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QUANTIZATION OF A DIMENSIONALLY REDUCED SEIBERG-WITTEN MODULI SPACE

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Abstract. In this paper we apply Quillen's determinant line bundle construction to construct a prequantum line bundle on the moduli space of solutions $\mathcal{N}$ of the dimensionally reduced Seiberg-Witten equations with a Higgs field. The Quillen curvature of the line bundle is shown to be proportional to a symplectic form on the moduli space.

1. Introduction

The problem of quantization of symplectic manifolds can often be related to geometry. Geometric prequantization is a construction of a Hilbert space $\mathcal{H}$, namely, sections of a prequantum line bundle on a symplectic manifold $(\mathcal{M}, \Omega)$ and a correspondence between classical observables - functions on $\mathcal{M}$ - and operators on $\mathcal{H}$ such that the Poisson bracket of the functions corresponds to the commutator of the operators. The latter is ensured by the fact that the curvature of the prequantum line bundle is precisely the symplectic form [18]. A relevant example would be geometric quantization of the moduli space of flat connections. The moduli space of flat connections of a principal G-bundle on a Riemann surface has been quantized by Axelrod, Della Pietra and Witten by a construction of the the determinant line bundle of the Cauchy-Riemann operator, namely, $\mathcal{L} = det(Ker\bar{\partial}_A)^* \otimes det(Coker\bar{\partial}_A)$, [2].

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It carries the Quillen metric such that the canonical unitary connection has a curvature form which coincides with the natural Kähler form on the moduli space of flat connections on vector bundles over M of a given rank [14].

Inspired by [2], we apply Quillen's determinant line bundle construction to construct a prequantum line bundle on the moduli space of solutions to the dimensionally reduced Seiberg-Witten equations in two dimensions, with a Higgs field [5].In [5] we dimensionally reduced the Seiberg-Witten equations with a Higgs field and showed that the moduli space $\mathcal{N}$ of solutions admits a symplectic form. We repeat the construction of the symplectic form here which uses a moment map and Marsden-Weinstein quotient construction. In our case the determinant line bundles are parametrized by unitary connections on a unitary line bundle on a compact Riemann surface of genus $g \geq 1$, sections of the line bundle and a Higgs field. We show that the Quillen curvature of this line bundle is precisely the symplectic form.

2. The moduli space of the dimensional reduction

Let M be a compact Riemann surface of genus $g \geq 1$ with a conformal metric $ds^2 = h^2 dz \otimes d\bar{z}$ and let $\omega = ih^2 dz \wedge d\bar{z}$ be a real form proportional to the induced Kähler form. Let L be a unitary line bundle with a Hermitian metric H such that $\bar{L} = L^{-1}$. Let ψ_1, ψ_2 be sections of the line bundle L i.e., $\psi_1, \psi_2 \in \Gamma(M, L)$. We denote $\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$. We assume that Ψ is not identically zero.

We have an inner product $< \psi_1, \psi_2 >_H$ and norm $|\psi|_H \in C^\infty(M)$ of the sections of L. Since $\bar{L} = L^{-1}$, we can write these without H since $< \psi_1, \psi_2 >$ and $|\psi|$ are sections of the trivial bundle and hence are functions on the surface M. Let $A - \bar{A}$ be a unitary connection on L and $\Phi = \phi dz - \bar{\phi} d\bar{z} \in \Omega^1(M, i\mathbb{R})$.

Dimensional reduction of the Seiberg-Witten equations are written as follows [5]:

$$F(A) = i\frac{(|\psi_1|^2 - |\psi_2|^2)}{2}\omega, \tag{1.1}$$

$$2\bar{\partial}\Phi = -i < \psi_1, \psi_2 > \omega, \tag{1.2}$$

$$\begin{bmatrix} -\frac{1}{2}\bar{\phi} d\bar{z} & \bar{\partial} - \bar{A} \\ \partial + A & -\frac{1}{2}\phi dz \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0. \tag{1.3}$$

Let $\mathcal{C} = \mathcal{A} \times (\Gamma(M, L) \oplus \Gamma(M, L)) \times \Omega^1(M, i\mathbb{R})$, where $\mathcal{A}$ is the space of connections on the line bundle L, $\Gamma(M, L)$ the space of sections of the line bundle and $\Omega^1(M, i\mathbb{R})$ is the space of Higgs fields. The gauge group $\mathcal{G} = Maps(M, U(1))$ acts on $\mathcal{C}$ as $(A, \Psi, \Phi) \to (A + id\tau, e^{-i\tau}\Psi, \Phi)$ and leaves the space of solutions to $(1.1) - (1.3)$ invariant. There are no fixed points of this action. Taking the quotient by the gauge group $\mathcal{G}$ of the solutions to $(1.1) - (1.3)$ we obtain a moduli space which we denote by $\mathcal{N}$, which has a symplectic structure [5].

We showed in [5] that when Ψ is not identically zero, the moduli space has dimension $2g + 2$. We show here that for a trivial bundle L on the torus, genus $g = 1$, the moduli space is non-empty. We repeat the proof here.

Proposition 2.1. *Let L be a trivial line bundle on a compact Riemann surface of genus $g = 1$ then $(1.1) - (1.3)$ has a solution with $\Psi \neq 0, \Phi \neq 0$. Thus $\mathcal{N}$ is non-empty.*

Proof. Let us solve for the case of the case of the torus, $g = 1$. Let our torus be thought of as $0 \leq x \leq 2\pi$ and $0 \leq y \leq 2\pi$ with the endpoints identified. We take the metric on the torus to be $ds^2 = dz \otimes d\bar{z}$, i.e. $h = 1$. The equations are then as follows

(1.1) $F(A) = -\frac{|\Psi_1|^2 - |\Psi_2|^2}{2} dz \wedge d\bar{z} = 0$

(1.2) $\bar{\partial}\Phi = \frac{-1}{2}\Psi_1\bar{\Psi}_2 d\bar{z} \wedge dz$

(1.3a) $\frac{\bar{\partial}\Psi_2}{\Psi_2} - \bar{A} - \frac{1}{2}(\bar{\phi}d\bar{z})\frac{\Psi_1}{\Psi_2} = 0$

(1.3b) $\frac{\partial\Psi_1}{\Psi_1} + A - \frac{1}{2}\phi dz \frac{\Psi_2}{\Psi_1} = 0.$

where $\Phi = \phi dz - \bar{\phi} d\bar{z}$

Since we took the line bundle to be trivial, one solution would be to take $\Psi_1 = c_1$, $\Psi_2 = c_1 e^{ic_2(z+\bar{z})}$, $\phi dz = -ic_2 e^{-ic_2(z+\bar{z})}dz$, $A = -\frac{ic_2}{2}dz$ where c_1 is a complex constant and c_2 is a real constant satisfying $|c_1| = \sqrt{2}c_2$. □

We show later, as we showed in [5], that $\mathcal{N}$ has a natural symplectic form. This form was mentioned in [22] as well. It arises from a form defined on $\mathcal{C}$ as follows

$$
\begin{aligned}
(1.4) \quad \Omega(\mathrm{X},\mathrm{Y}) \quad &= \quad -\int_M \alpha_1 \wedge \alpha_2 + \int_M \frac{1}{2}(-\beta_1\bar{\zeta}_1 + \beta_2\bar{\zeta}_2 + \bar{\beta}_1\zeta_1 - \bar{\beta}_2\zeta_2)h^2 dz \wedge d\bar{z} \\
&- \quad \int_M \gamma_1 \wedge \gamma_2
\end{aligned}
$$

where $I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, where $X = (\alpha_1, \beta, \gamma_1)$, $Y = (\alpha_2, \eta, \gamma_2) \in T_p\mathcal{C}$. It descends to the moduli space $\mathcal{N}$ by a moment map construction which we have elaborated on later in this paper.

3. Prequantum line bundle

For the rest of the paper we shall restrict ourselves to a genus $g \geq 1$ compact Riemann surface, L a unitary line bundle, i.e. $\bar{L} = L^{-1}$.

A very clear description of the determinant line bundle can be found in [14] and [3]. Here we mention the formula for the Quillen curvature of the determinant line bundle $det(Ker\bar{\partial}_A)^* \otimes det(Coker\bar{\partial}_A)$, given the canonical unitary connection ∇_Q, induced by the Quillen metric, [14]. Namely, recall that the affine space $\mathcal{A}$ (notation as in [14]) is an infinite-dimensional Kähler manifold. Here each connection is identified with its $(0,1)$ part. Since the total connection is unitary (i.e. of the form $A - \bar{A}$, where $A = A^{1,0}$, $-\bar{A} = A^{0,1}$) this identification is easy. The complex structure is the Hodge-star operator. In fact, for every $A \in \mathcal{A}$, $T'_A(\mathcal{A}) = \Omega^{0,1}(M, End(L))$ and the corresponding Kähler form is given by

$$F(\alpha, \beta) = \int_M (\alpha \wedge *\beta),$$

where $\alpha, \beta \in \Omega^{0,1}(M, EndL)$, and $* : \Omega^{0,1}(M, End(L)) \to \Omega^{1,0}(M, End(L))$ is the Hodge-star operator such that $*(\eta dz) = -\bar{\eta} d\bar{z}$ and $*(\eta d\bar{z}) = \bar{\eta} dz$. Then one has

$$F(\nabla_Q) = \frac{2i}{\pi}F.$$

Similarly, since the space $\mathcal{A}$ is the space of unitary connections, the ∂-deteminant line bundle $det(Ker\partial_A)^* \otimes det(Coker\partial_A)$ exist. $T''_A = \Omega^{1,0}(M, End(L))$ and the corresponding Kähler form is given by

$$F(\alpha,\beta)=\int_M(\alpha\wedge *\beta),$$

where $\alpha,\beta\in\Omega^{1,0}(M,End(L))$ and $F(\nabla_Q)=\frac{2i}{\pi}F$.

3.1. **Prequantization of the moduli space of solutions to $\mathcal{N}$ for a genus $g\geq 1$ Riemann surface.** Let $\psi_1^0,\psi_2^0\in\Gamma(M,L)$ be two sections of the trival line bundle L, fixed upto gauge equivalence and such that $|\psi_1^0|^2=|\psi_2^0|^2=1$. This is possible to choose since $|\psi_1^0|^2$ and $|\psi_2^0|^2$ are sections of a trivial bundle $(L\oplus\bar{L})$, L being unitary, i.e. $\bar{L}=L^{-1}$. Let a $(1,0)$-form θ be such that $\theta\wedge\bar{\theta}=h^2dz\wedge d\bar{z}$, the Kähler form on the compact Riemann surface [7]. We assume that $|\psi_1|,|\psi_2|$ and $<\psi_1,\psi_2>$ are smooth functions on M.

Theorem 3.1. *The moduli space $\mathcal{N}$ of solutions to* (1.1) $-$ (1.3) *admits a prequantum line bundle P whose Quillen curvature $\mathcal{F}=-\frac{4i}{\pi}\Omega$ where Ω is the natural symplectic form on $\mathcal{N}$ as in* (1.4)*.*

Before proving 3.1 we need some definitions and lemmas. First we note that to the connection $A-\bar{A}$ $(A=A^{1,0},-\bar{A}=A^{0,1})$ one can add any one form such that its $(1,0)$ part is negative of the bar of the $(0,1)$ part and still obtain a derivative operator. We are going to let these new 1-forms be parametrised by components of $\Psi,\bar{\Psi}$ and Φ. In this way we will create 12 new determinant line bundles all parametrised by $\mathcal{C}$. These are constructed by adding new 1-forms, like $(\psi\bar{\psi}_0)\bar{\theta}$, or $\Phi^{0,1}$ to the connection 1-form $A^{0,1}$ to form new Cauchy-Reimann derivative operators. Note that terms like $\psi\bar{\psi}_0$ are functions on M since $\bar{L}=L^{-1}$, so such objects are well defined.

Next we will use a particular combinations of the 12 determinant bundles so constructed so that the Quillen curvature is exactly proportional the natural symplectic form on $\mathcal{N}$. The particular combination which will be useful is $\mathcal{P}=\tilde{\mathcal{L}}_+^1\otimes(\mathcal{L}_+^1)^{-1}\otimes\tilde{\mathcal{L}}_-^1\otimes(\mathcal{L}_-^1)^{-1}\otimes\tilde{\mathcal{L}}_+^2\otimes(\mathcal{L}_+^2)^{-1}\otimes\tilde{\mathcal{L}}_-^2\otimes(\mathcal{L}_-^2)^{-1}\otimes\tilde{\mathcal{R}}_+\otimes(\mathcal{R}_+)^{-1}\otimes\tilde{\mathcal{R}}_-\otimes(\mathcal{R}_-)^{-1}$., where each of these will be defined in what follows.

Definitions : Let us denote by $\mathcal{L}_\pm^1=det(\frac{\bar{\partial}+A^{0,1}}{\sqrt{3}}\pm\frac{\psi_1\bar{\psi}_2^0}{\sqrt{2}}\bar{\theta})$, (where the index 1 in $\mathcal{L}_\pm^1$ stands for ψ_1). This is a determinant line bundle on the affine space $\mathcal{B}^1=\{\frac{A^{0,1}+\alpha^{0,1}}{\sqrt{3}}\pm\frac{(\psi_1+\psi)\bar{\psi}_2^0}{\sqrt{2}}\bar{\theta}|\alpha\in\mathcal{A},\psi\in\Gamma(M,L)\}$. This is isomorphic to $\mathcal{A}^{0,1}\oplus\{f\bar{\theta}\}$ (where $f=\frac{\psi_1\bar{\psi}_2^0}{\sqrt{2}}$). The latter is isomorphic to $\{\alpha^{0,1},\psi_1\}$ which is isomorphic to a subspace of $\mathcal{C}$ where Φ and ψ_2 is kept fixed. Thus the determinant bundle is a line bundle on $\mathcal{C}$ by extending the same fiber for all Φ and ψ_2.

Let us denote by $\mathcal{L}_\pm^2=det(\frac{\bar{\partial}+A^{0,1}}{\sqrt{3}}\pm\frac{\bar{\psi}_2\psi_1^0}{\sqrt{2}}\bar{\theta})$, a determinant line bundle on the affine space $\mathcal{B}^2=\{\frac{A^{0,1}+\alpha^{0,1}}{\sqrt{3}}\pm\frac{(\bar{\psi}_2+\bar{\psi})\psi_1^0}{\sqrt{2}}\bar{\theta}|\alpha\in\mathcal{A},\psi\in\Gamma(L)\}$ which is again isomorphic to a subspace of $\mathcal{C}$. Thus one can define this to be a line bundle on $\mathcal{C}$ by extending the same fiber for all Φ and Ψ_1.

Let us denote by $\mathcal{R}_\pm=det(\frac{\bar{\partial}+A^{0,1}}{\sqrt{3}}\pm\Phi^{0,1})$ a determinant line bundle on $\mathcal{B}_3=\{\frac{A^{0,1}+\alpha^{0,1}}{\sqrt{3}}\pm\Phi^{0,1}|\ \alpha$ and Φ varying$\}$ which is isomorphic to a subspace of $\mathcal{C}$. The line bundle can be extended to be a line bundle on $\mathcal{C}$.

We define in parallel the ∂-determinant line bundles. We have $\tilde{\mathcal{R}}_\pm=det(\frac{\partial+A^{1,0}}{\sqrt{3}}\pm\Phi^{1,0})$, (the notation tilde stands for ∂-determinant line bundles). Similarly, we have

$\tilde{\mathcal{L}}^1_\pm = det(\frac{\partial + A^{1,0}}{\sqrt{3}} \pm \frac{\bar{\psi}_1\psi_2^0}{\sqrt{2}}\theta)$ and $\tilde{\mathcal{L}}^2_\pm = det(\frac{\partial + A^{1,0}}{\sqrt{3}} \pm \frac{\bar{\psi}_1^0\psi_2}{\sqrt{2}}\theta)$. In all these definitions α, ψ, Φ vary. Overlooking the isomorphisms and extending by the same fiber over the parameters which are not present we think of all these bundles as being determinant bundles on $\mathcal{C}$.

Curvature and symplectic form:

Using the definition of $*$ and that $\alpha_2^{1,0} = -\bar{\alpha}_2^{0,1}$ and $\gamma_2^{1,0} = -\bar{\gamma}_2^{0,1}$ the Quillen curvature of $\mathcal{L}^1_\pm$ is

$$\begin{aligned}\mathcal{F}_{\mathcal{L}^1_\pm}((\alpha_1,\beta,\gamma_1),(\alpha_2,\zeta,\gamma_2)) &= \frac{2i}{\pi}\int_M (\frac{\alpha_1^{0,1}}{\sqrt{3}} \pm \frac{\beta_1\bar{\psi}_2^0}{\sqrt{2}}\bar{\theta}) \wedge *(\frac{\alpha_2^{0,1}}{\sqrt{3}} \pm \frac{\zeta_1\bar{\psi}_2^0}{\sqrt{2}}\bar{\theta}) \\ &= \frac{2i}{\pi}\int_M (\frac{\alpha_1^{0,1}}{\sqrt{3}} \pm \frac{\beta_1\bar{\psi}_2^0}{\sqrt{2}}\bar{\theta}) \wedge (-\frac{\alpha_2^{1,0}}{\sqrt{3}} \pm \frac{\bar{\zeta}_1\psi_2^0}{\sqrt{2}}\theta)\end{aligned}$$

Similarly, the Quillen curvature of $\tilde{\mathcal{L}}^1_+$ is

$$\begin{aligned}\mathcal{F}_{\tilde{\mathcal{L}}^1_\pm}((\alpha_1,\beta,\gamma_1),(\alpha_2,\zeta,\gamma_2)) &= \frac{2i}{\pi}\int_M (\frac{\alpha_1^{1,0}}{\sqrt{3}} \pm \frac{\bar{\beta}_1\psi_2^0}{\sqrt{2}}\theta) \wedge *(\frac{\alpha_2^{1,0}}{\sqrt{3}} \pm \frac{\bar{\zeta}_1\psi_2^0}{\sqrt{2}}\theta) \\ &= \frac{2i}{\pi}\int_M (\frac{\alpha_1^{1,0}}{\sqrt{3}} \pm \frac{\bar{\beta}_1\psi_2^0}{\sqrt{2}}\theta) \wedge (\frac{\alpha_2^{0,1}}{\sqrt{3}} \mp \frac{\zeta_1\bar{\psi}_2^0}{\sqrt{2}}\bar{\theta})\end{aligned}$$

Changing β_1 to β_2 and ζ_1 to ζ_2 in the above formula we get $\mathcal{F}_{\mathcal{L}^2_\pm}$ and $\mathcal{F}_{\tilde{\mathcal{L}}^2_\pm}$.

Similarly, the Quillen curvature of $\mathcal{R}_\pm$ is

$$\begin{aligned}\mathcal{F}_{\mathcal{R}_\pm}((\alpha_1,\beta,\gamma_1),(\alpha_2,\zeta,\gamma_2)) &= \frac{2i}{\pi}\int_M (\frac{\alpha_1^{0,1}}{\sqrt{3}} \pm \gamma_1^{0,1}) \wedge *(\frac{\alpha_2^{0,1}}{\sqrt{3}} \pm \gamma_2^{0,1}) \\ &= \frac{2i}{\pi}\int_M (\frac{\alpha_1^{0,1}}{\sqrt{3}} \pm \gamma_1^{0,1}) \wedge (-\frac{\alpha_2^{1,0}}{\sqrt{3}} \mp \gamma_2^{1,0})\end{aligned}$$

and the Quillen curvature of $\tilde{\mathcal{R}}_\pm$ is

$$\begin{aligned}\mathcal{F}_{\tilde{\mathcal{R}}_\pm}((\alpha_1,\beta,\gamma_1),(\alpha_2,\zeta,\gamma_2)) &= \frac{2i}{\pi}\int_M (\frac{\alpha_1^{1,0}}{\sqrt{3}} \pm \gamma_1^{1,0}) \wedge *(\frac{\alpha_2^{1,0}}{\sqrt{3}} \pm \gamma_2^{1,0}) \\ &= \frac{2i}{\pi}\int_M (\frac{\alpha_1^{1,0}}{\sqrt{3}} \pm \gamma_1^{1,0}) \wedge (\frac{\alpha_2^{0,1}}{\sqrt{3}} \pm \gamma_2^{0,1})\end{aligned}$$

Let $\mathcal{P} = \tilde{\mathcal{L}}^1_+ \otimes (\mathcal{L}^1_+)^{-1} \otimes \tilde{\mathcal{L}}^1_- \otimes (\mathcal{L}^1_-)^{-1} \otimes \tilde{\mathcal{L}}^2_+ \otimes (\mathcal{L}^2_+)^{-1} \otimes \tilde{\mathcal{L}}^2_- \otimes (\mathcal{L}^2_-)^{-1} \otimes \tilde{\mathcal{R}}_+ \otimes (\mathcal{R}_+)^{-1} \otimes \tilde{\mathcal{R}}_- \otimes (\mathcal{R}_-)^{-1}$.

Then we can check that

Lemma 3.2. *The Quillen curvature of $\mathcal{P}$ is $-\frac{4i}{\pi}\Omega$ on $\mathcal{C}$.*

Proof. $\mathcal{F}_{\tilde{\mathcal{L}}^1_+} - \mathcal{F}_{\mathcal{L}^1_+} + \mathcal{F}_{\tilde{\mathcal{L}}^1_-} - \mathcal{F}_{\mathcal{L}^1_-} + \mathcal{F}_{\tilde{\mathcal{L}}^2_+} - \mathcal{F}_{\mathcal{L}^2_+} + \mathcal{F}_{\tilde{\mathcal{L}}^2_-} - \mathcal{F}_{\mathcal{L}^2_-} + \mathcal{F}_{\tilde{\mathcal{R}}_+} - \mathcal{F}_{\mathcal{R}_+} + \mathcal{F}_{\tilde{\mathcal{R}}_-} - \mathcal{F}_{\mathcal{R}_-}$ $= -\frac{4i}{\pi}\Omega$. $\square$

Quotient by the gauge group:

$(A,\Psi,\Phi) \to (A_\tau,\Psi_\tau,\Phi) = (A + id\tau, e^{-i\tau}\Psi, \Phi)$ by a gauge transformation and $\Psi^0 \to \Psi^0_\tau = e^{-i\tau}\Psi^0$, such that $\psi_1\bar{\psi}_2^0$ and $\psi_1^0\bar{\psi}_2$ remain invariant under gauge transformations. Over $(A_\tau, \psi_\tau, \Phi)$ we consider the fiber $F_\tau = det(\bar{\partial}_{A_\tau,\Psi_\tau})$ and analogous fibers for the other $\bar{\partial}$ and ∂-determinant bundles. Let us rename $\mathcal{P}$ to be the bundle constructed by identifying $F \equiv F_\tau$, when (A,Ψ) is gauge equivalent to (A_τ,Ψ_τ) and analogously for all the other terms.

Lemma 3.3. *$\mathcal{P}$ is a well defined line bundle over $\mathcal{N} \subset \mathcal{C}/G$*

Proof. Let us consider first the Cauchy-Riemann operator $\frac{\bar{\partial}+A^{0,1}}{\sqrt{3}} + \frac{\psi_1 \bar{\psi}_2^0}{\sqrt{2}}\bar{\theta}$ which appears in the first term of the tensor product in $\mathcal{P}$ parametrized by the A, Ψ. The same argument works for the other terms in the tensor product. The term $\psi_1 \bar{\psi}_2^0$ does not change under the gauge group. If $A \to A_\tau$, under the gauge transformation τ, it is easy to show that the operators $\frac{\bar{\partial}+A^{0,1}}{\sqrt{3}} + \frac{\psi_1 \bar{\psi}_2^0}{\sqrt{2}}\bar{\theta}$ and $\frac{\bar{\partial}+A_\tau^{0,1}}{\sqrt{3}} + \frac{(\psi_1 \bar{\psi}_2^0)}{\sqrt{2}}\bar{\theta}$ have isomorphic kernels and cokernels and their corresponding Laplacians have the same spectrum and the eigenspaces are of the same dimension. There is a canonical isomorphism given by: $s \to e^{-i\tau}s$. Thus when one identifies $\wedge^{top}(Ker\bar{\partial}_{A,\Psi})^* \otimes \wedge^{top}(Coker\bar{\partial}_{A,\Psi})$ with $\wedge^{top}((K^a_{A,\Psi}))^* \otimes \wedge^{top}\bar{\partial}_{A,\Psi}K^a_{A,\psi}$, where $K^a_{A,\Psi}$ is the direct sum of eigenspaces of the operator $D_{A,\Psi}D^\dagger_{A,\Psi}$ of eigenvalues $< a$, over the subset $U^a = \{(A,\Psi) | a \notin Spec\Delta_{A,\Psi}\}$ of the affine space (see [14] or [3] for more details), there is an isomorphism of the fibers as $(A,\Psi) \to (A_\tau, \Psi_\tau)$. Thus one can identify

$$\wedge^{top}((K^a_{A,\psi}))^* \otimes \wedge^{top}\bar{\partial}_{A,\psi}K^a_{A,\Psi} \equiv \wedge^{top}((K^a_{A_\tau,\Psi_\tau}))^* \otimes \wedge^{top}\bar{\partial}_{A_\tau,\Psi_\tau}K^a_{A_\tau,\Psi_\tau}$$

and thus we can define the fiber over the quotient space $\frac{\mathcal{C}}{G}$ to be the equivalence class of this fiber. Thus $\mathcal{P}$ is well defined on $\frac{\mathcal{C}}{G}$. Then we restrict it to $\mathcal{N} \subset \frac{\mathcal{C}}{G}$. □

Lemma 3.4. *Ω is a symplectic form on $\mathcal{N}$.*

Proof. Let $\mathcal{C} = \mathcal{A} \times (\Gamma(M,L) \oplus \Gamma(M,L)) \times \Omega^1(M, i\mathbb{R})$ be the space on which equations (1.1) − (1.3) are imposed. Let $p = (A, \Psi, \Phi) \in \mathcal{C}$, $X = (\alpha_1, \beta, \gamma_1)$, $Y = (\alpha_2, \eta, \gamma_2) \in T_p\mathcal{C}$.

On $\mathcal{C}$ one can define a metric

$$g(X,Y) = \int_M *\alpha_1 \wedge \alpha_2 + \int_M Re < \beta, \eta > \omega + \int_M *\gamma_1 \wedge \gamma_2$$

and an almost complex structure $\mathcal{I} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & * \end{bmatrix} : T_p\mathcal{C} \to T_p\mathcal{C}$ where $* : \Omega^1 \to \Omega^1$ is the Hodge star operator on M which is different from the previous one (since it takes dx forms to type dy and dy forms to $-dx$ so that $*(\eta dz) = -i\eta dz$ and $*(\eta d\bar{z}) = i\eta d\bar{z}$).

We define

$$\Omega(X,Y) = -\int_M \alpha_1 \wedge \alpha_2 + \int_M Re < I\beta, \eta > \omega - \int_M \gamma_1 \wedge \gamma_2$$

where $I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

such that $g(\mathcal{I}X, Y) = \Omega(X,Y)$. It can be checked that the metric g, the symplectic form Ω, and the almost complex structure $\mathcal{I}$ are invariant under the gauge group action on $\mathcal{C}$.

The equation (1.1) can be realised as a moment map $\mu = 0$ with respect to the action of the gauge group and the symplectic form Ω. The reason is as follows. Let $\zeta \in \Omega(M, i\mathbb{R})$ be the Lie algebra of the gauge group (the gauge group element being $u = e^{i\tau}$, where $\zeta = i\tau$); it generates a vector field X_τ on $\mathcal{C}$ as follows :

$$X_\zeta(A, \Psi, \Phi) = (d\zeta, -\zeta\Psi, 0) \in T_p\mathcal{C}, p = (A, \Psi, \Phi) \in \mathcal{C}.$$

We show next that X_ζ is Hamiltonian. Namely, define $H_\zeta : \mathcal{C} \to \mathbb{C}$ as follows:

$$H_\zeta(p) = \int_M \zeta \cdot (F_A - i\frac{(|\psi_1|^2 - |\psi_2|^2)}{2}\omega).$$

Let $X = (\alpha, \beta, \gamma) \in T_p\mathcal{C}$.

$$\begin{aligned} dH_\zeta(X) &= \int_M \zeta d\alpha - i\int_M \zeta Re(\psi_1\bar{\beta}_1 - \psi_2\bar{\beta}_2)\omega \\ &= \int_M (-d\zeta)\wedge\alpha - \int_M Re < I\zeta \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \omega \\ &= \Omega(X_\zeta, X), \end{aligned}$$

where we use that $\bar{\zeta} = -\zeta$.

Thus we can define the moment map $\mu : \mathcal{C} \to \Omega^2(M, i\mathbb{R}) = \mathcal{G}^*$ (the dual of the Lie algebra of the gauge group) to be

$$\mu(A, \Psi) \doteq (F(A) - i\frac{(|\psi_1|^2 - |\psi_2|^2)}{2}\omega).$$

Thus equation (1.1)) is $\mu = 0$.

Next we show that if S be the solution spaces to equation (1.1) – (1.3), $X \in T_p\mathcal{S}$. Then $\mathcal{I}X \in T_p\mathcal{S}$ if and only if X is orthogonal to the gauge orbit $O_p = G \cdot p$. The reason is as follows. We let $X_\zeta \in T_pO_p$, where $\zeta \in \Omega^0(M, i\mathbb{R})$, $g(X, X_\zeta) = -\Omega(\mathcal{I}X, X_\zeta) = -\int_M \zeta \cdot d\mu(\mathcal{I}X)$, and therefore $\mathcal{I}X$ satisfies the linearization of equation (1.1) iff $d\mu(\mathcal{I}X) = 0$, i.e., iff $g(X, X_\zeta) = 0$ for all ζ. Second, it is easy to check that $\mathcal{I}X$ satisfies the linearization of equation (1.2), (1.3) whenever X does.

Now we are ready to show that $\mathcal{N}$ has a natural symplectic structure and an almost complex structure compatible with the symplectic form Ω and the metric g.

First we show that the almost complex structure descends to $\mathcal{N}$. Then using this and the symplectic quotient construction we will show that Ω gives a symplectic structure on $\mathcal{N}$. To show that $\mathcal{I}$ descends as an almost complex structure we let $pr : \mathcal{S} \to \mathcal{S}/G = \mathcal{N}$ be the projection map and set $[p] = pr(p)$. Then we can naturally identify $T_{[p]}\mathcal{N}$ with the quotient space $T_p\mathcal{S}/T_pO_p$, where $O_p = G \cdot p$ is the gauge orbit. Using the metric g on $\mathcal{S}$ we can realize $T_{[p]}\mathcal{N}$ as a subspace in $T_p\mathcal{S}$ orthogonal to T_pO_p. Then by what is said before, this subspace is invariant under $\mathcal{I}$. Thus $I_{[p]} = \tau|_{T_p(O_p)^\perp}$, gives the desired almost complex structure. This construction does not depend on the choice of p since $\mathcal{I}$ is G-invariant.

The symplectic structure Ω descends to $\mu^{-1}(0)/G$, (by what we said before and by the Marsden-Wienstein symplectic quotient construction [8], [9]) since the leaves of the characteristic foliation are the gauge orbits. Now, as a 2-form Ω descends to $\mathcal{N}$ and so does the metric g. We check that equation (1.2), (1.3) does not give rise to new degeneracy of Ω (i.e. the only degeneracy of Ω is due to (1.1) but along gauge orbits). Thus Ω is symplectic on $\mathcal{N}$. □

The proof of the theorem 3.1 follows from the lemmas.

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