beta_q^\pm$, $q>0$. For instance, if H has a pure point spectrum then $\beta_0^\pm = 0$, because minimal carriers remain bounded in time; still, moments may display nonvanishing growth exponents.

The lower growth exponents β_q^- characterize the minimal spreading of the wave packet: for all $T\geq 0$, $M_q(T) \geq C(\delta)T^{q(\beta_q^- -\delta)}$ holds for all $\delta>0$ with appropriate constants $C(\delta)$. The upper exponents β_q^+ give the fastest possible spreading on sequences of times: for all $\delta>0$, $M_q(T)\leq C(\delta)T^{q(\beta_q^+ +\delta)}$ holds at all times T, but there exists a diverging sequence $(T_k)_{k\geq 1}$ such that $M_q(T_k)\geq C'(\delta)T_k^{q(\beta_q^+ -\delta)}$ for all $k\geq 1$.

3 Spectral dimensions.

Given a (Borel) probability measure μ on $\mathbf{R}$, we define its lower and upper pointwise dimensions at $E\in \mathrm{supp}(\mu)$ as follows:

$$\underline{d}_\mu(E) \;=\; \liminf_{\epsilon\to 0} \frac{\log(\mu([E-\epsilon,E+\epsilon]))}{\log(\epsilon)} \;, \qquad \overline{d}_\mu(E) \;=\; \limsup_{\epsilon\to 0} \frac{\log(\mu([E-\epsilon,E+\epsilon]))}{\log(\epsilon)} \;, \tag{4}$$

while for $E \notin \mathrm{supp}(\mu)$, $\underline{d}_\mu(E) = \overline{d}_\mu(E) = \infty$.

The (upper) Hausdorff dimension $\dim_{\mathrm{H}}^+(\mu)$ and the (upper) packing dimension $\dim_{\mathrm{P}}^+(\mu)$ of the measure μ can then be defined as follows:

$$\dim_{\mathrm{H}}^+(\mu) \;=\; \mu\text{-}\operatorname*{ess\,sup}_{E\in\mathbf{R}} \underline{d}_\mu(E) \;, \qquad \dim_{\mathrm{P}}^+(\mu) \;=\; \mu\text{-}\operatorname*{ess\,sup}_{E\in\mathbf{R}} \overline{d}_\mu(E) \;, \tag{5}$$

where, for a real function f, $\mu\text{-}\operatorname{ess\,sup}_{E\in\mathbf{R}} f(E)$ denotes the μ-essential supremum of f, *i.e.* the infimum over all sets Δ of full μ-measure of the quantity $\sup_{E\in\Delta} f(E)$. Although the above definitions are optimally suited to our present purposes, dimensions of Borel measures are more properly defined and discussed in the appendix.

In the rest of this article, we shall use dyadic partitions of the real axis in intervals $I_j^N = ((j-1)2^{-N}, j2^{-N}]$, $j \in \mathbf{Z}$. Of course, any other hierarchic partition could be used. For $E \in \mathbf{R}$, we shall denote $I_{j(E)}^N$ the dyadic interval of the Nth generation to which E belongs. We shall in particular make use of the fact that

$$\underline{d}_\mu(E) \;=\; \liminf_{N\to\infty} \frac{\log(\mu(I_{j(E)}^N))}{\log(|I_{j(E)}^N|)} \;=\; \liminf_{N\to\infty} \frac{-\log_2(\mu(I_{j(E)}^N))}{N} \;. \tag{6}$$

Similar equalities hold for the upper pointwise dimension $\overline{d}_\mu(E)$, with $\liminf$ replaced by $\limsup$.

4 Lower bounds on growth exponents.

The following proposition expresses in a quantitative way the fact that the time evolution up to time T does not resolve details of the spectrum on scales smaller than $1/T$ ($\hbar = 1$). Up to time T it is thus possible to work with an approximate Hamiltonian with discrete spectrum the eigenvalues of which have a spacing of order $1/T$.

We use the notation $|\chi_j^N\rangle = \chi_j^N(H)|\psi\rangle$, where χ_j^N is the characteristic function of I_j^N.

Proposition 1 *Given $\epsilon > 0$, we associate to any time T a generation index N so that:*

$$2^{(N-1)} < \frac{T}{\sqrt{\epsilon}} \leq 2^N \;. \tag{7}$$

Then for any family of indices $\mathcal{F} \subset \mathbf{N}$ and $\psi \in \mathcal{H}$ with $\|\psi\| \leq 1$, the following estimate holds true:

$$\sum_{n\in\mathcal{F}} \int_0^T \frac{dt}{T} |\langle n|e^{-\imath Ht}|\psi\rangle|^2 \;\leq\; 2\,\epsilon + \frac{8\pi}{\sqrt{\epsilon}} \sum_{n\in\mathcal{F}} \sum_{j\in\mathbf{Z}} |\langle n|\chi_j^N\rangle|^2 \;. \tag{8}$$

Proof. Throughout this proof, we understand that N and T are related to each other via (7). For $0 \leq t \leq T$, we approximate $e^{-\imath Ht}|\psi\rangle$ by:

$$|\psi_T(t)\rangle \;=\; \sum_{j\in\mathbf{Z}} e^{-\imath tE_j^N}|\chi_j^N\rangle \;,$$

where $E_j^N = j2^{-N}$. Then we have:

$$\|\,|\psi_T(t)\rangle - e^{-\imath tH}|\psi\rangle\|^2 \;\leq\; \sum_{j\in\mathbf{Z}}\int_{I_j^N} d\mu(E)\, t^2|E-E_j^N|^2 \;.$$

As $|E-E_j^N| \leq 2^{-N}$ for $E \in I_j^N$, it follows from (7) that the latter expression is less than ϵ as long as $0 \leq t \leq T$. Thus

$$\sum_{n\in\mathcal{F}}\int_0^T \frac{dt}{T}|\langle n|e^{-\imath tH}|\psi\rangle|^2 \;\leq\; 2\,\epsilon + 2\sum_{n\in\mathcal{F}}\int_0^T \frac{dt}{T}|\langle n|\psi_T(t)\rangle|^2 \;. \tag{9}$$

Now the integral can be bounded as follows:

$$\begin{aligned}\int_0^T \frac{dt}{T}|\langle n|\psi_T(t)\rangle|^2 \;&\leq\; \int_0^{\pi 2^{N+1}} \frac{dt}{T}\,|\langle n|\psi_T(t)\rangle|^2 \\ &=\; \sum_{j\in\mathbf{Z}}\sum_{l\in\mathbf{Z}} \langle n|\chi_{I_j^N}\rangle\langle\chi_{I_l^N}|n\rangle\,\frac{1}{T}\int_0^{\pi 2^{N+1}} dt\, e^{-i(E_j^N-E_l^N)t} \;.\end{aligned}$$

The latter integral yields $2^{N+1}\pi\delta_{jl}$. Putting this into (9) and recalling (7), we directly get inequality (8). $\square$

Proposition 2 *For integer N and $\alpha \in (0,1)$, let $A_{N,\alpha}$ be the union of all the dyadic intervals of the Nth dyadic generation which have measure $\mu(I_j^N) < 2^{-N\alpha}$. Set $b_N = \mu(A_{N,\alpha})$. If $b_N > 0$, then the following holds true for T satisfying $b_N 2^{N-1} < 9T \leq b_N 2^N$:*

$$\overline{n}\left(\frac{b_N}{2}, T\right) \;>\; C(\alpha)\, b_N^{3-\alpha}\, T^\alpha \;, \tag{10}$$

where $C(\alpha) > 0$ is only dependent on α.

Proof. Let us define $|\psi_N\rangle = \chi_{A_{N,\alpha}}(H)|\psi\rangle$, so $\|\,|\psi_N\rangle\|^2 = b_N > 0$. We wish to apply Proposition 1, with the following specifications: replace $|\psi\rangle$ by $|\psi_N\rangle$ and choose $\mathcal{F}$ as the integers smaller than a given m, finally set $\epsilon = (b_N/9)^2$. Condition (7), which makes Proposition 1 applicable, then becomes precisely $b_N 2^{N-1} < 9T \leq b_N 2^N$.

The spectral measure of $|\psi_N\rangle$ is $d\mu_N(E) = \chi_{A_{N,\alpha}}(E)d\mu(E)$. Hence the sum on the right-hand side of (8) is restricted to the set $\mathcal{J}_{N,\alpha}$ of indices $j \in \mathbf{Z}$ for which $\mu(I_j^N) < 2^{-N\alpha}$. The sum can be estimated as follows:

$$
\begin{aligned}
\sum_{n\in\mathcal{F}}\sum_{j\in\mathcal{J}_{N,\alpha}} |\langle n|\chi_j^N\rangle|^2 &\leq \sum_{n\in\mathcal{F}}\sum_{j\in\mathcal{J}_{N,\alpha}} \|\chi_j^N\|^2\,\|\chi_j^N(H)|n\rangle\|^2 \\
&= \sum_{n\in\mathcal{F}}\sum_{j\in\mathcal{J}_{N,\alpha}} \mu_N(I_j^N)\|\chi_j^N(H)|n\rangle\|^2 \\
&\leq m \max_{j\in\mathcal{J}_{N,\alpha}} \mu_N(I_j^N) \\
&\leq m\,2^{-N\alpha}\,.
\end{aligned}
$$

We choose m as follows:

$$
m = \frac{1}{4\pi}\left(\frac{b_N}{9}\right)^3 2^{N\alpha}\,, \tag{11}
$$

and denote $P_m = \sum_{n<m}|n\rangle\langle n|$. Substituting all the above in Proposition 1, we obtain

$$
\int_0^T \frac{dt}{T}\|P_m(t)\psi_N\|^2 \leq \left(\frac{2b_N}{9}\right)^2\,,
$$

where $P_m(t) = e^{\imath Ht}P_m e^{-\imath Ht}$. Now we continue as in [La, Theorem 6.1]. Let $|\psi'_N\rangle = |\psi\rangle - |\psi_N\rangle$. Then $|\psi'_N\rangle$ is orthogonal to $|\psi_N\rangle$, and

$$
\begin{aligned}
\int_0^T \frac{dt}{T}\,\|P_m(t)|\psi\rangle\|^2 &\leq \int_0^T \frac{dt}{T}\,\|P_m(t)|\psi_N\rangle\|^2 + 2\|\,|\psi'_N\rangle\|\int_0^T \frac{dt}{T}\,\|P_m(t)|\psi_N\rangle\| + \|\,|\psi'_N\rangle\|^2 \\
&\leq \left(\frac{2b_N}{9}\right)^2 + \|\,|\psi'_N\rangle\|^2 + \frac{4b_N}{9}\,.
\end{aligned}
$$

Whence, recalling $\|\,|\psi_N\rangle\|^2 + \|\,|\psi'_N\rangle\|^2 = 1$, we get:

$$
\int_0^T \frac{dt}{T}\,\|(1-P_m(t))|\psi\rangle\|^2 \geq \|\,|\psi_N\rangle\|^2 - \left(\frac{2b_N^2}{9}\right)^2 - \frac{4b_N}{9} \geq \frac{b_N}{2}\,.
$$

because $\|\,|\psi_N\rangle\|^2 = b_N < 1$. It therefore follows from the definition of a minimal carrier that $\overline{n}(b_N/2, T) \geq m$. The proof is concluded on recalling the definition of m and ϵ as well as the connection between N and T. □

A link to spectral dimensions is provided by (we also use lim sup and lim inf in their set-theoretic meaning; further we suppress the subscripts $N \to \infty$):

$$
\begin{aligned}
\limsup A_{N,\alpha} &= \limsup\left\{E\in\mathbf{R}\,|\,\mu(I_{j(E)}^N) < 2^{-N\alpha}\right\} \\
&= \left\{E\in\mathbf{R}\,\middle|\,\limsup\frac{-\log_2\mu(I_{j(E)}^N)}{N} > \alpha\right\} \\
&= \left\{E\in\mathbf{R}\,|\,\overline{d}_\mu(E) > \alpha\right\}
\end{aligned} \tag{12}
$$

In a completely analogous way, we get

$$\liminf A_{N,\alpha} \;=\; \left\{E \in \mathbf{R} \,|\, \underline{d}_\mu(E) > \alpha\right\} \tag{13}$$

At this point, we derive lower bounds. First of all, if $\alpha < \dim_{\mathrm{H}}(\mu)$, then (5) and (13) show that $\liminf \mu(A_{N,\alpha}) \geq \mu(\liminf A_{N,\alpha}) > 0$, so b_N is larger than some b_0 for all N. Inserting such a b_0 in Proposition 2, we immediately get $\overline{n}(b_0/2, T) \geq \mathrm{const}\, T^\alpha$ for all T. This proves the already known lower bound, $\beta_0^- \geq \dim_{\mathrm{H}}(\mu)$. In addition, we can now prove the existence of a sequence of times on which the large packing dimension of the "thin" part of the spectral measure forces the wave packet to travel possibly farther than imposed by the Hausdorff dimension.

Theorem 1 *For all positive q, the upper growth exponent of the q-th moment of the position operator satisfies $\beta_q^+ \geq \dim_{\mathrm{P}}^+(\mu)$.*

Proof. It is obviously sufficient to consider the case $\dim_{\mathrm{P}}^+(\mu) > 0$. Then, if $\alpha < \dim_{\mathrm{P}}(\mu)$, it follows from the definition (5) of the packing dimension and from (12) that $\mu(\limsup A_{N,\alpha}) > 0$. Therefore, from the Borel-Cantelli lemma we get that $\sum_N \mu(A_{N,\alpha}) = \infty$. This in turn implies that there is a sequence of integers $N_k \to \infty$ such that $b_{N_k} = \mu(A_{N_k,\alpha}) > N_k^{-2}$.

In this situation, Proposition 2 says that

$$\overline{n}\left(\frac{b_{N_k}}{2}, T_k\right) \;>\; \frac{C(\alpha)}{N_k^{6-2\alpha}}\, T_k^\alpha \;.$$

In this inequality, $k \in \mathbf{N}$ is arbitrary, $C(\alpha)$ is a numerical factor only depending on α, the sequence of times T_k satisfies $b_{N_k} 2^{N_k - 1} < 9T_k \leq b_{N_k} 2^{N_k}$ for all k, and $b_{N_k} > N_k^{-2}$. From all that it follows that $T_k \to \infty$ as $k \to \infty$ and that $N_k < f^{-1}(T_k)$ where f is the function $f(x) = 2^x/(18\, x^2)$. Hence we get that, for all $k \in \mathbf{N}$,

$$\overline{n}\left(\frac{b_{N_k}}{2}, T_k\right) \;>\; \frac{C(\alpha)}{(f^{-1}(T_k))^{6-2\alpha}}\, T_k^\alpha \;.$$

We denote the right-hand side by C_k. Since $\alpha > 0$, C_k will be eventually larger than 1. From the definition (1) of a minimal carrier, we obtain that, at all large enough k, the total probability supported by basis states $|n\rangle$ with $n > C_k - 1$ is larger than $b_{N_k}/2$. For such k's, the Chebyshev inequality yields:

$$M_q(T_k) \;\geq\; \frac{b_{N_k}}{2}\,(C_k - 1)^q \;.$$

Whence, replacing C_k and using $b_{N_k} > N_k^{-2}$, we get

$$M_q(T_k) \;>\; \frac{1}{2(f^{-1}(T_k))^2}\left(\frac{C(\alpha) T_k^\alpha}{(f^{-1}(T_k))^{6-2\alpha}} - 1\right)^q \;.$$

Since $\lim_{T\to\infty} \log(f^{-1}(T))/\log(T) \;=\; 0$, Theorem 1 follows from the definition of the exponent β_q^+. □

Remark 1 The present proof does not allow to bound β_0^+ below by means of $\dim_P^+(\mu)$. In order to do that one needs to know that $\limsup \mu(A_{N,\alpha}) > 0$. We do not know whether that follows from $\dim_P^+(\mu) > \alpha$. Nevertheless we can prove the weaker result given in Proposition 3 below.

For our next result, let us define

$$\mathcal{D}(\mu) \ = \ \sup_{\epsilon} \ \limsup_{N\to\infty} \frac{\log_2(\mathcal{N}(\mu, N, \epsilon))}{N} \tag{14}$$

where, for given N and ϵ, we define $\mathcal{N}(\mu, N, \epsilon)$ as the minimal number of dyadic intervals of generation N which support more than $1 - \epsilon$ of the measure μ. It is shown in the appendix that $\mathcal{D}(\mu)$ is bigger than or equal to the box-counting information dimension of μ and smaller than or equal to its packing dimension.

Proposition 3 $\beta_0^+ \geq \mathcal{D}(\mu)$.

Proof. It is enough to show that, if α is less than the expression on the rhs of (14), then $\limsup \mu(A_{N,\alpha}) > 0$. Let us choose ϵ and a sequence N_k so that

$$\alpha \ < \ \lim_{k\to\infty} \frac{\log_2(\mathcal{N}(\mu, N_k, \epsilon))}{N_k} \ . \tag{15}$$

If $\lim \mu(A_{N,\alpha}) = 0$, then the sequence of characteristic functions $\chi_{A_{N_k,\alpha}}$ converges to 0 in μ-measure. So there is a sequence N_{k_j} such that $\chi_{A_{N_{k_j},\alpha}}$ converges to 0 μ-almost everywhere. We can then find a compact K, with $\mu(K) > 1 - \epsilon$, so that $\chi_{A_{N_{k_j},\alpha}}$ converges uniformly to 0 in K. Hence, K is eventually a subset of all the $A^c_{N_{k_j},\alpha}$. Thus, for all sufficiently large j, K has a covering by dyadic intervals of the N_{k_j}th generation, everyone of which is not smaller than $2^{-\alpha N_{k_j}}$ in μ-measure. There cannot be more than $(1 - \epsilon)2^{\alpha N_{k_j}}$ intervals in those coverings. Therefore, $\mathcal{N}(\mu, N_{k_j}, \epsilon) \leq (1 - \epsilon)2^{\alpha N_{k_j}}$, which contradicts (15). □

Remark 2 An explicit albeit abstract illustration of the lower bounds proven above is given in [G3]. There H is multiplication by E in $L^2((0, 2\pi), \mu)$, $|n\rangle = \exp(2\pi\imath nF(E))$, where $F(E) = \mu((0, E))$, and the measure μ has $\dim_H^+(\mu) = 0 = \dim_I^-(\mu)$, $\dim_P^+(\mu) = 1 = \dim_I^+(\mu)$. That measure is constructed following [RJLS]. The motion in this model is ballistic, in the sense that $\beta_q^+ = 1$ for all $q > 0$.

5 Lower bound for covariant lattice Hamiltonians.

Here we transpose the results of the last section to quantum diffusion of dynamics governed by covariant Hamiltonians on the lattice. Because disordered media and quasicrystals can be described by these models, this situation is of particular physical interest. We do not furnish the technical proofs for the various statements in this section because they can be completed along the lines of [SB].

Let the space of disorder or quasicrystaline configurations Ω be a compact and metrizable space on which is given an action T of the group $\mathbf{Z}^D$. We suppose that to each configuration $\omega \in \Omega$ there is a bounded operator $H_\omega : \ell^2(\mathbf{Z}^D) \to \ell^2(\mathbf{Z}^D)$ and that this operator family is strongly continuous in ω and covariant with respect to a projective representation U of the translation group $\mathbf{Z}^D$ on $\ell^2(\mathbf{Z}^D)$, that is

$$U(a) H_\omega U(a)^* \;=\; H_{T^a\omega} \;, \qquad a \in \mathbf{Z}^D \;.$$

Finally we fix an invariant and ergodic probability measure $\mathbf{P}$ on Ω. Let now $\psi_\omega \in \ell^2(\mathbf{Z}^D)$ be a cyclic vector for H_ω and μ_ω its spectral measure. It can then be shown by the same techniques as in [SB] that the packing dimension $\dim_{\mathrm{P}}(\mu_\omega, \Delta)$ in the Borel set $\Delta \subset \mathbf{R}$ (see the appendix for the definition) is $\mathbf{P}$-almost surely constant and thus defines the packing dimension of the local density of states in Δ. It is smaller than or equal to the packing dimension of the density of states $\mathcal{N}$ which is defined to be the disorder average of the spectral measure $\mu_{\omega,|0\rangle}$ of the state $|0\rangle$ localized at the origin, namely we have

$$\dim_{\mathrm{P}}(\mu_\omega, \Delta) \;\leq\; \dim_{\mathrm{P}}(\mathcal{N}, \Delta) \;, \qquad \mathbf{P}\text{-a.s.} \;.$$

Next we define the time averaged moments of the position operator $\vec{X}$ on $\ell^2(\mathbf{Z}^D)$ by

$$\hat{M}_{q,\Delta,\omega}(T) \;=\; \int_0^T \frac{dt}{T} \langle 0|\, \Pi_\omega(\Delta)\, |e^{\imath H_\omega t} \vec{X} e^{-\imath H_\omega t} - \vec{X}|^q \,\Pi_\omega(\Delta)\, |0\rangle \;, \qquad q \neq 0 \;,$$

where $\Pi_\omega(\Delta)$ is the spectral projection of H_ω to Δ. The corresponding growth exponents $\hat{\beta}^{\pm}_{q,\omega}(\Delta)$ are defined as in (3). Modification of the proof of Theorem 1 (for fixed H_ω) leads to

$$\dim_{\mathrm{P}}(\mu_\omega, \Delta) \;\leq\; D\, \hat{\beta}^{+}_{q,\omega}(\Delta) \;, \qquad \forall\; \omega \in \Omega \;.$$

Appendix: various dimensions of Borel measures.

The aim of this appendix is to review and extend known results about lower and upper pointwise dimensions, Hausdorff and packing dimensions, as well as fractal and box-counting information dimensions of Borel measures on the real line (the extension to $\mathbf{R}^d$ is immediate). Links between lower pointwise dimensions and Hausdorff dimensions were first established by Rogers and Taylor (see [Ro]). The corresponding theory connecting upper pointwise dimensions and packing dimensions was given by Cutler [Cu], whose results we extend here to a completely dual treatment to that in [Ro, Chapters 3.2 and 3.3]. Fractal dimensions of Borel measures were studied by one of the authors [G2] and we complete here the results given in the latter reference. We also review box-counting information dimensions and establish relations to Hausdorff and packing dimensions. Throughout this appendix, μ and ν are Borel probability measures on $\mathbf{R}$, Δ is a Borel subset of $\mathbf{R}$ and $\gamma \in \mathbf{R}$.

The lower and upper pointwise dimensions of μ at a point $E \in \mathbf{R}$ were defined in equation (4) of Section 3. For various alternative definitions, see [RJLS, SB]. In [SB] it is proven that $\underline{d}_\mu(E) \leq \underline{d}_\nu(E)$ μ-almost surely, by a similar proof $\overline{d}_\mu(E) \leq \overline{d}_\nu(E)$ μ-almost surely. These results imply [SB] that $E \mapsto \underline{d}_\mu(E)$ and $E \mapsto \overline{d}_\mu(E)$ are Borel functions in $L^\infty(\mathbf{R}, \mu)$ taking values in $[0,1]$ and only depending on the measure class of μ. Let us further introduce as in [Ro] the upper and lower γ-derivative of μ at E by

$$\overline{D}^\gamma_\mu(E) \;=\; \limsup_{\epsilon\to 0} \frac{\mu([E-\epsilon, E+\epsilon])}{\epsilon^\gamma}\;, \qquad \underline{D}^\gamma_\mu(E) \;=\; \liminf_{\epsilon\to 0} \frac{\mu([E-\epsilon, E+\epsilon])}{\epsilon^\gamma}\;.$$

Links between lower pointwise dimensions and upper γ-derivatives are given by

$$\underline{d}_\mu(E) < \gamma \;\;\Rightarrow\;\; \overline{D}^\gamma_\mu(E) = \infty \;\;\Rightarrow\;\; \underline{d}_\mu(E) \leq \gamma\;, \tag{16}$$

$$\gamma < \underline{d}_\mu(E) \;\;\Rightarrow\;\; \overline{D}^\gamma_\mu(E) = 0 \;\;\Rightarrow\;\; \gamma \leq \underline{d}_\mu(E)\;. \tag{17}$$

Analogous relations hold between upper pointwise dimensions and lower γ-derivatives. Next we denote the γ-Hausdorff and the γ-packing measure by M^γ_{H} and M^γ_{P} respectively. The Hausdorff and packing dimension of Δ are denoted by $\dim_{\mathrm{H}}(\Delta)$ and $\dim_{\mathrm{P}}(\Delta)$. As we shall use it in the proofs below, let us recall the definition of the packing measure [TT]. A δ-packing of an arbitrary set $D \subset \mathbf{R}$ is a countable disjoint collection $(B(E_k, r_k))_{k\in\mathbf{N}}$ of closed balls centered at $E_k \in D$ and with radius $r_k \leq \delta/2$. A positive set function is defined by

$$M^{\gamma,\delta}_{\mathrm{P}}(D) \;=\; \sup\left\{ \sum_{k\in\mathbf{N}} (2r_k)^\gamma \;\middle|\; (B(E_k,r_k))_{k\in\mathbf{N}} \;\; \delta\text{-packing of } D \right\}\;. \tag{18}$$

The γ-packing measure is constructed in two steps:

$$\tilde{M}^\gamma_{\mathrm{P}}(D) \;=\; \lim_{\delta\to 0} M^{\gamma,\delta}_{\mathrm{P}}(D)\;, \tag{19}$$

$$M^\gamma_{\mathrm{P}}(D) \;=\; \inf\left\{ \sum_{n\in\mathbf{N}} \tilde{M}^\gamma_{\mathrm{P}}(\Delta_n) \;\middle|\; \Delta_n \text{ Borel}, \; \bigcup_{n\in\mathbf{N}} \Delta_n = D \right\}\;, \tag{20}$$

that is, M^γ_{P} is an metric outer measure in the sense of Caratheodory. The corresponding Borel measure is also denoted by M^γ_{P}. The packing dimension $\dim_{\mathrm{P}}(\Delta)$ of a Borel set Δ is defined as the infimum of all γ such that $M^\gamma_{\mathrm{P}}(\Delta) = 0$.

For the case of the upper γ-derivative and the γ-Hausdorff measure (the inequalities on the left hand-side of (21) and (22)), the following theorem summarizes the main technical results of [Ro, Chapter 3.2 and 3.3]. For the case of the lower γ-derivative and the γ-packing dimension, it is strictly speaking new, but the proof uses similar techniques as in [Cu]. Not only the results, but also the proofs show some kind of duality between Hausdorff dimensions and lower pointwise dimensions on one side and packing dimensions and upper pointwise dimensions on the other: for the Hausdorff measure case, the proof of (21) is based on a covering lemma, and (22) follows directly from the definitions; for the packing measure case, the situation is just the converse.

Theorem 2 *For $\lambda > 0$ we have*

$$M_{\mathrm{H}}^{\gamma}(\{E \in \Delta \mid \overline{D}_{\mu}^{\gamma}(E) > \lambda\}) \leq \frac{6^{\gamma}}{\lambda}, \qquad M_{\mathrm{P}}^{\gamma}(\{E \in \Delta \mid \underline{D}_{\mu}^{\gamma}(E) > \lambda\}) \leq \frac{2^{\gamma}}{\lambda}, \tag{21}$$

and

$$\mu(\{E \in \Delta \,|\, \overline{D}_{\mu}^{\gamma}(E) < \lambda\}) \leq 2{-}^{\gamma}\lambda\, M_{\mathrm{H}}^{\gamma}(\Delta), \quad \mu(\{E \in \Delta \,|\, \underline{D}_{\mu}^{\gamma}(E) < \lambda\}) \leq 2^{-\gamma}\lambda\, M_{\mathrm{P}}^{\gamma}(\Delta). \tag{22}$$

Proof. The first inequalities in (21) and (22) being proven in [Ro], we here only prove the packing dimension part. Let $R_{\mu}^{\gamma}(\delta, \lambda) = \{E \in \Delta \,|\, \inf_{\epsilon<\delta} \mu([E-\epsilon, E+\epsilon])\epsilon^{-\gamma} \geq \lambda\}$. Then for any $E \in R_{\mu}^{\gamma}(\delta,\lambda)$, $\mu([E-\epsilon, E+\epsilon]) \geq \lambda\epsilon^{\gamma} \quad \forall\ \epsilon < \delta$. Therefore, if $(B(E_k, r_k))_{k\in\mathbf{N}}$ is a δ-packing of $R_{\mu}^{\gamma}(\delta,\lambda)$, we have $\sum_k (2r_k)^{\gamma} \leq 2^{\gamma}/\lambda$ because the elements of the δ-packing are disjoint. Consequently, $M_{\mathrm{P}}^{\gamma}(R_{\mu}^{\gamma}(\delta,\lambda)) \leq M_{\mathrm{P}}^{\gamma,\delta}(R_{\mu}^{\gamma}(\delta,\lambda)) \leq 2^{\gamma}\mu(\Delta)/\lambda$. Moreover, $\bigcup_{n\in\mathbf{N}} R_{\mu}^{\gamma}(1/n,\lambda) = \{E \in \Delta \mid \underline{D}_{\mu}^{\gamma}(E) > \lambda\}$ so that $M_{\mathrm{P}}^{\gamma}(\{E \in \Delta \mid \underline{D}_{\mu}^{\gamma}(E) > \lambda\}) = \sup_n M_{\mathrm{P}}^{\gamma}(R_{\mu}^{\gamma}(1/n,\lambda)) \leq 2^{\gamma}\mu(\Delta)/\lambda$ due to the σ-additivity of M_{P}^{γ}.

To prove (22), we shall use (as in [Cu]) the fact that any Borel measure on $\mathbf{R}$ possesses the centered Vitali covering property [Be]. A centered Vitali covering of Δ is a set of closed balls containing for any $E \in \Delta$ and $\delta > 0$ some closed ball $B(E,r)$ with $r \leq \delta$. The centered covering property of μ means that every centered Vitali covering of Δ contains a countable set of disjoint balls B_k such that $\mu(\Delta \setminus \bigcup_k B_k) = 0$.

Let then $\Delta_n \subset \{E \in \Delta \,|\, \underline{D}_{\mu}^{\gamma}(E) < \lambda\}$. For any $E \in \Delta_n$ and $\delta > 0$, there exists $r < \delta$ such that $\mu([E-r, E+r]) \leq \lambda r^{\gamma}$. Hence the set of balls $B(E,r)$ such that $r \leq \delta$, $E \in \Delta_n$ and that $\mu([E-r, E+r]) \leq \lambda r^{\gamma}$ is a centered Vitali covering of Δ_n. Let $(B(E_k, r_k))_{k\in\mathbf{N}}$ be the associated δ-packing satisfying $\mu(\Delta_n \setminus \bigcup_k B(E_k, r_k)) = 0$ as given by the centered Vitali covering property. Then $\mu(\Delta_n) \leq \sum_k \mu(B(E_k, r_k)) \leq \lambda \sum_k r_k^{\gamma}$. As this holds for any $\delta > 0$, we have $\mu(\Delta_n) \leq 2^{-\gamma}\lambda \tilde{M}_{\mathrm{P}}^{\gamma}(\Delta_n)$. As the decomposition $\Delta = \bigcup_n \Delta_n$ in (20) can be chosen disjoint, the result follows. □

Before drawing the for us interesting consequences of Theorem 2, let us introduce some further notations. We define $\underline{d}_{\mu}^{+}(\Delta) = \mu\text{-}\operatorname{ess\,sup}_{E\in\Delta} \underline{d}_{\mu}(E)$ as well as $\underline{d}_{\mu}^{-}(\Delta) = \mu\text{-}\operatorname{ess\,inf}_{E\in\Delta} \underline{d}_{\mu}(E)$, and similarly $\overline{d}_{\mu}^{+}(\Delta)$ and $\overline{d}_{\mu}^{-}(\Delta)$. By the above remarks, these quantities only depend on the measure class of μ [SB].

The upper Hausdorff dimension $\dim_{\mathrm{H}}^{+}(\mu, \Delta)$ of μ in Δ is defined by the infimum of the Hausdorff dimensions of all Borel subsets $\Delta' \subseteq \Delta$ satisfying $\mu(\Delta') = \mu(\Delta)$ [Yo]. The lower Hausdorff dimension $\dim_{\mathrm{H}}^{-}(\mu, \Delta)$ of μ in Δ is defined as the supremum of the α's such that $\Delta' \subset \Delta$, $\dim_H(\Delta') \leq \alpha$ imply $\mu(\Delta') = 0$. The packing dimensions $\dim_{\mathrm{P}}^{\pm}(\mu, \Delta)$ are once more defined similarly. If $\Delta = \mathbf{R}$, we further drop the specification.

For the case of the Hausdorff dimension and lower pointwise dimensions, the first of the following corollaries already appears in [Ro]. A version of Corollary 2 can be found in [Cu]. Corollaries 4 and 5 appear in [G2, Co] and in [G2], respectively, for the case of the Hausdorff dimension; here we give a different proof.

Corollary 1 $\dim_{\rm H}(\Delta) < \underline{d}_\mu^-(\Delta)$ *implies* $\mu(\Delta) = 0$. $\dim_{\rm P}(\Delta) < \overline{d}_\mu^-(\Delta)$ *implies* $\mu(\Delta) = 0$.

Corollary 2 *The following identities hold:*

$$\dim_{\rm H}(\{E \in \Delta \mid \underline{d}_\mu(E) \leq \underline{d}_\mu^+(\Delta)\}) = \underline{d}_\mu^+(\Delta) \ ,$$

$$\dim_{\rm P}(\{E \in \Delta \mid \overline{d}_\mu(E) \leq \overline{d}_\mu^+(\Delta)\}) = \overline{d}_\mu^+(\Delta) \ .$$

Corollary 3
i) *Let* $\underline{\Delta}^+ = \{E \in \Delta \mid \underline{d}_\mu(E) = \underline{d}_\mu^+(\Delta)\}$. *Either* $\mu(\underline{\Delta}^+) = 0$ *or* $\dim_{\rm H}(\underline{\Delta}^+) = \underline{d}_\mu^+(\Delta)$.
ii) *Let* $\overline{\Delta}^+ = \{E \in \Delta \mid \overline{d}_\mu(E) = \overline{d}_\mu^+(\Delta)\}$. *Either* $\mu(\overline{\Delta}^+) = 0$ *or* $\dim_{\rm P}(\overline{\Delta}^+) = \overline{d}_\mu^+(\Delta)$.

Corollary 4 $\dim_{\rm H}^+(\mu,\Delta) = \underline{d}_\mu^+(\Delta)$ *and* $\dim_{\rm P}^+(\mu,\Delta) = \overline{d}_\mu^+(\Delta)$.

Corollary 5 $\dim_{\rm H}^-(\mu,\Delta) = \underline{d}_\mu^-(\Delta)$ *and* $\dim_{\rm P}^-(\mu,\Delta) = \overline{d}_\mu^-(\Delta)$.

Due to the complete symmetry of the results in Theorem 2, it is sufficient to prove the corollaries for the Hausdorff dimension case.

Proof of Corollary 1. Let γ be such that $\dim_{\rm H}(\Delta) < \gamma < \underline{d}_\mu^-(\Delta)$. Then $M_{\rm H}^\gamma(\Delta) = 0$ and by (17) and (22) one has

$$\mu(\Delta) = \mu(\{E \in \Delta \,|\, \gamma < \underline{d}_\mu(E)\}) \leq \mu(\{E \in \Delta \,|\, \overline{D}_\mu^\gamma(E) = 0\}) = 0 \ .$$

□

Proof of Corollary 2. First note that, for any $d > 0$, the set $\Delta_d = \{E \in \Delta \mid \underline{d}_\mu(E) < d\}$ has a Hausdorff dimension less or equal than d: in fact, by equations (16) and (21) one has $M_{\rm H}^d(\Delta_d) \leq M_{\rm H}^d(\{E \in \Delta \mid D_\mu^d = \infty\}) = 0$. Now for $n \in \mathbf{N}$ let $d_n = \underline{d}_\mu^+(\Delta) + 1/n$. Then $\Delta_0 = \bigcap_{n\in\mathbf{N}} \Delta_{d_n} = \{E \in \Delta | \underline{d}_\mu(E) \leq \underline{d}_\mu^+(\Delta)\}$ satisfies according to the above $\dim_{\rm H}(\Delta_0) \leq \underline{d}_\mu^+(\Delta)$. Now suppose there exists a γ such that $\dim_{\rm H}(\Delta_0) < \gamma < \underline{d}_\mu^+(\Delta)$. Then $M_{\rm H}^\gamma(\Delta_0) = 0$ and because of $\mu(\Delta_0) = \mu(\Delta)$, (17) and (22) we have

$$\mu(\{E \in \Delta \,|\, \gamma < \underline{d}_\mu(E)\}) \ = \ \mu(\{E \in \Delta_0 \,|\, \gamma < \underline{d}_\mu(E)\}) \ \leq \ \mu(\{E \in \Delta \,|\, \overline{D}_\mu^\gamma(E) = 0\}) \ = \ 0 \ , \tag{23}$$

which is in contradiction to $\gamma < \underline{d}_\mu^+(\Delta)$. Therefore $\dim_{\rm H}(\Delta_0) = \underline{d}_\mu^+(\Delta)$. □

Proof of Corollary 3. If $E \in \underline{\Delta}^+$, then $\overline{D}_\mu^\gamma(E) = 0$ for all $\gamma < \underline{d}_\mu^+(\Delta)$. Hence $\underline{\Delta}^+ \subset \{E \in \underline{\Delta}^+ | \overline{D}_\mu^\gamma(E) < \lambda\}$ for any $\lambda \geq 0$. Now by (22), $\mu(\underline{\Delta}^+) \leq 2^{-\gamma}\lambda M_{\rm H}^\gamma(\underline{\Delta}^+)$. Therefore either $\mu(\underline{\Delta}^+) = 0$ or $M_{\rm H}^\gamma(\underline{\Delta}^+) = \infty$ which by Corollary 2 implies $\dim_{\rm H}(\underline{\Delta}^+) = \underline{d}_\mu^+(\Delta)$. □

Proof of Corollary 4. We have $\dim_{\rm H}(\mu,\Delta) \leq \dim_{\rm H}(\Delta_0) = \underline{d}_\mu^+(\Delta)$ with Δ_0 as above. Suppose a γ exists such that $\dim_{\rm H}(\mu,\Delta) < \gamma < \underline{d}_\mu^+(\Delta)$. Then there exists a set $\Delta' \subset \Delta$ with $M_{\rm H}^\gamma(\Delta') = 0$ and $\mu(\Delta') = \mu(\Delta)$. As in (23), this is impossible. □

Proof of Corollary 5. Corollary 1 directly implies $\underline{d}_\mu^-(\Delta) \leq \dim_H^-(\mu,\Delta)$. Furthermore, by the definition of essential infimum there must exist a set $\Delta' \subseteq \Delta$ with $\mu(\Delta') > 0$

such that $\underline{d}_\mu^+(\Delta') \geq \underline{d}_\mu^-(\Delta)$. Then Corollary 2 implies $\dim_H(\Delta') \geq \underline{d}_\mu^-(\Delta)$. Therefore $\underline{d}_\mu^-(\Delta) \geq \dim_H^-(\mu,\Delta)$ by the definition of the latter dimension. □

A measure μ is said to have *exact (Hausdorff) dimension* d in a Borel set Δ if $\dim_H^-(\mu,\Delta) = \dim_H^+(\mu,\Delta)$, that is, if its lower pointwise dimension is μ-a.e. equal to d in Δ. In parallel, one can say that μ has *exact (packing) dimension* d' in Δ if its upper pointwise dimension is μ -a.e. equal to d' in Δ. In ref. [G2] an *exactly scaling* measure μ was defined as one for which the upper and lower local dimensions have a common μ-a.e. constant value. For exactly scaling measures the Hausdorff and the packing dimension coincide, but the converse is not true.

In the physical literature, box-counting dimensions have found wide use because of their easy numerical implementation. It is interesting to note that packing dimensions can also be computed by a box-counting procedure. First we recall that the upper and lower box-counting dimension of a compact set K are defined as

$$\dim_{\mathrm{B}}^+(K) \;=\; \limsup_{\delta\to 0} \frac{\log(N_\delta(K))}{-\log(\delta)} \;, \qquad \dim_{\mathrm{B}}^-(K) \;=\; \liminf_{\delta\to 0} \frac{\log(N_\delta(K))}{-\log(\delta)} \;,$$

where $N_\delta(K)$ denotes the minimal number of closed intervals of size δ needed to cover K (or equivalently, by the box-counting theorem [Fa], the number of elements of a grid cover of size δ which overlap K). The upper and lower fractal dimensions of μ in $\Delta \subset \mathbf{R}$ are now defined as in [G2] by

$$\dim_{\mathrm{F}}^{\pm}(\mu,\Delta) \;=\; \sup_{\eta>0}\ \inf_{K\subset\Delta} \{\, \dim_{\mathrm{B}}^{\pm}(K) \mid \mu(K) > \mu(\Delta) - \eta \;,\; K \text{ compact} \,\} \;.$$

Theorem 3 $\dim_{\mathrm{H}}^+(\mu,\Delta) \leq \dim_{\mathrm{F}}^-(\mu,\Delta) \leq \dim_{\mathrm{F}}^+(\mu,\Delta) = \dim_{\mathrm{P}}^+(\mu,\Delta)$.

Proof. To prove the first inequality, we replace in the definition of $\dim_{\mathrm{F}}^-(\mu,\Delta)$ the value of $\dim_{\mathrm{F}}^-(K)$ by $\dim_{\mathrm{H}}(K)$ and call the result $\tilde d$. Then there exist a sequence $(K_n)_{n\geq 1}$ of compact subsets of Δ satisfying $\mu(K_n) > \mu(\Delta) - 1/n$ and $\lim_{n\to\infty} \dim_{\mathrm{H}}(K_n) = \tilde d$. Now $\bigcup_n K_n$ supports μ in Δ and has Hausdorff dimension $\tilde d$ such that $\tilde d \geq \dim_{\mathrm{H}}^+(\mu,\Delta)$. On the other hand, $\tilde d \leq \dim_{\mathrm{F}}^-(\mu,\Delta)$ because $\dim_{\mathrm{H}}(K) \leq \dim_{\mathrm{F}}^-(K)$ [Fa].

Next let us recall from [G2] that $\dim_{\mathrm{F}}^+(\mu,\Delta) \leq \dim_{\mathrm{P}}^+(\mu,\Delta)$, so we only need to prove the converse inequality. Let $\eta_n \in \mathbf{R}$ and $K_n \subset \Delta$, $n \in \mathbf{N}$, be such that $\mu(K_n) > \mu(\Delta) - \eta_n$, $\lim_{n\to\infty} \eta_n = 0$ and $\sup_{n\geq 1} \dim_{\mathrm{B}}^+(K_n) = \dim_{\mathrm{F}}(\mu,\Delta)$. We set $K_\infty = \bigcup_{n\geq 1} K_n$ so that $\mu(K_\infty) = \mu(\Delta)$. Hence, by definition of $\dim_{\mathrm{P}}^+(\mu,\Delta)$ and countable stability of packing dimensions,

$$\dim_{\mathrm{P}}^+(\mu,\Delta) \;\leq\; \dim_{\mathrm{P}}(K_\infty) \;=\; \sup_{n\geq 1} \dim_{\mathrm{P}}(K_n) \;\leq\; \sup_{n\geq 1} \dim_{\mathrm{B}}^+(K_n) \;=\; \dim_{\mathrm{F}}(\mu,\Delta) \;,$$

where we used the fact that $\dim_{\mathrm{P}}(K) \leq \dim_{\mathrm{B}}^+(K)$ for any set $K \subset \mathbf{R}$ [Fa]. □

We finally examine the dimension $\mathcal{D}(\mu)$ introduced in (14) as well as box-counting information dimensions defined next. Given $N \in \mathbf{N}$, the Shannon entropy ("missing information") of the measure μ relative to the partition of the real line in dyadic intervals I_j^N, $j \in \mathbf{Z}$), of the Nth generation is given by:

$$S_N(\mu) \;=\; -\sum_{j\in\mathbf{Z}} \mu(I_j^N)\log_2(\mu(I_j^N)) \;, \tag{24}$$

where by convention those j's for which $\mu(I_j^N) = 0$ do not contribute in the sum. We consider the class $\mathcal{E}$ of measures μ for which $S_1(\mu)$ is finite. Then $S_N(\mu)$ is also finite for all N (see below) and the upper and lower information dimensions of μ are defined by

$$\dim_I^+(\mu) \;=\; \limsup_{N\to\infty} \frac{S_N(\mu)}{N} \;, \qquad \dim_I^-(\mu) \;=\; \liminf_{N\to\infty} \frac{S_N(\mu)}{N} \;.$$

Then we have the following:

Theorem 4 *If $\mu \in \mathcal{E}$, then $\dim_H^-(\mu) \le \dim_I^-(\mu)$ and $\dim_I^+(\mu) \le \mathcal{D}(\mu) \le \dim_P^+(\mu)$.*

Proof. Let $A_{N,\alpha}$ be defined as in Proposition 2. It is immediate that:

$$\dim_I^-(\mu) \;\ge\; \alpha \liminf \mu(A_{N,\alpha}) \;\ge\; \alpha\,\mu(\liminf A_{N,\alpha}) \;,$$

which is not less than α if $\alpha < \dim_H^-(\mu)$ because of (13) and of Corollary 5. This proves the first inequality. To prove the second one, let $\mathcal{N}(\mu, N, \epsilon)$ be as in equation (14), and let $B_N \subset \mathbf{R}$ be the union of exactly $\mathcal{N}(\mu, N, \epsilon)$ dyadic intervals of the Nth generation, with $\mu(B_N) > 1 - \epsilon$. Let us fix $N_0 \in \mathbf{N}$ so that

$$-\sum_{|j|\ge N_0} \mu(I_j^1)\log_2(\mu(I_j^1)) \;<\; \epsilon \;.$$

We denote $K = (-N_0, N_0]$ and we note that, if ϵ is small enough, then $\mu(K^c) < \epsilon$. Now we split the sum in (24) in three terms $S_i(\mu, N, \epsilon)$, $i = 1, 2, 3$, which result of summing over different sets J_i of indices j, namely $J_1 = \{j \in \mathbf{Z} | I_j^N \subset K^c\}$, $J_2 = \{j \in \mathbf{Z} | I_j^N \cap B_N \cap K \neq \emptyset\}$ and $J_3 = \{j \in \mathbf{Z} | I_j^N \subset K \setminus B_N\}$. We denote P_i, $i = 1, 2, 3$, the total measure of the intervals whose label belongs in J_i. From well-known properties of the conditional entropy it follows that

$$S_1(N+1) \;\le\; S_1(N) + P_1\log_2(2) \;\le\; S_1(N) + \epsilon \;\le\; (N+1)\epsilon \;.$$

Again from conditioning we also get that

$$S_2(N) \;\le\; P_2\log_2(\#J_2) - P_2\log_2(P_2) \;\le\; \log_2(\mathcal{N}(\epsilon, \mu, N)) + c \;,$$

where c is the maximum of $-x\log_2 x$ in $(0, 1)$. In the same way,

$$S_3(N) \;\le\; P_3\log_2(\#J_3) - P_3\log_2(P_3) \;\le\; \epsilon\log_2(N_0 2^{N+1}) + \epsilon\log_2\left(\frac{1}{\epsilon}\right) \;.$$

Putting all these estimates together, we get

$$\frac{S_N(\mu)}{N} \leq \frac{\log_2(\mathcal{N}(\epsilon,\mu,N))}{N} + 2\epsilon + \mathcal{O}\left(\frac{1}{N}\right) ,$$

whence:

$$\dim_I^+(\mu) \leq 2\epsilon + \mathcal{D}(\mu) \leq 2\epsilon + \dim_F^+(\mu) ,$$

because a Nth generation dyadic covering of a compact K with $\mu(K) > 1-\epsilon$ cannot be obtained with less than $\mathcal{N}(\epsilon,\mu,N)$ intervals, due to the very definition of the latter quantity. As ϵ is arbitrary and $\dim_F^+(\mu) = \dim_P^+(\mu)$, we get the second inequality in the thesis. □

References

[Be] A. S. Besicovitch, *A general form of the covering principle and relative differentiation of additive functions*, Proc. Camb. Phil. Soc. **41**, 103-110 (1945).

[Co] J.-M. Combes, in *Differential Equations with applications to Mathematical Physics*, edited by W.F.Ames, E.M.Harrel and I.V.Herod (Academic, Boston 1993); J.-M. Barbaroux, J.-M. Combes, R. Montcho, *Remarks on the relation between quantum dynamics and fractal spectra*, J. Math. Anal. and Appl. 213, 698-722 (1997).

[Cu] C. D. Cutler, *Measure disintegrations with respect to σ-stable monotone indices and the pointwise representation of packing dimensions*, Supp. Rend. Circ. Mat. Palermo **28**, serie II, 319-339 (1992).

[Fa] K. Falconer, *Fractal Geometry - Mathematical Foundations and Applications*, John Wiley & Sons, 1990.

[G1] I. Guarneri, *Spectral properties of quantum diffusion on discrete lattices*, Europhys. Lett., **10**, 95-100, (1989); *On an estimate concerning quantum diffusion in the presence of a fractal spectrum* , Europhys. Lett., **21**, 729-733, (1993); and I. Guarneri, G. Mantica, *On the asymptotic properties of quantum dynamics in the presence of fractal spectrum*, Ann. Inst. H. Poincaré, **61**, 369-379, (1994).

[G2] I. Guarneri, *Singular continuous spectra and discrete wave packet dynamics*, J. Math. Phys. **37**, 5195-5206, (1996).

[G3] I. Guarneri, *On the Dynamical Meaning of Spectral Dimensions*, to appear in Ann. Inst. H. Poincaré, 1998.

[GM] I. Guarneri, G. Mantica, *Multifractal Energy Spectra and Their Dynamical Implications*, Phys. Rev. Lett. **73**, 3379-3383 (1994).

[GS] I. Guarneri, H. Schulz-Baldes, *Upper bounds for quantum dynamics governed by Jacobi matrices with self-similar spectra*, preprint, Como 1998.

[KKK] R. Ketzmerick, K. Kruse, S. Kraut, T. Geisel, *What determines the spreading of a wave packet?*, Phys. Rev. Lett. **79**, 1959-1962 (1997).

[La] Y. Last, *Quantum dynamics and decomposition of singular continuous spectra*, J. Funct. Analysis **142**, 402-445 (1996).

[RJLS] R. del Rio, S. Jitomirskaya, Y. Last, B. Simon, *Operators with singular continuous spectrum: IV. Hausdorff dimension, rank-one perturbations and localization*, J. d'Analyse Math. **69**, 153-200 (1996).

[Ro] C. A. Rogers, *Hausdorff measures*, (Cambridge University Press, Cambridge, 1970).

[SB] H. Schulz-Baldes, J. Bellissard, *Anomalous transport: a mathematical framework*, Rev. Math. Phys. **10**, 1-46 (1998).

[TT] S. J. Taylor, C. Tricot, *Packing measure and its evaluation for a Brownian path*, Trans. Amer. Math. Soc. **288**, 679-699 (1985); S. J. Taylor, C. Tricot, *The packing measure of rectifiable subsets of the plane*, Math. Proc. Camb. Phil. Soc. **99**, 285-296 (1986).

[Yo] L. S. Young, *Dimension, Entropy, and Lyapunov exponents*, Erg. Th. Dyn. Sys. **2**, 109-124 (1982).