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Quantum Stochastic Dynamics I : Spin Systems on a Lattice

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Abstract:
In the context of non-commutative $\mathbb{L}_p$ spaces we discuss the conditions for existence and ergodicity of translation invariant stochastic spin flip and diffusion dynamics for quantum spin systems with finite range interactions on a lattice.

Key words: *Non-commutative $\mathbb{L}_p$ spaces, stochastic spin flip and diffusion dynamics, quantum spins, systems on a lattice, finite range interactions.*

1. Introduction

The analysis in the interpolating family of $\mathbb{L}_p$ spaces associated to a probability measure plays an essential role in the study of the classical Markov semigroups. In general it is important for their construction as well as for the investigation of the ergodicity properties. It is especially useful if the underlying configuration space is infinite dimensional. In this paper we introduce some basic ideas concerning the application of interpolating $\mathbb{L}_p$ spaces to study Markov semigroups in the noncommutative context of quantum spin systems on a lattice. In Section 2 we show that using the idea of thermodynamic limit, it is possible to give a very natural and very explicit construction of an interpolating family of spaces $\mathbb{L}_p$, $p \in [1, \infty)$, associated to a quantum Gibbs state on the algebra of quantum spins on a lattice. In the noncommutative setting such family is no longer unique. In Section 3 we show that in this framework one can define in a natural way a Markov generator of quantum spin flip stochastic dynamics which satisfies detailed balance condition in a judiciously chosen L_2 space associated to a Gibbs states corresponding to a given interaction at some given inverse temperature $\beta \in (0, \infty)$. As a consequence of that, such stochastic dynamics leaves this Gibbs state invariant. In that section we restrict ourselves to a finite volume theory to make the ideas and constructions as explicit as possible. The infinite volume case is considered in Section 4 where we provide an abstract sufficient condition for the existence of an infinite volume translation invariant stochastic dynamics. Under our conditions the stochastic dynamics can be constructed as the thermodynamic limit of the corresponding finite volume stochastic dynamics with an appropriate control of the convergence (called an approximation property). They are also sufficient for the infinite volume Markov semigroup to possess a Feller property in the sense of mapping the inductive limit of local algebras into itself. We also show that under appropriate finite volume condition (similar to the classical one [**AH**]) we have a strong exponential decay to equilibrium (proven along the lines of [**SZ**]).

Section 5 is devoted to a complete description of a construction of an infinite volume translation invariant stochastic dynamics of the diffusion type with generator built of elementary completely positive generators introduced in [**QSV**].

The study of Markov semigroup in noncommutative setting is relatively more complicated than in classical case and the progress in this domain is much slower. We would like to mention few recent works in this subject. In particular the works [**Ma3**], [**FNW**], [**N**], where the completely positive hamiltonian semigroup in ground state representation has been considered. A first (very special) example of translation invariant stochastic dynamics satisfying a detailed balance condition has been constructed in [**GM**], where the authors used a clever representation associated to a classical Gibbs measure at a finite temperature. One should also mention more recent interesting construction of [**Ma1,2**] where some translation invariant dynamics has been constructed, although in general without characterizing the set of corresponding invariant states.

An interesting dual approach involving a construction of quantum analog of a Markov process has been also developed recently with a growing number of works. The interested reader can find a more detailed references for example in the recent interesting work [**BGW**].

2. Non-commutative $\mathbb{L}_p$ Spaces Associated to a Gibbs State.

Let $\mathbb{Z}^d$ be a d-dimensional integer lattice and let $\mathcal{F}$ denote the family of all its finite subsets. By $\mathcal{F}_0$ we will denote an increasing sequence of finite volumes invading all the lattice $\mathbb{Z}^d$. Given a sequence $\{F_\Lambda\}_{\Lambda \in \mathcal{F}_0}$, we will denote its limit as $\Lambda \to \mathbb{Z}^d$ through the sequence $\mathcal{F}_0$ by $\lim_{\mathcal{F}_0} F_\Lambda$.

Let $\mathcal{A}$ be a $\mathbf{C}^*$algebra with norm $||\cdot||$ defined as the inductive limit over a finite dimensional complex matrix algebra $\mathbf{M}$. Later it will be natural to view $\mathcal{A}$ as a noncommutative analog of the space of bounded continuous functions. For a set $X \in \mathcal{F}$, let $\mathcal{A}_X$ denote a subalgebra of operators localized in the set X, i.e. the subalgebra in $\mathcal{A}$ isomorphic to $\mathbf{M}^X$. For an arbitrary subset $\Lambda \subset \mathbb{Z}^d$ we define $\mathcal{A}_\Lambda$ to be the smallest (closed) subalgebra of $\mathcal{A}$ containing $\bigcup\{\mathcal{A}_X : X \in \mathcal{F},\, X \subset \Lambda\}$. An operator $f \in \mathcal{A}$ will be called local if there is some $Y \in \mathcal{F}$ such that $f \in \mathcal{A}_Y$. By $\mathcal{A}_0$ we denote the subset of $\mathcal{A}$ consisting of all local operators. We will use notation $\mathcal{A}_0^+$ and $\mathcal{A}^+$, respectively, for the corresponding subsets of nonnegative elements.

By $\mathbf{Tr}_X$, $X \in \mathcal{F}$, we denote a normalised partial trace on $\mathcal{A}$, i.e. the unique completely positive map

$$\mathbf{Tr}_X : \mathcal{A} \longrightarrow \mathcal{A}_{X^c} \tag{2.1}$$

which satisfies the following conditions
(i)

$$\forall f \in \mathcal{A},\ g,h \in \mathcal{A}_{X^c} \qquad \mathbf{Tr}_X(gfh) = g(\mathbf{Tr}_X f)h \tag{2.2}$$

(ii)

$$\mathbf{Tr}_X \mathbf{1} = \mathbf{1} \tag{2.3}$$

(iii)

$$\forall f,g \in \mathcal{A}_X \qquad \mathbf{Tr}_X fg = \mathbf{Tr}_X gf \tag{2.4}$$

From (i) and (ii) the following property follows

$$\mathbf{Tr}_X (\mathbf{Tr}_X f) = \mathbf{Tr}_X f \tag{2.5}$$

Let us recall that a map satisfying properties (i) and (ii) is called a conditional expectation. Let $\mathbf{Tr} \equiv \lim_{\mathcal{F}_0} \mathbf{Tr}_\Lambda$ be the normalized trace on $\mathcal{A}$. We have

$$\mathbf{Tr}\,(\mathbf{Tr}_X(f^*)g) = \mathbf{Tr}\,(f^* \mathbf{Tr}_X(g)) \tag{2.6}$$

(A detailed account of matricial algebras can be found in [**KR**].)
Let $\Phi \equiv \{\Phi_X \in \mathcal{A}_X\}_{X\in\mathcal{F}}$ be a (Gibbsian) potential, i.e. a family of selfadjoint operators such that

$$\|\Phi\|_1 \equiv \sup_{\mathbf{i}\in\mathbb{Z}^d} \sum_{\substack{X\in\mathcal{F}\\ X\ni\mathbf{i}}} \|\Phi_X\| < \infty \tag{2.7}$$

A potential $\Phi \equiv \{\Phi_X\}_{X\in\mathcal{F}}$ is of *finite range* $R \geq 0$, iff $\Phi_X = 0$ for all $X \in \mathcal{F}$, $\ diam(X) > R$. We define a corresponding Hamiltonian H_Λ and the interaction energy U_Λ in $\Lambda \in \mathcal{F}$, by setting

$$H_\Lambda \equiv H_\Lambda(\Phi) \equiv \sum_{X\subset\Lambda} \Phi_X \tag{2.8}$$

and

$$U_\Lambda \equiv U_\Lambda(\Phi) \equiv \sum_{X\cap\Lambda\neq\emptyset} \Phi_X, \tag{2.9}$$

respectively. Let ρ_Λ be a density matrix given by

$$\rho_\Lambda \equiv \frac{e^{-\beta H_\Lambda}}{\mathbf{Tr} e^{-\beta H_\Lambda}} \tag{2.10}$$

with $\beta \in (0,\infty)$. We define a finite volume Gibbs state ω_Λ as follows

$$\omega_\Lambda(f) \equiv \mathbf{Tr}\,(\rho_\Lambda f) \tag{2.11}$$

It is known, see e.g. [**BR**], that for sufficiently small $\beta \in (0,\infty)$ the following limit state on $\mathcal{A}$ exists and is faithful

$$\omega \equiv \lim_{\mathcal{F}_0} \omega_\Lambda \tag{2.12}$$

Let

$$\alpha_t^\Lambda(f) \equiv e^{+itH_\Lambda} f e^{-itH_\Lambda} \tag{2.13}$$

denote the finite volume automorphism group associated to potential Φ. One has the following **KMS** condition for the finite volume state ω_Λ

$$\omega_\Lambda(f^* g) = \omega_\Lambda(\alpha^\Lambda_{-i\beta}(g) f^*) \tag{2.14}$$

Suppose the potential $\Phi \equiv \{\Phi_X\}_{X\in\mathcal{F}}$ satisfies also

$$||\Phi||_{exp,\varepsilon} \equiv \sup_{\mathbf{i}\in\mathbb{Z}^d} \sum_{\substack{X\in\mathcal{F}\\ X\ni\mathbf{i}}} e^{\varepsilon|X|}||\Phi_X|| < \infty \tag{2.15}$$

for some $\varepsilon > 0$. Then the following limit exists, [**BR**],

$$\alpha_t(f) \equiv \lim_{\mathcal{F}_0} \alpha_t^\Lambda(f) \tag{2.16}$$

for every $f \in \mathcal{A}_0$ and defines the automorphisms group associated to the infinite volume state ω. In fact every operator $f \in \mathcal{A}_0$ is an analytic element for α_t, in the sense that for all β, such that $|\beta| \in [0, \beta_0)$, with some $\beta_0 \in (0,\infty)$ sufficiently small dependent only on the potential Φ, the following series converges in the norm of the algebra $\mathcal{A}$

$$\alpha_{i\beta}(f) \equiv \sum_{n=0}^{\infty} \frac{(-i\beta)^n}{n!} \delta_\Phi^n(f) \tag{2.17}$$

where δ_Φ is the generator of the automorphism group α_t given on the local elements by

$$\delta_\Phi(f) \equiv -\frac{d}{dt}\alpha_t(f)_{|t=0} \equiv \lim_{\mathcal{F}_0} i[H_\Lambda(\Phi), f] \tag{2.18}$$

where $[F_1, F_2] \equiv F_1F_2 - F_2F_1$ denotes the commutator of two operators F_1 and F_2. Given Φ satisfying (2.15), the biggest $\beta_0 \equiv \beta_0(\Phi)$ for which the series (2.17) is convergent for every $f \in \mathcal{A}_0$ is called *the radius of analyticity* (of the modular dynamic).
Let us mention that the infinite volume state ω satisfies the following **KMS** condition

$$\omega(f^* g) = \omega(\alpha_{-i\beta}(g) f^*) \tag{2.19}$$

Therefore it is called an (α_t, β) - **KMS** state.
For later purposes we need to recall, [**Se**], [**Ku**], [**Dix**], [**Ye**], [**Ne**], some properties of norms $||\cdot||_{\mathbb{L}_p(\mathbf{Tr})}$, $p \in [1,\infty)$, associated to a normalised trace $\mathbf{Tr}$, defined on $\mathcal{A}_0$ as follows

$$||f||_{\mathbb{L}_p(\mathbf{Tr})} \equiv (\mathbf{Tr}|f|^p)^{\frac{1}{p}} \tag{2.20}$$

where $|f| \equiv (f^*f)^{\frac{1}{2}}$. First of all we note that for $p = 2$ the corresponding norm is associated to the following scalar product

$$< f, g >_{\mathbf{Tr}} \equiv \mathbf{Tr} f^* g \tag{2.21}$$

and one has the following Hölder inequalities with $\mathbb{L}_r(\mathbf{Tr})$-norms, see e.g. [**Se**], [**Dix**],

$$| < f, g >_{\mathbf{Tr}} | \leq ||f||_{\mathbb{L}_p(\mathbf{Tr})} ||g||_{\mathbb{L}_q(\mathbf{Tr})} \tag{2.22}$$

with $p, q \in (1,\infty)$ such that $p^{-1} + q^{-1} = 1$, and for $r \in [1,\infty)$, see e.g. [**Dix**] (Corollary 3 p.23),

$$||fg||_{\mathbb{L}_r(\mathbf{Tr})} \leq ||f||_{\mathbb{L}_p(\mathbf{Tr})} ||g||_{\mathbb{L}_q(\mathbf{Tr})} \tag{2.23}$$

provided that $p^{-1} + q^{-1} = r^{-1}$.

Applying (2.22) with $g = \mathbf{1}$ and f replaced by $|f|^r$ for some $r \in [1, \infty)$, and $\frac{p}{r} > 1$ instead of p, or simply taking $g = \mathbf{1}$ in (2.23), we get the following important special case

$$||f||_{\mathbb{L}_r(\mathbf{Tr})} \leq ||f||_{\mathbb{L}_p(\mathbf{Tr})} \tag{2.24}$$

when $r \leq p$. Clearly we have also

$$||f||_{\mathbb{L}_p(\mathbf{Tr})} \leq ||f|| \tag{2.25}$$

and therefore one can naturally regard $|| \cdot ||$ as an analog of the uniform norm on the space of bounded continuous functions.
Let us mention also that one has

$$||f||_{\mathbb{L}_p(\mathbf{Tr})} \equiv \sup_{||g||_{\mathbb{L}_q(\mathbf{Tr})}=1} |\mathbf{Tr} g^* f| \tag{2.26}$$

where q is the dual index given by $p^{-1} + q^{-1} = 1$.
Given (2.26) one can easily get the Minkowski inequality. One can get it also using the Hölder inequality by the following elementary arguments for the case $p \geq 2$, for which we have

$$||f+g||^p_{\mathbb{L}_p(\mathbf{Tr})} = \mathbf{Tr}|f+g|^p = \mathbf{Tr}(f^*+g^*)(f+g)|f+g|^{p-2} =$$

$$= \mathbf{Tr} f^* f |f+g|^{p-2} + \mathbf{Tr} f^* g |f+g|^{p-2} + \mathbf{Tr} g^* f |f+g|^{p-2} + \mathbf{Tr} g^* g |f+g|^{p-2} \tag{2.27}$$

If $p > 2$, we use Hölder inequality with the functions $f^* f$ and $|f+g|^{p-2}$ and norms $\frac{p}{2}$ and $\frac{p}{p-2}$, respectively, to get

$$0 \leq \mathbf{Tr} f^* f |f+g|^{p-2} \leq ||f||^2_{\mathbb{L}_p(\mathbf{Tr})} \cdot ||f+g||^{p-2}_{\mathbb{L}_p(\mathbf{Tr})} \tag{2.28}$$

and similarly for the last term on the right hand side of (2.27). For a term containing a product of f^* and g, by trace property and the Schwartz inequality, we have first

$$\left|\mathbf{Tr} f^* g |f+g|^{p-2}\right| = \left|\mathbf{Tr}(f|f+g|^{\frac{p-2}{2}})^* g |f+g|^{\frac{p-2}{2}}\right| \leq \left(\mathbf{Tr} f^* f |f+g|^{p-2}\right)^{\frac{1}{2}} \left(\mathbf{Tr} g^* g |f+g|^{p-2}\right)^{\frac{1}{2}} \tag{2.29}$$

Now the right hand side can be estimated with use of (2.28). Similarly we handle the other term involving g^* and f. Combining all that we obtain

$$||f+g||^p_{\mathbb{L}_p(\mathbf{Tr})} \leq \left(||f||_{\mathbb{L}_p(\mathbf{Tr})} + ||g||_{\mathbb{L}_p(\mathbf{Tr})}\right)^2 ||f+g||^{p-2}_{\mathbb{L}_p(\mathbf{Tr})} \tag{2.30}$$

Hence, by simple algebra, we arrive at the following <u>Minkowski inequality for $\mathbb{L}_p(\mathbf{Tr})$ norms</u>

$$||f+g||_{\mathbb{L}_p(\mathbf{Tr})} \leq ||f||_{\mathbb{L}_p(\mathbf{Tr})} + ||g||_{\mathbb{L}_p(\mathbf{Tr})} \tag{2.31}$$

See e.g. [**Dix**], [**Se**], for the general case $p \in [1, \infty)$.
Given a finite volume Gibbs state ω_Λ, we define the following $\mathbb{L}_p(\omega_\Lambda)$, $p \in [1, \infty)$, norms on $\mathcal{A}$

$$||f||_{\mathbb{L}_p(\omega_\Lambda)} \equiv \left(\mathbf{Tr}|\rho_\Lambda^{\frac{1}{2p}} f \rho_\Lambda^{\frac{1}{2p}}|^p\right)^{\frac{1}{p}} \tag{2.32}$$

In particular for $p = 2$ we see that the corresponding norm is given by the following scalar product

$$<f, g>_{\omega_\Lambda} \equiv \mathbf{Tr}\left(\rho_\Lambda^{\frac{1}{2}} f^* \rho_\Lambda^{\frac{1}{2}} g\right) = \mathbf{Tr}\left((\rho_\Lambda^{\frac{1}{4}} f \rho_\Lambda^{\frac{1}{4}})^* (\rho_\Lambda^{\frac{1}{4}} g \rho_\Lambda^{\frac{1}{4}})\right) \tag{2.33}$$

Using the information about the $\mathbb{L}_p(\mathbf{Tr})$-norms, one can get the following important for us lemma.

Lemma 2.1:
For any $f, g \in \mathcal{A}_0$ and any $p, q \in [1, \infty)$ we have:
(i) For any $c \in \mathbb{C}$

$$0 \leq ||cf||_{\mathbb{L}_p(\omega_\Lambda)} = |c| \cdot ||f||_{\mathbb{L}_p(\omega_\Lambda)} \tag{2.34}$$

with the equality on the left hand side iff $cf = \mathbf{0}$,
(ii) Hölder inequalities

$$|< f, g >_{\omega_\Lambda} | \leq ||f||_{\mathbb{L}_p(\omega_\Lambda)} ||g||_{\mathbb{L}_q(\omega_\Lambda)} \tag{2.35}$$

with $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$, and if $p \leq q$ we have

$$||f||_{\mathbb{L}_p(\omega_\Lambda)} \leq ||f||_{\mathbb{L}_q(\omega_\Lambda)} \leq ||f|| \tag{2.36}$$

(iii) Duality
For $p \in (1, \infty)$

$$||f||_{\mathbb{L}_p(\omega_\Lambda)} = \sup_{||g||_{\mathbb{L}_q(\omega_\Lambda)}=1} |< g, f >_{\omega_\Lambda} | \tag{2.37}$$

where q is the dual index given by $p^{-1} + q^{-1} = 1$
(iv) Minkowski inequality

$$||f + g||_{\mathbb{L}_p(\omega_\Lambda)} \leq ||f||_{\mathbb{L}_p(\omega_\Lambda)} + ||g||_{\mathbb{L}_p(\omega_\Lambda)} \tag{2.38}$$

$\circ$

Proof: The proof of (2.34) is trivial. Since by definition (2.10) we have $\rho_\Lambda > \mathbf{0}$, we can get equality on the left hand side of (2.34) iff $cf = \mathbf{0}$. To get (2.35) we use the following arguments, (see e.g. [**Tr**]), with $p, q \in (1, \infty)$ satisfying $p^{-1} + q^{-1} = 1$

$$|< f, g >_{\omega_\Lambda} | = |\mathbf{Tr} \left(\rho_\Lambda^{\frac{1}{2}} f^* \rho_\Lambda^{\frac{1}{2}} g \right) | = |\mathbf{Tr} \left(\rho_\Lambda^{\frac{1}{2q} + \frac{1}{2p}} f^* \rho_\Lambda^{\frac{1}{2p} + \frac{1}{2q}} g \right) | = |\mathbf{Tr} \left(\left(\rho_\Lambda^{\frac{1}{2p}} f^* \rho_\Lambda^{\frac{1}{2p}} \right) \left(\rho_\Lambda^{\frac{1}{2q}} g \rho_\Lambda^{\frac{1}{2q}} \right) \right) | \tag{2.39}$$

where we have used the trace property. Applying to the right hand side the Hölder inequality (2.22) for trace, we get the Hölder inequality (2.35) for the case of finite volume sate ω_Λ.
To get the inequality on the left hand side of (2.36), (the second Hölder inequality), we observe first that if $p < q$, we have

$$||f||_{\mathbb{L}_p(\omega_\Lambda)} = ||\rho_\Lambda^{\frac{1}{2p}} f \rho_\Lambda^{\frac{1}{2p}} ||_{\mathbb{L}_p(\mathbf{Tr})} = ||\rho_\Lambda^{\frac{1}{2s}} \left(\rho_\Lambda^{\frac{1}{2q}} f \rho_\Lambda^{\frac{1}{2q}} \right) \rho_\Lambda^{\frac{1}{2s}} ||_{\mathbb{L}_p(\mathbf{Tr})} \tag{2.40}$$

where $s^{-1} + q^{-1} = p^{-1}$. Now by (double) application of the Hölder inequality (2.23) for traces with use of the normalisation condition $\mathbf{Tr} \rho_\Lambda = 1$, we arrive at the left hand side inequality of (2.36).
The right hand side inequality of (2.36) is proven in Appendix 1 by elementary inductive arguments.
The Minkowski inequality (2.38) follows from the corresponding inequality (2.31) for the trace with the function $\rho_\Lambda^{\frac{1}{2p}} f \rho_\Lambda^{\frac{1}{2p}}$.

$\diamondsuit$

Let us note that for $p \in \mathbb{N}$ we have the following useful representation of the $\mathbb{L}_p(\omega_\Lambda)$-norms for nonnegative elements $f \in \mathcal{A}_0$:

Lemma 2.2 *Let $f \in \mathcal{A}_0^+$.*
If $p = 1$, then

$$||f||_{\mathbb{L}_1(\omega_\Lambda)} = \omega_\Lambda(f) =< f, \mathbf{1} >_{\omega_\Lambda} =< \mathbf{1}, f >_{\omega_\Lambda} \tag{2.41}$$

If $p \in \mathbb{N}$, $p > 1$, then

$$||f||^p_{\mathbb{L}_p(\omega_\Lambda)} = \omega_\Lambda \left(\alpha^\Lambda_{\frac{1}{2p} i\beta}(f) \alpha^\Lambda_{\frac{3}{2p} i\beta}(f) ... \alpha^\Lambda_{\frac{(2p-1)}{2p} i\beta}(f) \right) \tag{2.42}$$

with the following (shift invariance) property

$$\omega_\Lambda\left(\alpha^\Lambda_{\frac{1}{2p}i\beta}(f)\alpha^\Lambda_{\frac{3}{2p}i\beta}(f)...\alpha^\Lambda_{\frac{(2p-1)}{2p}i\beta}(f)\right) = \omega_\Lambda\left(\alpha^\Lambda_{(\frac{1}{2p}-a)i\beta}(f)\alpha^\Lambda_{(\frac{3}{2p}-a)i\beta}(f)...\alpha^\Lambda_{(\frac{(2p-1)}{2p}-a)i\beta}(f)\right) \tag{2.43}$$

for any $a \in [-1, +1]$.
If $p \in I\!\!N$ *is even, then for any* $f \in \mathcal{A}_0$ *we have*

$$||f||^p_{I\!\!L_p(\omega_\Lambda)} = \omega_\Lambda\left(\alpha^\Lambda_{\frac{1}{2p}i\beta}(f^*)\alpha^\Lambda_{\frac{3}{2p}i\beta}(f)...\alpha^\Lambda_{\frac{(2p-3)}{2p}i\beta}(f^*)\alpha^\Lambda_{\frac{(2p-1)}{2p}i\beta}(f)\right) \tag{2.44}$$

$\circ$

It is clear that if $\beta \in (0, \beta_0)$, with $\beta_0 \in (0, \infty)$ being not bigger than the radius of analyticity and such that the limit (2.12) exists, the above formula has a limit and we can define on $\mathcal{A}_0$ the $I\!\!L_p(\omega)$-norms corresponding to the (α_t, β) - **KMS** state ω. Similarly one can expect that the corresponding sequences of other norms on $\mathcal{A}_0$ also converges in the thermodynamic limit. We have the following theorem.

Theorem 2.4:
Let ω *be an* (α_t, β)-**KMS** *state. There is a family of norms* $I\!\!L_p(\omega)$, $p \in [1, \infty)$ *on* $\mathcal{A}_0$ *such that the following conditions hold*
(i) For any $f \in \mathcal{A}_0^+$ *and any* $p \in I\!\!N$ *we have*

$$||f||^p_{I\!\!L_p(\omega)} = \omega\left(\alpha_{\frac{1}{2p}i\beta}(f)\alpha_{\frac{3}{2p}i\beta}(f)...\alpha_{\frac{(2p-1)}{2p}i\beta}(f)\right) \tag{2.45}$$

(ii)

$$||f||^2_{I\!\!L_2(\omega)} = \omega\left((\alpha_{\frac{1}{4}i\beta}(f))^*\alpha_{\frac{1}{4}i\beta}(f)\right) \tag{2.46}$$

(iii) For any $p, q \in [1, \infty)$ *such that* $p^{-1} + q^{-1} = 1$

$$|< g, f >_\omega| \leq ||f||_{I\!\!L_p(\omega)}||g||_{I\!\!L_q(\omega)} \tag{2.47}$$

and if $p \leq q$

$$0 \leq ||f||_{I\!\!L_p(\omega)} \leq ||f||_{I\!\!L_q(\omega)} \leq ||f|| \tag{2.48}$$

(iv)

$$||f||_{I\!\!L_p(\omega)} = \sup_{||g||_{I\!\!L_q(\omega)}=1} |< g, f >_\omega| \tag{2.49}$$

$\circ$

Remark: *In general one could define the following norms*

$$\limsup_{\mathcal{F}_0} ||f||_{I\!\!L_p(\omega_\Lambda)}$$

see also [**Ha**],[**T2**], [**Ko**], [**MZ**]. $\circ$

Using a norm $||\cdot||_{I\!\!L_p(\omega)}$ introduced above, we can define the corresponding set $I\!\!L_p(\omega)$ of equivalence classes of Cauchy's sequences $\{f_n \in \mathcal{A}_0\}_{n\in I\!\!N}$ modulo the class of the zero operator. It follows from the corresponding Minkowski inequality that one can introduce in this space the structure of complex linear space. In this way we arrive at the following definition.

Definition 2.4:
The linear space $\mathbb{L}_p(\omega)$, $p \in [1,\infty)$, defined above is called the <u>$\mathbb{L}_p$*-space*</u> *associated to an (α_t, β)-***KMS** *state ω.* $\circ$

Whenever it will not cause a confusion we will use a short notation $\mathbb{L}_p$ for the space $\mathbb{L}_p(\omega)$.
Let us note the following fact.

Proposition 2.5
If $p, q \in [1,\infty)$, and $p \le q$, then

$$\mathcal{A} \subseteq \mathbb{L}_q \subseteq \mathbb{L}_p \tag{2.50}$$

$\circ$

Let us mention that it is also possible to define different family of interpolating norms and spaces. For example one could define an $\mathbb{L}_2$ space by taking the following natural choice of the scalar product

$$< f, g >' \equiv \omega(f^* g) \tag{2.51}$$

The reason why we prefer to make the other choice will become clear later in the next section where we introduce the stochastic dynamics.
Finally we would like to say that our construction of $\mathbb{L}_p$ spaces is similar to the corresponding construction in the semifinite case considered in [**Tr**], [**Zo**], [**Sh**]. Although, let us stress, that by taking the thermodynamic limit we are able to define our $\mathbb{L}_p$ spaces in a more general setting, i.e. in the thermodynamic limit we do not use integration with respect to a tracial state. Let us recall that the existence of a faithful trace excludes the von Neumann algebras of type **III** associated to an infinite volume Gibbs state corresponding to a potential Φ, [**Po**], [**BR**]. For general von Neumann algebras a rather involved construction of $\mathbb{L}_p$ spaces has been completed in the following works: [**Ha**], [**Co**], [**ArM**], [**Hi**], [**T1**], and of the interpolating spaces in [**Ko**], [**T2**]. In our work, having some concrete applications in mind, we have applied a pragmatic functional analytic approach, instead of the wise measure theoretic one. Let us say however that it is useful to use both constructions.

3. <u>Markov Generators and Markov Semigroups of a Finite System</u>

In this section we introduce a family of Markov generators and semigroups corresponding to a block spin flip stochastic dynamics of a quantum spin system on a lattice.
For $X \in \mathcal{F}$, let $E_{X,\Lambda} : \mathcal{A} \longrightarrow \mathcal{A}$ be a map defined as follows

$$E_{X,\Lambda}(f) = \mathbf{Tr}_X \left(\gamma^*_{X,\Lambda} f \gamma_{X,\Lambda}\right) \tag{3.1}$$

where $\gamma_{X,\Lambda} \equiv \gamma_{X,\Lambda}(\frac{1}{2})$, with

$$\gamma_{X,\Lambda}(s) \equiv \rho^s_\Lambda \left(\mathbf{Tr}_X \rho_\Lambda\right)^{-s} \tag{3.2}$$

where $\mathbf{Tr}_X$ is the partial trace and ρ_Λ the density matrix of a finite volume Gibbs state ω_Λ. The map $E_{X,\Lambda}$ has the following properties.

Proposition 3.1:
(i)

$$E_{X,\Lambda}(\mathcal{A}) \subseteq \mathcal{A}_{X^c} \tag{3.3}$$

(ii)
$E_{X,\Lambda}$ is <u>*completely positive*</u> *, i.e.* [**St**] *for any $n \in \mathbb{N}$ the map $E^{(n)}_{X,\Lambda}$ on the space $n \times n$ matrices $\{a_{kl} \in \mathcal{A}\}_{k,l=1,..,n}$ defined by*

$$E^{(n)}_{X,\Lambda}(\{a_{kl}\}) \equiv \{E_{X,\Lambda}(a_{kl})\} \tag{3.4}$$

is positive.
(iii)

$$E_{X,\Lambda}(\mathbf{1}) = \mathbf{1} \tag{3.5}$$

$\circ$

Remark: *Note that in general we have <u>not</u>*

$$E_{X,\Lambda}(gfh) = g(E_{X,\Lambda}f)h \tag{3.6}$$

for $g, h \in \mathcal{A}_{X^c}$, *and therefore in general*

$$E_{X,\Lambda}\left(E_{X,\Lambda}(f)\right) \neq E_{X,\Lambda}(f) \tag{3.7}$$

Proof: The property (i) follows from the definition of $E_{X,\Lambda}$ and the property of the partial trace. The complete positivity property is a consequence of the fact that $E_{X,\Lambda}$ is defined as a composition of two obviously completely positive maps: the partial trace and the map

$$\mathcal{A} \ni f \longmapsto \gamma^*_{X,\Lambda} f \gamma_{X,\Lambda} \tag{3.8}$$

To see the unit preserving property, we use definitions (3.1) and (3.2) from which we have

$$E_{X,\Lambda}(\mathbf{1}) = \mathbf{Tr}_X\left(\gamma^*_{X,\Lambda}\gamma_{X,\Lambda}\right) = \mathbf{Tr}_X\left(\left(\rho_\Lambda^{\frac{1}{2}}\left(\mathbf{Tr}_X\rho_\Lambda\right)^{-\frac{1}{2}}\right)^* \rho_\Lambda^{\frac{1}{2}}\left(\mathbf{Tr}_X\rho_\Lambda\right)^{-\frac{1}{2}}\right) =$$

$$= \mathbf{Tr}_X\left(\left(\mathbf{Tr}_X\rho_\Lambda\right)^{-\frac{1}{2}}\rho_\Lambda\left(\mathbf{Tr}_X\rho_\Lambda\right)^{-\frac{1}{2}}\right) = \left(\mathbf{Tr}_X\rho_\Lambda\right)^{-\frac{1}{2}}\left(\mathbf{Tr}_X\rho_\Lambda\right)\left(\mathbf{Tr}_X\rho_\Lambda\right)^{-\frac{1}{2}} = \mathbf{1} \tag{3.9}$$

This ends the proof.

$\diamondsuit$

For later purposes let us mention the following particular consequences of Proposition 3.1.

Proposition 3.2
(i) <u>Positivity</u>

$\forall f \in \mathcal{A}^+$
$$E_{X,\Lambda}(f) \geq 0 \tag{3.10}$$

(ii) <u> - Invariance</u>*

$\forall f \in \mathcal{A}$
$$\left(E_{X,\Lambda}(f)\right)^* = E_{X,\Lambda}(f^*) \tag{3.11}$$

(iii) <u>Boundedness</u>

$\forall f \in \mathcal{A}$
$$||E_{X,\Lambda}(f)|| \leq ||f|| \tag{3.12}$$

(iv) <u>The Kadison - Schwarz inequality</u>

$\forall f \in \mathcal{A}$
$$E_{X,\Lambda}(f)^* E_{X,\Lambda}(f) \leq E_{X,\Lambda}(f^* f) \tag{3.13}$$

$\circ$

The proof of (i) and (ii) easily follows from $n = 1$ positivity, and (iii) follows from (3.5) and (3.10), while (iv) is a consequence of $n = 2$ positivity; see e.g. [**BR**] vol.2, [**Ta**].
Another important consequence of our definition (3.1) of the map $E_{X,\Lambda}$ is the following property.

Proposition 3.3

The map $E_{X,\Lambda}$ is a positive, symmetric and bounded operator in $\mathbb{L}_2(\omega_\Lambda)$ with

$$||E_{X,\Lambda}||_{\mathbb{L}_2(\omega_\Lambda)\to\mathbb{L}_2(\omega_\Lambda)} = 1 \tag{3.14}$$

$\circ$

Proof: First of all let us note that, by the boundedness property (3.12), the operator $E_{X,\Lambda}$ is well defined as an operator in $\mathbb{L}_2(\omega_\Lambda)$ for any finite set $\Lambda \in \mathcal{F}$. Using the definition of $\mathbb{L}_2(\omega_\Lambda)$ - scalar product, the $*$ - invariance of the map $E_{X,\Lambda}$ and a property of the trace, we have

$$< E_{X,\Lambda}(f), g >_{\omega_\Lambda} = \mathbf{Tr}\left(\rho_\Lambda^{\frac{1}{2}}(E_{X,\Lambda}(f))^*\rho_\Lambda^{\frac{1}{2}}g\right) = \mathbf{Tr}\left(E_{X,\Lambda}(f^*)\rho_\Lambda^{\frac{1}{2}}g\rho_\Lambda^{\frac{1}{2}}\right) \tag{3.15}$$

Now from the definition (3.1) of $E_{X,\Lambda}$, we get

$$\mathbf{Tr}\left(E_{X,\Lambda}(f^*)\rho_\Lambda^{\frac{1}{2}}g\rho_\Lambda^{\frac{1}{2}}\right) = \mathbf{Tr}\left(\mathbf{Tr}_X\left(\gamma_{X,\Lambda}^* f^*\gamma_{X,\Lambda}\right)\mathbf{Tr}_X\left(\rho_\Lambda^{\frac{1}{2}}g\rho_\Lambda^{\frac{1}{2}}\right)\right) = \tag{3.16}$$

and using the definition of $\gamma_{X,\Lambda}(s)$ in (3.2), we arrive at

$$= \mathbf{Tr}\left(\mathbf{Tr}_X\left(\gamma_{X,\Lambda}^*(\frac{1}{4})\rho_\Lambda^{\frac{1}{4}}f^*\rho_\Lambda^{\frac{1}{4}}\gamma_{X,\Lambda}(\frac{1}{4})\right)\mathbf{Tr}_X\left(\gamma_{X,\Lambda}^*(\frac{1}{4})\rho_\Lambda^{\frac{1}{4}}g\rho_\Lambda^{\frac{1}{4}}\gamma_{X,\Lambda}(\frac{1}{4})\right)\right) \tag{3.17}$$

From this the symmetry as well as positivity of the operator $E_{X,\Lambda}$ in $\mathbb{L}_2(\omega_\Lambda)$ follows. The proof of (3.14) will be given later, (see Proposition 3.6iv).

$\diamondsuit$

Let $\mathcal{L}_{X,\Lambda}$ be an operator on $\mathcal{A}$ defined by

$$\mathcal{L}_{X,\Lambda}f \equiv E_{X,\Lambda}(f) - f \tag{3.18}$$

It has the following properties.

Proposition 3.4:

(i)

$$\mathcal{L}_{X,\Lambda}\mathbf{1} = \mathbf{0} \tag{3.19}$$

(ii) <u>* - Invariance</u>

$$(\mathcal{L}_{X,\Lambda}f)^* = \mathcal{L}_{X,\Lambda}(f^*) \tag{3.20}$$

(iii) <u>*Dissipativity*</u>
For any $f \in \mathcal{A}$

$$\mathcal{L}_{X,\Lambda}(f^*f) - \mathcal{L}_{X,\Lambda}(f^*)f - f^*\mathcal{L}_{X,\Lambda}(f) \geq 0 \tag{3.21}$$

(iv) <u>*Symmetry*</u>
For any $f, g \in \mathcal{A}$ we have

$$< \mathcal{L}_{X,\Lambda}(f), g >_{\omega_\Lambda} = < f, \mathcal{L}_{X,\Lambda}(g) >_{\omega_\Lambda} \tag{3.22}$$

(v) <u>*Boundedness*</u>

$$||\mathcal{L}_{X,\Lambda}(f)|| \leq 2||f|| \tag{3.23}$$

and

$$||\mathcal{L}_{X,\Lambda}(f)||_{\mathbb{L}_2(\omega_\Lambda)} \leq 2||f||_{\mathbb{L}_2(\omega_\Lambda)} \tag{3.24}$$

$\circ$

Proof: We shall proof only the dissipativity property, as all the others easily follow from the definition of $\mathcal{L}_{X,\Lambda}$ and the corresponding properties of $E_{X,\Lambda}$. To get (iii), we observe that

$$\mathcal{L}_{X,\Lambda}(f^*f) - \mathcal{L}_{X,\Lambda}(f^*)f - f^*\mathcal{L}_{X,\Lambda}(f) = E_{X,\Lambda}(f^*f) - E_{X,\Lambda}(f^*)f - f^*E_{X,\Lambda}(f) + f^*f =$$

$$= (E_{X,\Lambda}(f^*f) - E_{X,\Lambda}(f^*)E_{X,\Lambda}(f)) + |E_{X,\Lambda}(f) - f|^2 \tag{3.25}$$

Hence using the Kadison - Schwarz inequality (3.13) for $E_{X,\Lambda}$, we conclude that

$$\mathcal{L}_{X,\Lambda}(f^*f) - \mathcal{L}_{X,\Lambda}(f^*)f - f^*\mathcal{L}_{X,\Lambda}(f) \geq 0 \tag{3.26}$$

This ends the proof of Proposition 3.4.

$\diamondsuit$

Remark: *After this Proposition the careful reader should understand the usefulness of our choice of* $\mathbb{L}_2(\omega_\Lambda)$ *space.*

Definition 3.5:
An operator $\mathcal{L}$ *defined on a dense subalgebra* $\mathcal{D}(\mathcal{L}) \subset \mathcal{A}$ *satisfying the conditions (i) - (iii), will be called a Markov pre - generator.* $\circ$

Remark: *The most complete abstract characterization of generators of norm continuous semigroups on* C^* *- algebras can be found in* [**EO**], *while a characterization of generators of positive* C_0 *semigroups is given in* [**BDR**].

Let $\mathcal{L}_{X+\mathbf{j},\Lambda}$ be the bounded symmetric Markov generators defined similarly as above for the translations $X + \mathbf{j}$ of a given finite set $X \in \mathcal{F}$. Using these operators we would like to introduce the following Markov generators

$$\mathcal{L}^{X,\Lambda}f \equiv \sum_{\mathbf{j}\in\Lambda} \mathcal{L}_{X+\mathbf{j},\Lambda}f \tag{3.27}$$

defined for any $\Lambda \in \mathcal{F}$ on the algebra $\mathcal{A}$. From Proposition 3.4 it is clear that $\mathcal{L}^{X,\Lambda}$ is a bounded operator on the algebra as well as bounded and symmetric in $\mathbb{L}_2(\omega_\Lambda)$. Let $P_t^{X,\Lambda} \equiv e^{t\mathcal{L}^{X,\Lambda}}$ be a corresponding semigroup. It has the following properties.

Proposition 3.6:
(i) *Positivity preserving*: *For any* $f \in \mathcal{A}^+$

$$P_t^{X,\Lambda}f \geq \mathbf{0} \tag{3.28}$$

(ii) *Unit preserving*

$$P_t^{X,\Lambda}\mathbf{1} = \mathbf{1} \tag{3.29}$$

(iii) $\mathbb{L}_2$ *- Symmetry*

$$< P_t^{X,\Lambda}(f), g >_{\omega_\Lambda} = < f, P_t^{X,\Lambda}(g) >_{\omega_\Lambda} \tag{3.30}$$

(iv)

$$||P_t^{X,\Lambda}||_{\mathbb{L}_2(\omega_\Lambda)\to\mathbb{L}_2(\omega_\Lambda)} \leq 1 \tag{3.31}$$

and

$$< \mathcal{L}^{X,\Lambda}(f), f >_{\omega_\Lambda} \leq 0 \tag{3.32}$$

(v) Invariance: For any $f \in \mathcal{A}$

$$\omega_\Lambda \left(P_t^{X,\Lambda}(f) \right) = \omega_\Lambda(f) \tag{3.33}$$

Equivalently we have

$$\omega_\Lambda \left(\mathcal{L}_{X,\Lambda}(f) \right) = 0 \tag{3.34}$$

$\circ$

Remark: *The inequality (3.32) implies*

$$< E_{X,\Lambda}(f), f >_{\omega_\Lambda} \leq < f, f >_{\omega_\Lambda} \tag{3.35}$$

which implies (3.14) of Proposition 3.3. (In fact one can prove it also in a more direct way.)

Proof: The positivity preserving property (i) follows by the following (actually stronger) property in the proof of which we use the standard arguments, see e.g. [**B**], [**BR**], based on the dissipativity property of the generator $\mathcal{L}^{X,\Lambda}$.

$$P_t^{X,\Lambda}(f^* f) - P_t^{X,\Lambda}(f^*) P_t^{X,\Lambda}(f) = -\int_0^t ds \frac{d}{ds} P_{t-s}^{X,\Lambda} \left(P_s^{X,\Lambda}(f^*) P_s^{X,\Lambda}(f) \right) =$$

$$\int_0^t ds P_{t-s}^{X,\Lambda} \left\{ \mathcal{L}^{X,\Lambda} \left(P_s^{X,\Lambda}(f^*) P_s^{X,\Lambda}(f) \right) - \left(\mathcal{L}^{X,\Lambda} P_s^{X,\Lambda}(f^*) \right) P_s^{X,\Lambda}(f) - P_s^{X,\Lambda}(f^*) \left(\mathcal{L}^{X,\Lambda} P_s^{X,\Lambda}(f^*) \right) \right\} \geq 0 \tag{3.36}$$

The properties (ii) and (iii) follow from the properties (i) and (iv) of generator $\mathcal{L}^{X,\Lambda}$ given in Proposition 3.4. To get (3.31) we use the symmetry of the operator $P_t^{X,\Lambda}$ and the fact that for any $f \in \mathcal{A}$ we have

$$||P_t^{X,\Lambda} f - \omega_\Lambda(f)||_{\mathbb{L}_2(\omega_\Lambda)} \leq ||P_t^{X,\Lambda} f - \omega_\Lambda(f)|| \leq ||f - \omega_\Lambda(f)|| \tag{3.37}$$

From this, by use of spectral theorem we conclude that (3.32) has to be true. To prove the invariance we observe that by the definition of $\mathbb{L}_2$-norm and properties (i) and (ii),we have

$$\omega_\Lambda \left(P_t^{X,\Lambda}(f) \right) = < \mathbf{1}, P_t^{X,\Lambda}(f) >_{\omega_\Lambda} = < P_t^{X,\Lambda} \mathbf{1}, f >_{\omega_\Lambda} = \omega_\Lambda(f) \tag{3.38}$$

This ends the proof of Proposition 3.6.

$\diamondsuit$

Let us recall the following definition.

Definition 3.7: *A strongly continuous semigroup P_t, $t \geq 0$, on a Banach algebra $\mathbb{B}$, is called Markov iff it is positivity and unit preserving. In case when $\mathcal{A} \subseteq \mathbb{B}$, we say that the semigroup P_t has a Feller property iff*

$$\forall f \in \mathcal{A}, \, t \geq 0, \qquad P_t f \in \mathcal{A} \tag{3.39}$$

$\circ$

Thus the semigroups $P_t^{X,\Lambda}$ constructed above are Markov semigroups on $\mathcal{A}$ and $\mathbb{L}_2(\omega_\Lambda)$, and obviously have the Feller property.
Given an automorphism group α_t on an algebra $\mathcal{A}$ (or some its closure) it is easy to construct semigroups which preserves all the (α_t, β) - **KMS** states, for all β's. Therefore the following feature of the stochastic dynamics introduced above is important.

Theorem 3.8:
Let $\mathcal{L}_{\mathbf{j},\Lambda} \equiv \mathcal{L}_{\mathbf{j},\Lambda,\beta\Phi}$ *be the Markov generator defined by (3.18) with* $X = \{\mathbf{j}\}$ *and let*

$$\mathcal{L}^{0,\Lambda} = \sum_{\mathbf{j}\in\Lambda} \mathcal{L}_{\mathbf{j},\Lambda}$$

If for some state ν *on* $\mathcal{A}_\Lambda$ *we have for any* $f \in \mathcal{A}_\Lambda$

$$\nu\mathcal{L}^{0,\Lambda}(f) = 0 \tag{3.40}$$

then ν *is* $(\alpha_t^\Lambda, \beta)$ - **KMS** . $\circ$

Remark: *Similar result is true with the operator* $\sum_{X+\mathbf{j}\subset\Lambda} \mathcal{L}_{X+\mathbf{j},\Lambda}$ *and arbitrary set* $X \in \mathcal{F}$*, provided that the union of* $X + \mathbf{j} \subset \Lambda$ *covers the set* Λ*.*

Proof: Suppose a state ν on $\mathcal{A}_\Lambda$ has a density $\rho_\nu > 0$ with respect to the normalized trace **Tr**. Then the condition (3.40) implies that for every $f \in \mathcal{A}_\Lambda$ we have

$$0 = \mathbf{Tr}\left(\rho_\nu \sum_{\mathbf{j}\in\Lambda} \mathcal{L}_{\mathbf{j},\Lambda}(f)\right) = \sum_{\mathbf{j}\in\Lambda} < \rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}}, \mathcal{L}_{\mathbf{j},\Lambda}(f) >_{\omega_\Lambda} = \sum_{\mathbf{j}\in\Lambda} < \mathcal{L}_{\mathbf{j},\Lambda}(\rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}}), f >_{\omega_\Lambda} \tag{3.41}$$

In particular choosing $f = \rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}}$ we get

$$\sum_{\mathbf{j}\in\Lambda} < \mathcal{L}_{\mathbf{j},\Lambda}(\rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}}), \rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}} >_{\omega_\Lambda} = 0 \tag{3.42}$$

This can be written in the following form

$$\sum_{\mathbf{j}\in\Lambda} < E_{\mathbf{j},\Lambda}(\rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}}), \rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}} >_{\omega_\Lambda} = |\Lambda| \cdot ||\rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}}||^2_{\mathbb{L}_2(\omega_\Lambda)} \tag{3.43}$$

Since the operators $E_{\mathbf{j},\Lambda}$ are all contractive in $\mathbb{L}_2(\omega_\Lambda)$, the above equality can only be true if for every $\mathbf{j} \in \Lambda$, we in fact have

$$E_{\mathbf{j},\Lambda}(\rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}}) = \rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}} \tag{3.44}$$

The left hand side of this equality, by definition of $E_{\mathbf{j},\Lambda}$, commutes with every element of $\mathcal{A}_\mathbf{j}$. Since this is true for every $\mathbf{j} \in \Lambda$ and $\{\mathcal{A}_\mathbf{j}\}_{\mathbf{j}\in\Lambda}$ generates $\mathcal{A}_\Lambda$, we conclude that

$$\rho_\Lambda^{-\frac{1}{2}}\rho_\nu\rho_\Lambda^{-\frac{1}{2}} = 1\,, \tag{3.45}$$

i.e. $\rho_\nu = \rho_\Lambda$, which clearly implies our theorem.

$\diamondsuit$

Finally let us mention that as shown in Appendix II the infinite volume limit of γ_Λ make sense as operators (possibly in some larger algebra). This motivates the general considerations of the next section, in which we formulate some general conditions for existence and ergodicity of infinite volume translation invariant spin flip stochastic dynamics.

4. Quantum Stochastic Dynamics: The Spin Flip Case

Let

$$\partial_{\mathbf{j}} f \equiv f - \mathbf{Tr}_{\mathbf{j}} f \tag{4.1}$$

We define the following seminorm $|||\cdot|||$ in $\mathcal{A}$

$$|||f||| \equiv \sum_{\mathbf{j}\in \mathbb{Z}^d} ||\partial_{\mathbf{j}} f|| \tag{4.2}$$

One can see that the seminorm $|||\cdot|||$ is finite on a subalgebra $\mathcal{A}_1 \subset \mathcal{A}$ containing $\mathcal{A}_0$ and it vanishes only on constants.
For $X \in \mathcal{F}$, let

$$\mathcal{L}_{X+\mathbf{j}}(f) \equiv E_{X+\mathbf{j}}(f) - f \tag{4.3}$$

where $E_{X+\mathbf{j}}$ is a two - positive unit preserving map on $\mathcal{A}$, such that $E_{X+\mathbf{j}}(\mathcal{A}) \subset \mathcal{A}_{X^c+\mathbf{j}}$. We define a finite volume generator $\mathcal{L}_\Lambda^X$ as follows

$$\mathcal{L}_\Lambda^X \equiv \sum_{\mathbf{j}\in\Lambda} \mathcal{L}_{X+\mathbf{j}} \tag{4.4}$$

The generator $\mathcal{L}_\Lambda^X$ is well defined bounded operator on all the algebra $\mathcal{A}$. Let $P_{t,\Lambda}^X \equiv e^{t\mathcal{L}_\Lambda^X}, \quad t \geq 0$ be the corresponding finite volume semigroup.
We would like to consider also an infinite volume generator $\mathcal{L}^X$ defined formally by the formula (4.4) with $\Lambda = \mathbb{Z}^d$. To ensure that it is defined on a sufficiently large domain, we will require that the elementary generators $\mathcal{L}_{X+\mathbf{j}}$ satisfy the following regularity property.

Definition 4.1:
The operator $\mathcal{L}_{X+j}$ is called <u>regular</u> if and only if there are nonnegative constants $b_{\mathbf{jk}}^X$, $\mathbf{j},\mathbf{k} \in \mathbb{Z}^d$, such that

$$||\mathcal{L}_{X+\mathbf{j}} f|| \leq \sum_{\mathbf{k}\in\mathbb{Z}^d} b_{\mathbf{jk}}^X ||\partial_{\mathbf{k}} f|| \tag{4.5}$$

and

$$\sup_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{j}\in\mathbb{Z}^d} b_{\mathbf{jk}}^X = b^X < \infty \tag{4.6}$$

$\circ$

Remark: *Let us mention that all block spin flip generators for classical discrete spin systems are regular.*

It is easy to see that under the assumption of regularity, the finite as well as infinite volume generators are well defined on the domain containing the subalgebra $\mathcal{A}_1$, which is dense in $\mathcal{A}$. If the elementary generators $\mathcal{L}_{X+\mathbf{j}}$ would be additionally symmetric in the space $\mathbb{L}_2(\omega)$, for some state ω, the operator $\mathcal{L}^X$, as a nonpositive densely defined symmetric operator in $\mathbb{L}_2(\omega)$, could be extended to a selfadjoint operator, (although possibly not in a unique way). Using this extension we could define in $\mathbb{L}_2(\omega)$ an infinite volume semigroup which in general need not to have the Feller property, (i.e. it would not need to map $\mathcal{A}$ into $\mathcal{A}$). Our first aim will be to formulate a condition which allows to construct an infinite volume semigroup P_t^X, $t \geq 0$ as a limit of finite volume semigroups in a way which ensures also the Feller property. It will be useful also to study the ergodicity properties of the semigroup P_t^X. To formulate our condition formally we use an idea from statistical mechanics of classical spin system on a lattice.

Definition 4.2:
*The elementary generators $\mathcal{L}_{X+\mathbf{j}}$, $\mathbf{j} \in \mathbb{Z}^d$ satisfy <u>**CX condition**</u> if and only if there are nonnegative constants $a_{\mathbf{kl}}^{X+\mathbf{j}}$, for $\mathbf{k},\mathbf{l} \in \mathbb{Z}^d$, such that $a_{\mathbf{k-i},\mathbf{l-i}}^{X+\mathbf{j-i}} = a_{\mathbf{kl}}^{X+\mathbf{j}}$, for any $\mathbf{i} \in \mathbb{Z}^d$ and for any $f \in \mathcal{A}_1$ we have*

(i)

$$||[\partial_{\mathbf{k}}, \mathcal{L}_{X+\mathbf{j}}](f)|| \leq \sum_{\mathbf{l}\in \mathbb{Z}^d} a_{\mathbf{kl}}^{X+\mathbf{j}} ||\partial_{\mathbf{l}} f|| \tag{4.7}$$

with

$$\frac{1}{|X|} \sum_{\mathbf{k},\mathbf{l}\in \mathbb{Z}^d} a_{\mathbf{kl}}^{X+\mathbf{j}} < \infty \tag{4.8}$$

(ii)

$$\frac{1}{|X|} \sum_{\mathbf{k}\in X^c+\mathbf{j},\mathbf{l}\in \mathbb{Z}^d} a_{\mathbf{kl}}^{X+\mathbf{j}} \leq \kappa < 1 \tag{4.9}$$

$\circ$

Remark: *For the construction of the infinite volume Markov-Feller semigroup we will need only the condition* **CX** (i). *The condition* **CX** (ii) *is similar to the famous uniqueness condition of Dobrushin and Shlosman. One can expect that also in the case of quantum spin systems one could use it to develop a uniqueness theory along the lines of* [**DS1-3**], *(or better to say its dual version of Aizenman and Lieb). In our paper we will use it in a similar way as in* [**SZ**], *(see also* [**AH**]*), to prove the corresponding strong ergodicity property of infinite volume semigroup.*

The interest in the condition **CX** is motivated by the following implications.

Theorem 4.3:

Suppose that the operators $\mathcal{L}_{X+\mathbf{j}}$ are regular and that the condition **CX** (i) *is satisfied. Then the following limit exists and defines a Markov semigroup on $\mathcal{A}$*

$$P_t^X \equiv e^{t\mathcal{L}^X} \equiv \lim_{\mathcal{F}_0} P_{t,\Lambda}^X \tag{4.10}$$

and we have the following <u>*approximation property*</u>*:*
There are positive functions φ and D satisfying $\varphi(t) \to_{t\to\infty} 0$ and $D(t) \to_{t\to\infty} \infty$, such that for any $f \in \mathcal{A}_Y$, $Y \in \mathcal{F}$, we have

$$||P_t^X f - P_{t,\Lambda}^X f|| \leq C(Y)\varphi(t)|||f||| \tag{4.11}$$

with some constant $C(Y) \in (0,\infty)$ independent of f, provided

$$d(f,\Lambda^c) \geq D(t) \tag{4.12}$$

$\circ$

Theorem 4.4:

If also **CX** (ii) *is satisfied then we have*

$$|||P_t^X f||| \leq e^{-(1-\kappa)|X|t}|||f||| \tag{4.13}$$

and therefore the semigroup P_t^X is strongly ergodic in the sense that there is unique P_t^X-invariant state ω for which we have

$$||P_t f - \omega f|| \leq 2e^{-(1-\kappa)|X|t}|||f||| \tag{4.14}$$

for every $f \in \mathcal{A}_1$. $\circ$

The proof of Theorem 4.3 is rather standard. We include it for the readers convenience.

Proof of Theorem 4.3: For $\Lambda_i \in \mathcal{F}$, $i = 1, 2$, let $P_t^i \equiv e^{t\mathcal{L}_i} \equiv P_{t,\Lambda_i}^X$. Then we have

$$\frac{d}{ds}(P_s^2 f - P_s^1 f) = \mathcal{L}_2(P_s^2 f - P_s^1 f) + (\mathcal{L}_2 - \mathcal{L}_1) P_s^1 f \tag{4.15}$$

Hence

$$\frac{d}{ds} P_{t-s}^2 (P_s^2 f - P_s^1 f) = P_{t-s}^2 (\mathcal{L}_2 - \mathcal{L}_1) P_s^1 f \tag{4.16}$$

Integration of this equation from 0 to t and use of contractivity property of Markov semigroup, yield

$$||P_t^2 f - P_t^1 f|| = || \int_0^t ds P_{t-s}^2 (\mathcal{L}_2 - \mathcal{L}_1) P_s^1 f|| \leq \int_0^t ds ||(\mathcal{L}_2 - \mathcal{L}_1) P_s^1 f|| \tag{4.17}$$

Therefore we need to study the expression $(\mathcal{L}_2 - \mathcal{L}_1) P_s^1 f$. As the difference of two Markov operators of interest to us equals to a sum of elementary generators $\mathcal{L}_{X+\mathbf{j}}$, it is sufficient to study $\mathcal{L}_{X+\mathbf{j}} P_s^1 f$ for $\mathbf{j} \in \mathbb{Z}^d$. Since however by our regularity assumption we have

$$||\mathcal{L}_{X+\mathbf{j}} P_t^1 f|| \leq \sum_{\mathbf{k}} b_{\mathbf{jk}}^X ||\partial_{\mathbf{k}} P_t^1 f|| \tag{4.18}$$

we shall study the behavior of $\partial_{\mathbf{k}} P_t^1 f$. For this we use the following differential equation

$$\frac{d}{d\tilde{s}} \partial_{\mathbf{k}} P_{\tilde{s}}^1 f = \partial_{\mathbf{k}} \mathcal{L}_1 P_{\tilde{s}}^1 f = \mathcal{L}_1 \left(\partial_{\mathbf{k}} P_{\tilde{s}}^1 f\right) + [\partial_{\mathbf{k}}, \mathcal{L}_1] P_{\tilde{s}}^1 f \tag{4.19}$$

Hence we get

$$\frac{d}{d\tilde{s}} P_{s-\tilde{s}}^1 \left(\partial_{\mathbf{k}} P_{\tilde{s}}^1 f\right) = P_{s-\tilde{s}}^1 [\partial_{\mathbf{k}}, \mathcal{L}_1] P_{\tilde{s}}^1 f = \sum_{\mathbf{i} \in \Lambda_1} P_{s-\tilde{s}}^1 \left([\partial_{\mathbf{k}}, \mathcal{L}_{X+\mathbf{i}}] P_{\tilde{s}}^1 f\right) \tag{4.20}$$

Integration of this equation and use of contractivity property of the Markov semigroups, give the following bound

$$||\partial_{\mathbf{k}} P_s^1 f|| \leq ||\partial_{\mathbf{k}} f|| + \sum_{\mathbf{i} \in \Lambda_1} \int_0^s d\tilde{s} ||[\partial_{\mathbf{k}}, \mathcal{L}_{X+\mathbf{i}}] P_{\tilde{s}}^1 f|| \tag{4.21}$$

If the condition **CX** (i) is satisfied, the right hand side of (4.21) can be bounded by

$$||\partial_{\mathbf{k}} P_s^1 f|| \leq ||\partial_{\mathbf{k}} f|| + \int_0^s d\tilde{s} \sum_{\mathbf{l}} \mathbf{G}_X(\mathbf{k} - \mathbf{l}) ||\partial_{\mathbf{l}} P_{\tilde{s}}^1 f|| \tag{4.22}$$

with a translation invariant matrix

$$\mathbf{G}_X(\mathbf{k} - \mathbf{l}) \equiv \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{kl}}^{X+\mathbf{i}} \tag{4.23}$$

Since by our assumptions about $a_{\mathbf{kl}}^{X+\mathbf{i}}$ we have

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \mathbf{G}_X(\mathbf{k}) = \sum_{\mathbf{k}} \sum_{\mathbf{i}} a_{\mathbf{kl}}^{X+\mathbf{i}} = \sum_{\mathbf{k}} \sum_{\mathbf{l}} a_{\mathbf{kl}}^X \leq \kappa |X| < \infty \tag{4.24}$$

we can solve the inequality (4.22) by iteration and we obtain the following bound

$$||\partial_{\mathbf{k}} P_s^1 f|| \leq \sum_{\mathbf{l}} \left(e^{s\mathbf{G}_X}\right)_{\mathbf{kl}} ||\partial_{\mathbf{l}} f|| \tag{4.25}$$

Combining (4.17), (4.18) and (4.25), we arrive at the following estimate

$$||P_t^2 f - P_t^1 f|| \leq t \sum_{\mathbf{j} \in \Lambda_2 \setminus \Lambda_1} \sum_{\mathbf{k},\mathbf{l}} b_{\mathbf{jk}}^X \left(e^{t\mathbf{G}_X}\right)_{\mathbf{kl}} ||\partial_{\mathbf{l}} f|| \tag{4.26}$$

Hence for any $\Lambda_2 \subset \mathbb{Z}^d$ containing a set Λ_1 we have

$$||P_t^2 f - P_t^1 f|| \leq t \sum_{\mathbf{j}\in\Lambda_1{}^c} \sum_{\mathbf{k},\mathbf{l}} b_{\mathbf{jk}}^X \left(e^{t\mathbf{G}_X}\right)_{\mathbf{kl}} ||\partial_{\mathbf{l}} f|| \tag{4.27}$$

Using the summability properties (4.6) and (4.24) of the matrices involved on the right hand side of (4.27), one can easily conclude that the limit

$$P_t^X f \equiv \lim_{\mathcal{F}_0} P_{t,\Lambda}^X f \tag{4.28}$$

exists for all local elements $f \in \mathcal{A}_0$. Hence, by continuity in the norm $||\cdot||$, it exists also for any $f \in \mathcal{A}$. From the estimate (4.27) one gets also the approximation property (4.11-12) with the appropriate functions $\varphi(t)$ and $D(t)$, (the second dependent only on the choice of the former). This ends the proof of Theorem 4.3. $\diamondsuit$

Proof of Theorem 4.4: To proof Theorem 4.4 we follow closely [**SZ**]. First we note that by our assumption $E_{X+\mathbf{j}}\mathcal{A} \subset \mathcal{A}_{X^c+\mathbf{j}}$ and therefore for any $\mathbf{k} \in X+\mathbf{j}$ we have

$$\partial_{\mathbf{k}} \mathcal{L}_{X+\mathbf{j}} f = \partial_{\mathbf{k}}(E_{X+\mathbf{j}} f - f) = -\partial_{\mathbf{k}} f \tag{4.29}$$

Using this we get

$$\frac{d}{ds}\partial_{\mathbf{k}} P_s^X f = \partial_{\mathbf{k}} \mathcal{L}^X P_s^X f = -|X|\partial_{\mathbf{k}} P_s^X f + \mathcal{L}^{X,\mathbf{k}} \partial_{\mathbf{k}} P_s^X f + \sum_{\mathbf{j}:X^c+\mathbf{j}\ni\mathbf{k}} [\partial_{\mathbf{k}}, \mathcal{L}_{X+\mathbf{j}}] P_s^X f \tag{4.30}$$

where we have set

$$\mathcal{L}^{X,\mathbf{k}} = \sum_{\mathbf{j}:X^c+\mathbf{j}\ni\mathbf{k}} \mathcal{L}_{X+\mathbf{j}} \tag{4.31}$$

Setting $P_t^{X,\mathbf{k}} \equiv e^{t\mathcal{L}^{X,\mathbf{k}}}$, we get

$$\frac{d}{ds}\left(e^{s|X|} P_{t-s}^{X,\mathbf{k}} \partial_{\mathbf{k}} P_s^X f\right) = \sum_{\mathbf{j}:X^c+\mathbf{j}\ni\mathbf{k}} e^{s|X|} P_{t-s}^{X,\mathbf{k}} \left([\partial_{\mathbf{k}}, \mathcal{L}_{X+\mathbf{j}}] P_s^X f\right) \tag{4.32}$$

Integrating this equation from 0 to t, and taking into account that $P_t^{X,\mathbf{k}}$ is a contraction semigroup on $\mathcal{A}$, we obtain the following bound

$$||\partial_{\mathbf{k}} P_t^X f|| \leq e^{-t|X|}||\partial_{\mathbf{k}} f|| + \int_0^t ds e^{-(t-s)|X|} \sum_{\mathbf{j}:X^c+\mathbf{j}\ni\mathbf{k}} ||[\partial_{\mathbf{k}}, \mathcal{L}_{X+\mathbf{j}}] P_s^X f|| \tag{4.33}$$

Applying to the last term on the right hand side the condition **CX** (i), we get

$$||\partial_{\mathbf{k}} P_t^X f|| \leq e^{-t|X|}||\partial_{\mathbf{k}} f|| + \int_0^t ds e^{-(t-s)|X|} \sum_{\mathbf{j}:X^c+\mathbf{j}\ni\mathbf{k}} \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{kl}}^{X+\mathbf{j}} ||\partial_{\mathbf{l}} P_s^X f|| \tag{4.34}$$

Summing this inequalities over $\mathbf{k} \in \mathbb{Z}^d$ and taking into the account that by translation invariance of $a_{\mathbf{kl}}^{X+\mathbf{j}}$ and the condition **CX** (ii) we have

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{j}:X^c+\mathbf{j}\ni\mathbf{k}} a_{\mathbf{kl}}^{X+\mathbf{j}} \leq \kappa|X| \tag{4.35}$$

we obtain

$$|||P_t^X f||| \leq e^{-t|X|}|||f||| + \kappa|X| \int_0^t ds e^{-(t-s)|X|} |||P_s^X f||| \tag{4.36}$$

From this the inequality (4.13) easily follows. To prove the strong ergodicity property we note first that, by weak compactness of the space of states on $\mathcal{A}$ and the fact that by our construction P_t^X has Feller property, the set of P_t^X-invariant states is nonempty. Let ω be one such state. Then we have

$$||P_t^X f - \omega f|| = ||P_t^X f - \omega(P_t^X f)|| = ||\Theta\left(P_t^X f \otimes \mathbf{1} - \mathbf{1} \otimes P_t^X f\right)|| \leq ||P_t^X f \otimes \mathbf{1} - \mathbf{1} \otimes P_t^X f|| \tag{4.37}$$

the norm on the right hand side means the norm on the injective tensor product algebra $\mathcal{A}$ by itself, while Θ is a completely positive map from $\mathcal{A} \otimes \mathcal{A}$ to $\mathcal{A}$ defined by $\Theta(x_1 \otimes x_2) \equiv \omega(x_1)x_2$, where $x_1, x_2 \in \mathcal{A}$, (c.f. Section IV.4 in [**Ta**], in particular Corollary 4.25). Choosing some lexicographic sequence $\{\mathbf{j}_n\}_{n\in\mathbb{N}}$ in $\mathbb{Z}^d$ and observing that

$$P_t^X f \otimes \mathbf{1} - \mathbf{1} \otimes P_t^X f = \mathbf{\Sigma} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{\Sigma} \tag{4.38}$$

with

$$\mathbf{\Sigma} \equiv (P_t^X f - \mathbf{Tr}_{\mathbf{j}_1} P_t^X f) + \sum_{n\in\mathbb{N}} \mathbf{Tr}_{\{\mathbf{j}_1,..,\mathbf{j}_n\}}(P_t^X f - \mathbf{Tr}_{\mathbf{j}_{n+1}} P_t^X f) \tag{4.39}$$

one easily arrives at the following inequality

$$||P_t^X f - \omega f|| \leq 2|||P_t^X f||| \tag{4.40}$$

Now the desired bound (4.14) follows from the first part of Theorem 4.4.

$\diamondsuit$

In the rest of this section we would like to consider the elementary operators defined by

$$\mathcal{L}_{X+\mathbf{j}}(f) = \mathbf{Tr}_{X+\mathbf{j}}(\gamma_{X+\mathbf{j}}^* f \gamma_{X+\mathbf{j}}) - f \tag{4.41}$$

with some $\gamma_{X+\mathbf{j}} \in \mathcal{A}$, such that $\mathbf{Tr}_{X+\mathbf{j}}(\gamma_{X+\mathbf{j}}^* \gamma_{X+\mathbf{j}}) = \mathbf{1}$. This assumption assures that the finite volume dynamics $P_t^\Lambda \equiv e^{t\mathcal{L}_\Lambda^X}$ have the Feller property. We would also like to formulate the general conditions implying the regularity and **CX** conditions. To get the first one, we will need the following simple lemma (in which we use a notation $\{x, y\}$ for the anticommutator of operators x and y, defined by $\{x, y\} \equiv xy + yx$).

Lemma 4.5:
The operators $\mathcal{L}_{X+\mathbf{j}}$ admits the following representation

$$\mathcal{L}_{X+\mathbf{j}}(f) = \frac{1}{2}\mathbf{Tr}_{X+\mathbf{j}}\left([\gamma_{X+\mathbf{j}}^*, f]\gamma_{X+\mathbf{j}}\right) + \frac{1}{2}\mathbf{Tr}_{X+\mathbf{j}}\left(\gamma_{X+\mathbf{j}}^*[f, \gamma_{X+\mathbf{j}}]\right) +$$

$$+ \frac{1}{2}\mathbf{Tr}_{X+\mathbf{j}}\left(\{(f - \mathbf{Tr}_{X+\mathbf{j}} f), (\gamma_{X+\mathbf{j}}^* \gamma_{X+\mathbf{j}} - 1)\}\right) + (\mathbf{Tr}_{X+\mathbf{j}} f - f) \tag{4.42}$$

and so

$$||\mathcal{L}_{X+\mathbf{j}}(f)|| \leq \frac{1}{2}\left(|||[\gamma_{X+\mathbf{j}}^*, f]||| \cdot ||\gamma_{X+\mathbf{j}}|| + ||\gamma_{X+\mathbf{j}}^*|| \cdot |||[f, \gamma_{X+\mathbf{j}}]|||\right) + \left(||\gamma_{X+\mathbf{j}}^* \gamma_{X+\mathbf{j}} - 1|| + 1\right) \sum_{\mathbf{k}\in X+\mathbf{j}} ||\partial_{\mathbf{k}} f|| \tag{4.43}$$

$\circ$

Proof: We have

$$\mathcal{L}_{X+\mathbf{j}}(f) = \mathbf{Tr}_{X+\mathbf{j}}(\gamma_{X+\mathbf{j}}^* f \gamma_{X+\mathbf{j}}) - f = \tag{4.44}$$

$$= \frac{1}{2}\mathbf{Tr}_{X+\mathbf{j}}\left([\gamma_{X+\mathbf{j}}^*, f]\gamma_{X+\mathbf{j}}\right) + \frac{1}{2}\mathbf{Tr}_{X+\mathbf{j}}\left(\gamma_{X+\mathbf{j}}^*[f, \gamma_{X+\mathbf{j}}]\right) + \frac{1}{2}\mathbf{Tr}_{X+\mathbf{j}}\left(\{f, \gamma_{X+\mathbf{j}}^* \gamma_{X+\mathbf{j}}\}\right) - f$$

Using the normalisation condition $\mathbf{Tr}_{X+\mathbf{j}}(\gamma^*_{X+\mathbf{j}}\gamma_{X+\mathbf{j}}) = 1$ and a property of the partial trace, we can represent the last part of the right hand side of (4.44) as follows

$$\frac{1}{2}\mathbf{Tr}_{X+\mathbf{j}}\left(\{f,\gamma^*_{X+\mathbf{j}}\gamma_{X+\mathbf{j}}\}\right) - f = \frac{1}{2}\mathbf{Tr}_{X+\mathbf{j}}\left(\{(f-\mathbf{Tr}_{X+\mathbf{j}}f),(\gamma^*_{X+\mathbf{j}}\gamma_{X+\mathbf{j}}-1\}\right)\right) + (\mathbf{Tr}_{X+\mathbf{j}}f - f) \tag{4.45}$$

Combining this with (4.44) we get the desired representation (4.42). The inequality (4.43) easily follows from (4.42).

$\diamondsuit$

Given Lemma 4.5 we obtain the following condition for the regularity.

Theorem 4.6:
Suppose

$$\sup_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{j}\in\mathbb{Z}^d}\sum_{\mathbf{l}:d(\mathbf{l},X+\mathbf{j})\geq d(\mathbf{k},X+\mathbf{j})}||\partial_{\mathbf{l}}\gamma_{X+\mathbf{j}}|| < \infty \tag{4.46}$$

Then the operators $\mathcal{L}_{X+\mathbf{j}}$ given by (4.41) satisfies the regularity condition. $\circ$

Proof: In view of Lemma 4.5 it is sufficient to prove that the following inequality is true

$$||[\gamma_{X+\mathbf{j}},f]|| \leq \sum_{\mathbf{k}}\tilde{b}^X_{\mathbf{jk}}\,||\partial_{\mathbf{k}}f|| \tag{4.47}$$

with some nonnegative constants $\tilde{b}^X_{\mathbf{jk}}$ such that

$$\sup_{\mathbf{k}}\sum_{\mathbf{j}}\tilde{b}^X_{\mathbf{jk}} < \infty \tag{4.48}$$

To do this let us choose a lexicographic sequence $\mathbf{l}_n, n\in\mathbb{N}$, such that for some countable exhaustion $\mathcal{F}^{\mathbf{j}} \equiv \{\Lambda_1 \equiv X+\mathbf{j}, \Lambda_{m+1}\supset\Lambda_m\}_{m\in\mathbb{N}}$ we have

$$\mathbf{l}_n\in\Lambda_m \text{ and } \mathbf{l}_{n'}\in\Lambda_{m+1}\setminus\Lambda_m \Longrightarrow n' > n \tag{4.49}$$

We observe now the following representation for a commutator, (which follows from the simple fact that a commutator vanishes if one of its entries equals to a multiple of the identity).

$$[g,f] = \sum_{m\in\mathbb{N}}\left[\mathbf{Tr}_{\{\mathbf{l}_1,..,\mathbf{l}_m\}^c}(\partial_{\mathbf{l}_m}g),f\right] = \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{Z}^+}\left[\mathbf{Tr}_{\{\mathbf{l}_1,..,\mathbf{l}_m\}^c}(\partial_{\mathbf{l}_m}g),\mathbf{Tr}_{\{\mathbf{l}_1,..,\mathbf{l}_n\}}(\partial_{\mathbf{l}_{n+1}}f)\right] \tag{4.50}$$

with the convention in the second term that $\mathbf{Tr}_{\mathbf{l}_0}\equiv I$. (Observe that the last partial trace on the right hand side of (4.50) is associated to the set $\{\mathbf{l}_1,..,\mathbf{l}_n\}$, not to its complement, as in the previous trace.) Using also the local structure of our algebra, we get

$$[g,f] = \sum_{n\in\mathbb{N}}\sum_{m\geq n\in\mathbb{N}}\left[\mathbf{Tr}_{\{\mathbf{l}_1,..,\mathbf{l}_m\}^c}(\partial_{\mathbf{l}_m}g),\mathbf{Tr}_{\{\mathbf{l}_1,..,\mathbf{l}_n\}}(\partial_{\mathbf{l}_{n+1}}f)\right] \tag{4.51}$$

Hence we obtain

$$||[g,f]|| \leq \sum_{n\in\mathbb{N}}\left(\sum_{m\geq n\in\mathbb{N}}||\partial_{\mathbf{l}_m}g||\right)\cdot||\partial_{\mathbf{l}_{n+1}}f|| \tag{4.52}$$

An application of this formula to the case studied by us leads to the following inequality (4.47) with the corresponding constants $\tilde{b}^X_{\mathbf{jk}}$ given by

$$\tilde{b}^X_{\mathbf{jk}} = \sum_{l_m:d(\mathbf{l}_m,X+\mathbf{j})\geq d(\mathbf{k},X+\mathbf{j})}||\partial_{\mathbf{l}_m}\gamma_{X+\mathbf{j}}|| \tag{4.53}$$

Thus the condition (4.46) implies that

$$\sup_{\mathbf{k}} \sum_{\mathbf{j}\in\mathbb{Z}^d} \tilde{b}^X_{\mathbf{jk}} < \infty \tag{4.54}$$

Similar considerations involving $\gamma^*_{X+\mathbf{j}}$ together with the use of (4.43) allows us to construct the constants $b^X_{\mathbf{jk}}$ such that

$$||\mathcal{L}_{X+\mathbf{j}}(f)|| \le \sum_{\mathbf{j}\in\mathbb{Z}^d} b^X_{\mathbf{jk}}\, ||\partial_{\mathbf{k}} f|| \tag{4.55}$$

and

$$\sup_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{j}\in\mathbb{Z}^d} b^X_{\mathbf{jk}} < \infty \tag{4.56}$$

This ends the proof of Theorem 4.6.

◇

Now we shall study the **CX** condition. We have the following result

Theorem 4.7:
Suppose that (4.46) is true. Then

$$||[\partial_{\mathbf{k}}, \mathcal{L}_X](f)|| \le \sum_{\mathbf{l}\in X} a_{\mathbf{kl}} \cdot ||\partial_{\mathbf{l}} f|| \tag{4.57}$$

with

$$a_{\mathbf{kl}} \le ||\partial_{\mathbf{k}}\gamma_X|| \cdot \tilde{b}^X_{\mathbf{0},\mathbf{l}} + \sum_{\substack{\mathbf{l}_m : d(\mathbf{l}_m, X) \ge d(\mathbf{l}, X) \\ d(\mathbf{l}_m, X) \ge d(\mathbf{k}, X)}} ||\gamma_X|| \cdot ||\partial_{\mathbf{l}_m}\gamma^*_X|| + \chi(\mathbf{l}\in X) ||\gamma_X|| \cdot ||\partial_{\mathbf{k}}\gamma_X|| \tag{4.58}$$

∘

Proof: Using Lemma 4.5 one can see that for any $\mathbf{k} \in \mathbb{Z}^d$ we have

$$[\partial_{\mathbf{k}}, \mathcal{L}_X](f) = \mathbf{A1} + \mathbf{A2} + \mathbf{A3} \tag{4.59}$$

where

$$\mathbf{A1} \equiv \frac{1}{2}\left(\mathbf{Tr}_X\left([\gamma^*_X, \mathbf{Tr}_{\mathbf{k}} f]\gamma_X\right) - \mathbf{Tr}_{\mathbf{k}}\mathbf{Tr}_X\left([\gamma^*_X, f]\gamma_X\right)\right) \tag{4.60a}$$

$$\mathbf{A2}(f) \equiv \mathbf{A1}(f^*)^* \tag{4.60b}$$

and

$$\mathbf{A3} \equiv \frac{1}{2}\Big(\mathbf{Tr}_X\left(\{(\mathbf{Tr}_{\mathbf{k}} f - \mathbf{Tr}_X\mathbf{Tr}_{\mathbf{k}} f), (\gamma^*_X\gamma_X - 1)\}\right) - \mathbf{Tr}_{\mathbf{k}}\mathbf{Tr}_X\left(\{(f - \mathbf{Tr}_X f), (\gamma^*_X\gamma_X - 1)\}\right)\Big) \tag{4.61}$$

Let us consider first **A1**. After simple calculations one gets

$$\begin{aligned} 2\mathbf{A1} &= \mathbf{Tr}_X\left([\partial_{\mathbf{k}}\gamma^*_X, \mathbf{Tr}_{\mathbf{k}} f]\gamma_X\right) + \mathbf{Tr}_X\left([\mathbf{Tr}_{\mathbf{k}}\gamma^*_X, \mathbf{Tr}_{\mathbf{k}} f]\partial_{\mathbf{k}}\gamma_X\right) \\ &\quad - \mathbf{Tr}_{\mathbf{k}}\mathbf{Tr}_X\left([\partial_{\mathbf{k}}\gamma^*_X, f]\gamma_X\right) - \mathbf{Tr}_{\mathbf{k}}\mathbf{Tr}_X\left([\mathbf{Tr}_{\mathbf{k}}\gamma^*_X, f]\partial_{\mathbf{k}}\gamma_X\right) \end{aligned} \tag{4.62}$$

Now we use similar idea as in (4.50) to expand γ^*_X and f. By this we get the following representation of the first term on the right hand side of (4.62)

$$\mathbf{Tr}_X\left([\partial_{\mathbf{k}}\gamma^*_X, \mathbf{Tr}_{\mathbf{k}} f]\gamma_X\right) = \sum_{\mathbf{l}_m}\sum_{\mathbf{l}_n} \mathbf{Tr}_X\left([\partial_{\mathbf{k}}\mathbf{Tr}_{\{\mathbf{l}_1,..,\mathbf{l}_m\}^c}\partial_{\mathbf{l}_m}\gamma^*_X, \mathbf{Tr}_{\mathbf{k}}\mathbf{Tr}_{\{\mathbf{l}_1,..,\mathbf{l}_n\}}\partial_{\mathbf{l}_{n+1}} f]\gamma_X\right) =$$

$$= \sum_{\mathbf{l}_n} \sum_{\substack{\mathbf{l}_m : d(\mathbf{l}_m, X) \geq d(\mathbf{l}_n, X) \\ d(\mathbf{l}_m, X) \geq d(\mathbf{k}, X)}} \mathbf{Tr}_X \left([\partial_{\mathbf{k}} \mathbf{Tr}_{\{\mathbf{l}_1, \ldots, \mathbf{l}_m\}^c} \partial_{\mathbf{l}_m} \gamma_X^*, \mathbf{Tr}_{\mathbf{k}} \mathbf{Tr}_{\{\mathbf{l}_1, \ldots, \mathbf{l}_n\}} \partial_{\mathbf{l}_{n+1}} f] \gamma_X \right) \tag{4.63}$$

Hence we obtain the following bound on the first term on the right hand side of (4.62)

$$||\mathbf{Tr}_X \left([\partial_{\mathbf{k}} \gamma_X^*, \mathbf{Tr}_{\mathbf{k}} f] \gamma_X \right)|| \leq \sum_{\mathbf{l}_n} \left(\sum_{\substack{\mathbf{l}_m : d(\mathbf{l}_m, X) \geq d(\mathbf{l}_n, X) \\ d(\mathbf{l}_m, X) \geq d(\mathbf{k}, X)}} ||\gamma_X|| \cdot ||\partial_{\mathbf{l}_m} \gamma_X^*|| \right) \cdot ||\partial_{\mathbf{l}_{n+1}} f|| \tag{4.64}$$

The similar estimate will remain true also for the third term on the right hand side of (4.62). The second (as well as the last) term from the right hand side of (4.62) can be bounded as follows

$$||\mathbf{Tr}_X \left([\mathbf{Tr}_{\mathbf{k}} \gamma_X^*, \mathbf{Tr}_{\mathbf{k}} f] \partial_{\mathbf{k}} \gamma_X \right)|| \leq \sum_{\mathbf{l} \in \mathbb{Z}^d} ||\partial_{\mathbf{k}} \gamma_X|| \cdot \tilde{b}^X_{\mathbf{0},\mathbf{l}} \, ||\partial_{\mathbf{l}} f|| \tag{4.65}$$

Combining (4.64) and (4.65) we get

$$||\mathbf{A1} + \mathbf{A2}|| \leq \sum_{\mathbf{l} \in \mathbb{Z}^d} \left(||\partial_{\mathbf{k}} \gamma_X|| \cdot \tilde{b}^X_{\mathbf{0},\mathbf{l}} + \sum_{\substack{\mathbf{l}_m : d(\mathbf{l}_m, X) \geq d(\mathbf{l}, X) \\ d(\mathbf{l}_m, X) \geq d(\mathbf{k}, X)}} ||\gamma_X|| \cdot ||\partial_{\mathbf{l}_m} \gamma_X^*|| \right) \cdot ||\partial_{\mathbf{l}} f|| \tag{4.66}$$

For the $\mathbf{A3}$ term we have

$$2\mathbf{A3} \equiv \mathbf{Tr}_X \left(\{ (\mathbf{Tr}_{\mathbf{k}} f - \mathbf{Tr}_X \mathbf{Tr}_{\mathbf{k}} f), \partial_{\mathbf{k}} (\gamma_X^* \gamma_X) \} \right) - \mathbf{Tr}_{\mathbf{k}} \mathbf{Tr}_X \left(\{ (f - \mathbf{Tr}_X f), \partial_{\mathbf{k}} (\gamma_X^* \gamma_X) \} \right) \tag{4.67}$$

whence we obtain

$$||\mathbf{A3}|| \leq \sum_{\mathbf{l} \in X} (||\gamma_X|| \cdot ||\partial_{\mathbf{k}} \gamma_X||) \cdot ||\partial_{\mathbf{l}} f|| \tag{4.68}$$

Combining (4.66) and (4.68), we obtain

$$||[\partial_{\mathbf{k}}, \mathcal{L}_X](f)|| \leq \sum_{\mathbf{l} \in X} a_{\mathbf{kl}} \cdot ||\partial_{\mathbf{l}} f|| \tag{4.69}$$

with

$$a_{\mathbf{kl}} \leq \left(||\partial_{\mathbf{k}} \gamma_X|| \cdot \tilde{b}^X_{\mathbf{0},\mathbf{l}} + \sum_{\substack{\mathbf{l}_m : d(\mathbf{l}_m, X) \geq d(\mathbf{l}, X) \\ d(\mathbf{l}_m, X) \geq d(\mathbf{k}, X)}} ||\gamma_X|| \cdot ||\partial_{\mathbf{l}_m} \gamma_X^*|| \right) + \chi(\mathbf{l} \in X) ||\gamma_X|| \cdot ||\partial_{\mathbf{k}} \gamma_X|| \tag{4.70}$$

This ends the proof of Theorem 4.7.

$\diamond$

Since also in this paper we would like to discuss the general strategy for the case of diffusion type stochastic dynamics, some specific applications of the above presented strategy will be studied elsewhere.

5. Quantum Stochastic Dynamics: The Diffusion Case

Let ω be an (α_t, β)-**KMS** state of a quantum lattice system. In this section we consider a family of $\mathbb{L}_2(\omega, \lambda)$ spaces with the following scalar product

$$< f, g >_{\omega,\lambda} \equiv \omega\left((\alpha_{i\lambda\beta/2}(f))^* \alpha_{i\lambda\beta/2}(g)\right), \qquad \lambda \in [0,1] \tag{5.1}$$

If $\beta \in (0, \infty)$ is sufficiently small, then for $f, g \in \mathcal{A}_0$ we have $\alpha_{i\lambda\beta/2}(f), \alpha_{i\lambda\beta/2}(g) \in \mathcal{A}$ and in this case the right hand side of (5.1) make sense. In general it has to be understood in the sense of analytic continuation of an appropriate function.
In particular for $\lambda = 0$ we have

$$< f, g >_{\omega,0} = \omega(f^* g)$$

whereas for $\lambda = 1$ we have

$$< f, g >_{\omega,1} = \omega(g f^*)$$

The case $\lambda = \frac{1}{2}$ has been considered in the previous sections. Let $|| \cdot ||_{L_2(\omega,\lambda)}$ denote the corresponding norm. The index λ will be frequently omitted, as all the claims of this section remain true for every $\mathbb{L}_2(\omega, \lambda)$ space. For $x \in \mathcal{A}$, let ∇_x denote the associated derivation given by $\nabla_x(f) \equiv i[x, f]$. Let $\tau_{\mathbf{j}}$ denote the translation automorphism on $\mathcal{A}$ corresponding to the translation of the lattice by a vector $\mathbf{j} \in \mathbb{Z}^d$. For a subalgebra $\mathcal{B}$ of $\mathcal{A}$, we define $\tau(\mathcal{B}) \equiv \bigcup_{\mathbf{j} \in \mathbb{Z}^d} \tau_{\mathbf{j}}(\mathcal{B})$.
For later purposes we would like to distinguish following Asymptotic Abelianess conditions.

Conditions AA :
There is $p \in [1, 2]$, a finite set $\mathbf{M}_0$ of selfadjoint elements in the single spin algebra $\mathbf{M}$ and a dense subalgebra $\tilde{\mathcal{A}}$ in $\mathcal{A}$ such that for any $x \in \tau(\mathbf{M}_0)$ and $f \in \tilde{\mathcal{A}}$, we have

Weak Asymptotic Abelianess:

$(\mathbf{WAA}_p)$
$$\int_{-\infty}^{\infty} ||\nabla_{\alpha_s(x)}(f)||^p_{L_2(\omega)} ds < \infty$$

Strong Asymptotic Abelianess

$(\mathbf{SAA}_p)$
$$\int_{-\infty}^{\infty} ||\nabla_{\alpha_s(x)}(f)||^p ds < \infty$$

$\circ$

Remark: *The choice of $\mathbf{M}_0$ seems to be natural (see discussion given later), although some other choices should not be a'priori excluded, as e.g. $\delta^n_H(x)$.* $\circ$

Let $\mathbf{K}_\lambda$ be a positive definite function belonging to $\mathbb{L}_q(\mathbb{R}, ds)$, for $q = \frac{p}{2(p-1)}$ and suppose a condition $\mathbf{WAA}_p$ is satisfied with some $p \in [1, 2]$. We introduce *an elementary quadratic form* $\mathcal{E}_x(\cdot\,,\cdot) \equiv \mathcal{E}_{x,\lambda}(\cdot\,,\cdot)$ in direction $x \in \tau(\mathbf{M}_0)$, with the domain $\mathcal{D} \equiv \mathcal{D}(\mathcal{E}_x) = \tilde{\mathcal{A}}$ as follows

$$\mathcal{E}_x(f, g) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr\, ds\, \mathbf{K}_\lambda(r - s) < \nabla_{\alpha_r(x)}(f), \nabla_{\alpha_s(x)}(g) >_\omega \equiv \tag{5.2}$$

$$\equiv \lim_{T\to\infty} \int_{-T}^{T}\int_{-T}^{T} dr\, ds\, \mathbf{K}_\lambda(r - s) < \nabla_{\alpha_r(x)}(f), \nabla_{\alpha_s(x)}(g) >_\omega \equiv \lim_{T\to\infty} \mathcal{E}_{x,T}(f, g)$$

If a condition **AA** is true, then (using Hölder's and Young's inequalities) one can see that $\mathcal{E}_x(\cdot, \cdot)$ is a well (densely) defined nonnegative form.

Suppose additionally that the function $\mathbf{K}_\lambda$ is analytic in an open strip containing $Im\, z \in [-(1-\lambda)\beta, \lambda\beta]$ and satisfies the following conditions

$$\mathbf{K}_\lambda(s - r - i\beta(1-\lambda)) = \mathbf{K}_\lambda(r - s + i\beta\lambda) \tag{5.3}$$

One can realize that by setting

$$\mathbf{K}_\lambda(s) = \int_{-\infty}^{\infty} dq\, e^{iqs} \hat{\mathbf{K}}_\lambda(q) \tag{5.4}$$

with

$$\hat{\mathbf{K}}_\lambda(q) = (1 + e^{-q\beta(1-2\lambda)})\hat{\mathbf{C}}(q) \tag{5.5}$$

where

$$0 \leq \hat{\mathbf{C}}(q) = \hat{\mathbf{C}}(-q) \tag{5.6}$$

is some sufficiently smooth and fast decreasing function. For later purposes we assume that $\hat{\mathbf{K}}_\lambda$ is bounded for every $\lambda \in [0,1]$ and that the following condition is satisfied

$$\sup_{\beta' \in [-\beta,\beta]} \int_{-\infty}^{\infty} dr |\mathbf{K}_\lambda(r + i\beta')| < \infty \tag{5.7}$$

For $T \in (0,\infty)$ we define on $\mathcal{A}$ the following bounded generators $\mathbf{L}_{x,T} \equiv \mathbf{L}_{x,\lambda,T}$ of (completely) positive semigroups

$$\mathbf{L}_{x,T}(f) \equiv \mathbf{L}_{x,\lambda,T}(f) \equiv \int_{-T}^{T}\int_{-T}^{T} dr\, ds\, \mathcal{K}_\lambda(r-s) i \left(\nabla_{\alpha_r(x)}(f)\alpha_s(x) - \alpha_r(x)\nabla_{\alpha_s(x)}(f)\right) \tag{5.8a}$$

where

$$\mathcal{K}_\lambda(r-s) \equiv \mathbf{K}_\lambda(r - s + i\beta\lambda) \tag{5.8b}$$

One has the following interesting fact.

Theorem 5.1:

If $\mathbf{SAA}_1$ *and (5.3) are satisfied with the positive definite kernel* $\mathbf{K}_\lambda \in \mathbb{L}_1(\mathbb{R}, dr)$*, then the following operator*

$$\mathbf{L}_x(f) \equiv -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dr\, ds\, \mathcal{K}_\lambda(r-s) i \Big(\nabla_{\alpha_r(x)}(f)\alpha_s(x) - \alpha_r(x)\nabla_{\alpha_s(x)}(f)\Big) \equiv \lim_{T\to\infty} \mathbf{L}_{x,T}(f) \tag{5.9}$$

is well defined as an operator $\mathbf{L} : \mathcal{D}_0 \longrightarrow \mathcal{A}$*, on a dense domain* $\mathcal{D}_0 \equiv \tilde{\mathcal{A}}$*, and if*

$$\sup_{\beta' \in [-\beta,\beta]} ||\alpha_{i\beta'}(x)|| \leq C_1 < \infty \quad , \tag{5.10}$$

its quadratic form in $\mathbb{L}_2(\omega,\lambda)$ *coincides with* $-\mathcal{E}_{x,\lambda}(\cdot,\cdot)$*. Moreover the operator* $\mathbf{L}_x$ *is:*
$*$-invariant ,i.e.

$$(\mathbf{L}_x f)^* = \mathbf{L}_x(f^*) \tag{5.11}$$

and dissipative, i.e. for any $f \in \tilde{\mathcal{A}}$ *we have*

$$\mathbf{\Gamma}_x(f,f) \equiv \frac{1}{2}\Big(\mathbf{L}_x(f^*f) - (\mathbf{L}_x f)^* f - f^*(\mathbf{L}_x f)\Big) \geq 0 \tag{5.12}$$

$\circ$

The operators (5.9) have been first introduced in **[QSV]** in the special case of $\mathbb{L}_2(\omega, \lambda = 0)$ space. The theorem says that one can introduce a similar well defined and symmetric operator in every $\mathbb{L}_2(\omega,\lambda)$ space.

For the proof of this theorem, as well as for some later purposes, we need to study the nonnegative quadratic form $\mathcal{E}_{x,T}(\cdot,\cdot)$ and the completely positive operator $\mathbf{L}_{x,T}$, both defined for $T \in (0,\infty)$ on all the algebra $\mathcal{A}$. The quadratic form $\mathcal{E}_{x,T,\lambda} \equiv \mathcal{E}_{x,T,\lambda}(\cdot,\cdot)$ defines a nonpositive symmetric operator $\tilde{\mathbf{L}}_{x,T} \equiv \tilde{\mathbf{L}}_{x,T,\lambda}$ on $\mathbb{L}_2(\omega,\lambda)$ described in the following lemma.

Lemma 5.2

$$\tilde{\mathbf{L}}_{x,T} = \mathbf{L}_{x,T} + \delta\mathbf{L}_{x,T} \tag{5.13}$$

with

$$\begin{aligned} -\delta\mathbf{L}_{x,T}(f) \equiv i\int_{-T}^{T} dr \int_0^\beta d\beta' \Big(&(1-\lambda)\left(\mathbf{K}_\lambda(T-r-i(\beta-\beta')(1-\lambda))\nabla_{\alpha_r(x)}(f)\alpha_{T+i\beta'(1-\lambda)}(x)\right. \\ &\left. -\mathbf{K}_\lambda(-T-r-i(\beta-\beta')(1-\lambda))\nabla_{\alpha_r(x)}(f)\alpha_{-T+i\beta'(1-\lambda)}(x)\right) \\ &+\lambda\left(\mathbf{K}_\lambda(T-r+i(\beta-\beta')\lambda)\alpha_{T-i\beta'\lambda}(x)\nabla_{\alpha_r(x)}(f) - \mathbf{K}_\lambda(-T-r+i(\beta-\beta')\lambda)\alpha_{-T-i\beta'\lambda}(x)\nabla_{\alpha_r(x)}(f)\right)\Big) \end{aligned} \tag{5.14}$$

$\circ$

Proof of Lemma 5.2: First of all, using the formula (5.1) together with the (α_t,β) **-KMS** condition, we note that

$$< \nabla_{\alpha_r(x)}f, \nabla_{\alpha_s(x)}g >_{\omega,\lambda} = < f, i\left(\nabla_{\alpha_s(x)}(g)\alpha_{r+i\beta(1-\lambda)}(x) - \alpha_{r-i\beta\lambda}(x)\nabla_{\alpha_s(x)}(g)\right) >_{\omega,\lambda} \tag{5.15}$$

Thus we have

$$\mathcal{E}_{x,T}(f,g) = < f, i\int_{-T}^{T}\int_{-T}^{T} dr\, ds\, \left(\mathbf{K}_\lambda(s-r)\nabla_{\alpha_r(x)}(g)\alpha_{s+i\beta(1-\lambda)}(x) - \mathbf{K}_\lambda(r-s)\alpha_{r-i\beta\lambda}(x)\nabla_{\alpha_s(x)}(g)\right) >_{\omega,\lambda} \tag{5.16}$$

To discuss the first and the second term under the double integral, we consider the following analytic functions in a strip containing $Im\, z \in [-i\beta(1-\lambda), i\beta\lambda]$

$$z \longrightarrow \mathbf{K}_\lambda(z - i\beta(1-\lambda) - r) < f, \nabla_{\alpha_r(x)}(g)\alpha_z(x) >_{\omega,\lambda} \tag{5.17}$$

and

$$z \longrightarrow \mathbf{K}_\lambda(z + i\beta\lambda - s) < f, \alpha_z(x)\nabla_{\alpha_s(x)}(g) >_{\omega,\lambda} \tag{5.18}$$

respectively. Then by Cauchy integral theorem, we obtain for the first term

$$\begin{aligned} \int_{-T}^{T} ds\mathbf{K}_\lambda(s-r) < f, \nabla_{\alpha_r(x)}(g)\alpha_{s+i\beta(1-\lambda)}(x) >_{\omega,\lambda} = \int_{-T}^{T} ds\mathbf{K}_\lambda(s-r-i\beta(1-\lambda)) < f, \nabla_{\alpha_r(x)}(g)\alpha_s(x) >_{\omega,\lambda} + \\ + \int_0^\beta d\beta'(1-\lambda)\Big(\mathbf{K}_\lambda(T-r-i(\beta-\beta')(1-\lambda)) < f, \nabla_{\alpha_r(x)}(g)\alpha_{T+i\beta'(1-\lambda)}(x) >_{\omega,\lambda} \\ - \mathbf{K}_\lambda(-T-r-i(\beta-\beta')(1-\lambda)) < f, \nabla_{\alpha_r(x)}(g)\alpha_{-T+i\beta'(1-\lambda)}(x) >_{\omega,\lambda}\Big) \end{aligned} \tag{5.19}$$

and for the second

$$\begin{aligned} \int_{-T}^{T} dr\mathbf{K}_\lambda(r-s) < f, \alpha_{r-i\beta\lambda}(x)\nabla_{\alpha_s(x)}(g) >_{\omega,\lambda} = \int_{-T}^{T} dr\mathbf{K}_\lambda(r-s+i\beta\lambda) < f, \alpha_r(x)\nabla_{\alpha_s(x)}(g) >_{\omega,\lambda} + \\ + \int_0^\beta d\beta'\lambda\Big(\mathbf{K}_\lambda(T-s+i(\beta-\beta')\lambda) < f, \alpha_{T-i\beta'\lambda}(x)\nabla_{\alpha_s(x)}(g) >_{\omega,\lambda} \ + \end{aligned} \tag{5.20}$$

$$- \mathbf{K}_\lambda(-T-s+i(\beta-\beta')\lambda) < f, \alpha_{-T-i\beta'\lambda}(x)\nabla_{\alpha_s(x)}(g) >_{\omega,\lambda} \Big)$$

Now applying the condition (5.3) to the $\mathbf{K}_\lambda$ in the first term on the right hand side of (5.19) together with (5.16) and setting $\mathcal{K}_\lambda(r-s) \equiv \mathbf{K}_\lambda(r-s+i\beta\lambda)$, we arrive at the following equality

$$\begin{aligned}
\mathcal{E}_{x,T}(f,g) = &< f, i\int_{-T}^{T}\int_{-T}^{T} dr\,ds\,\mathcal{K}_\lambda(r-s)\Big(\nabla_{\alpha_r(x)}(g)\alpha_s(x) - \alpha_r(x)\nabla_{\alpha_s(x)}(g)\Big) >_{\omega,\lambda} + \\
&+ < f, i\int_{-T}^{T} dr \int_0^\beta d\beta'(1-\lambda)\Big(\mathbf{K}_\lambda(T-r-i(\beta-\beta')(1-\lambda))\nabla_{\alpha_r(x)}(g)\alpha_{T+i\beta'(1-\lambda)}(x) \\
&\qquad - \mathbf{K}_\lambda(-T-r-i(\beta-\beta')(1-\lambda))\nabla_{\alpha_r(x)}(g)\alpha_{-T+i\beta'(1-\lambda)}(x)\Big) >_{\omega,\lambda} \\
&+ < f, i\int_{-T}^{T} ds \int_0^\beta d\beta'\lambda\Big(\mathbf{K}_\lambda(T-s+i(\beta-\beta')\lambda)\alpha_{T-i\beta'\lambda}(x)\nabla_{\alpha_s(x)}(g) \\
&\qquad - \mathbf{K}_\lambda(-T-s+i(\beta-\beta')\lambda)\alpha_{-T-i\beta'\lambda}(x)\nabla_{\alpha_s(x)}(g)\Big) >_{\omega,\lambda} \\
&\equiv < f, -\tilde{\mathbf{L}}_{x,T}(g) >_{\omega,\lambda} = < f, (-\mathbf{L}_{x,T} - \delta\mathbf{L}_{x,T})(g) >_{\omega,\lambda}
\end{aligned} \tag{5.21}$$

This ends the proof of Lemma 5.2.

$\diamondsuit$

The next useful fact is the following lemma.

Lemma 5.3

If

$$\sup_{\beta'\in[-\beta,\beta]} \int_{-\infty}^{\infty} dr |\mathbf{K}_\lambda(r+i\beta')| < \infty \tag{5.22}$$

and

$$\sup_{\beta'\in[-\beta,\beta]} ||\alpha_{i\beta'}(x)|| \le C_1 < \infty \tag{5.23}$$

then for any $f\in\tilde{\mathcal{A}}$ we have

$$\lim_{T\to\infty} ||\delta\mathbf{L}_{x,T}(f)|| = 0 \tag{5.24}$$

Moreover, if

$$\sup_{0\le\beta'\le\beta} \Big\{ ||\alpha_{i(\lambda\beta/2+\beta')}(x)||, ||\alpha_{i(\lambda\beta/2+\beta')}(x)\alpha_{i((1-\lambda/2)\beta-\beta')}(x)||^{\frac{1}{2}} \Big\} \le C_2 < \infty \tag{5.25}$$

then

$$\sup_T ||\delta\mathbf{L}_{x,T}||_{\mathbb{L}_2\to\mathbb{L}_2} \le C < \infty \tag{5.26}$$

$\circ$

Proof of Lemma 5.3: Suppose $f\in\tilde{\mathcal{A}}$. Then we have

$$\begin{aligned}
||\delta\mathbf{L}_{x,T}(f)|| \le \int_{-\infty}^{\infty} dr \int_0^\beta d\beta' \Big(&(1-\lambda)\left(|\mathbf{K}_\lambda(T-r-i\beta'(1-\lambda))| + |\mathbf{K}_\lambda(-T-r-i\beta'(1-\lambda))|\right) + \\
&+ \lambda\left(|\mathbf{K}_\lambda(T-r+i\beta'\lambda)| + |\mathbf{K}_\lambda(-T-r+i\beta'\lambda)|\right)\Big) \sup_{\beta''\in[-\beta,\beta]} (||\alpha_{i\beta''}(x)||)\cdot||\nabla_{\alpha_r(x)}(f)||
\end{aligned} \tag{5.27}$$

Since by our assumption the last factor on the right hand side of (5.27) is integrable, the conditions (5.22) and (5.23) together with (5.27) imply

$$\lim_{T\to\infty} ||\delta \mathbf{L}_{x,T}(f)|| = 0 \tag{5.28}$$

(From (5.27) one can also see that in fact $\sup_T ||\delta \mathbf{L}_{x,T}||_{\mathcal{A}\to\mathcal{A}} < \infty$.) If the condition (5.25) is satisfied, then the right,as well as the left, multiplication by $\alpha_{t+i\beta'}(x)$, for any $t \in I\!R$, $\beta' \in [-\beta,\beta]$ is a bounded operator in $I\!L_2(\omega,\lambda)$ with a norm not exceeding the left hand side of (5.25); see Lemma AIII.1 in Appendix III. Therefore we have

$$| < g, \delta \mathbf{L}_{x,T}(f) >_{\omega,\lambda} | \leq 2C_2^2 \int_{-\infty}^{\infty} dr \int_0^{\beta} d\beta' \Big((1-\lambda)|\mathbf{K}_\lambda(r-i\beta'(1-\lambda))| + \lambda|\mathbf{K}_\lambda(r+i\beta'\lambda)| \Big) ||g||_{I\!L_2(\omega,\lambda)} ||f||_{I\!L_2(\omega,\lambda)} \tag{5.29}$$

Hence (5.26) follows. This ends the proof of Lemma 5.3.

$\diamondsuit$

Proof of Theorem 5.1: Since for $f \in \tilde{A}$, we have

$$||\mathbf{L}_x f|| \leq 2||\mathcal{K}_\lambda||_{I\!L_1(I\!R)} ||x|| \int_{-\infty}^{\infty} ds ||\nabla_{\alpha_s(x)} f|| \tag{5.30}$$

so, if $\mathbf{SAA}_{\ 1}$ holds, the right hand side is finite, i.e. $\mathbf{L}_x$ is well defined on the dense domain $\mathcal{D}_0 = \tilde{\mathcal{A}}$. Using Lemma 5.2 and (5.24) from Lemma 5.3, one can easily see that

$$\mathcal{E}_x(f,g) = < f, -\mathbf{L}_x g >_\omega \tag{5.31}$$

This ends the proof of the first part of the theorem. The $*$-invariance condition follows from our assumption that $\hat{\mathbf{K}}_\lambda \geq 0$ (see (5.5)), which implies that $\hat{\mathcal{K}}_\lambda \geq 0$, and therefore we also have $\mathcal{K}_\lambda(r-s) = \mathcal{K}_\lambda^*(s-r)$. To prove the dissipativity let us first note that for any $f,g \in \mathcal{D}_0$ also $fg \in \mathcal{D}_0$. Then by direct calculations with $f,g \in \mathcal{D}_0$, we get

$$\mathbf{L}_x(f^*g) = f^*\mathbf{L}_x(g) + \mathbf{L}_x(f^*)g + 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} drds\, \mathcal{K}_\lambda(r-s) \nabla_{\alpha_r(x)}(f^*) \cdot \nabla_{\alpha_s(x)}(g) \tag{5.32}$$

whence, using the fact that $\mathcal{K}_\lambda$ is positive definite, we obtain

$$\mathbf{\Gamma}_x(f,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} drds\, \mathcal{K}_\lambda(r-s) \nabla_{\alpha_r(x)}(f^*) \cdot \nabla_{\alpha_s(x)}(f) \geq 0 \tag{5.33}$$

This ends the proof of Theorem 5.1.

$\diamondsuit$

Remark: *Let us note that the square of gradient form $\mathbf{\Gamma}_x$ is well defined under weaker* **AA** *condition than the one assumed in Theorem 5.1.*

It follows from Theorem 5.1 that $\mathbf{L}_x$ is a densely defined, symmetric and nonnegative operator in $I\!L_2$, i.e. a pre - generator of a completely positive Markov semigroup. Its closure in $I\!L_2$, which will be denoted later on by the same symbol $\mathbf{L}_x$, can be used to define a semigroup for which the (α_t,β)-**KMS** state is invariant. It will be made clear later that the corresponding semigroup is indeed a Markov semigroup. One can expect however that such semigroup would have rather poor ergodic properties. Therefore one would like to consider a translation invariant generator defined as a sum of all elementary generators. We define it as follows. Let $x^a \in \mathbf{M}_0$, $a = 1,..,D$, be a base consisting of selfadjoint elements of norm one and let

$\mathbf{x_j} \equiv \{x_{\mathbf{j}}^a = \tau_{\mathbf{j}}(x^a)\}_{a=1,..,D}$. We introduce a gradient at a point $\mathbf{j} \in \mathbb{Z}^d$ by $\nabla_{s,\mathbf{j}} f \equiv \left(\nabla_{\alpha_s(\tau_{\mathbf{j}} x^a)} f\right)_{a=1,..,D}$. With this notation we define an elementary generator $\mathbf{L_j}$ as follows

$$\mathbf{L_j}(f) \equiv \lim_{T\to\infty} \mathbf{L}_{\mathbf{j},T}(f) \tag{5.34}$$

with

$$\mathbf{L}_{\mathbf{j},T}(f) \equiv -\int_{-T}^{T}\int_{-T}^{T} drds\, \mathcal{K}_\lambda(r-s) i\Big(\nabla_{r,\mathbf{j}}(f)\cdot\alpha_s(\mathbf{x_j}) - \alpha_r(\mathbf{x_j})\cdot\nabla_{s,\mathbf{j}}(f)\Big) \tag{5.35}$$

where the dot means a scalar product of finite component vectors. As follows from the definition and Theorem 5.1, $\mathbf{L_j}$ is a selfadjoint nonpositive operator with a domain $\mathcal{D} \supseteq \tilde{\mathcal{A}}$. For $\Lambda \in \mathcal{F}$ we define a finite volume generator $\mathbf{L}_\Lambda$ as follows

$$\mathbf{L}_\Lambda \equiv \sum_{\mathbf{j}\in\Lambda} \mathbf{L_j} \tag{5.36}$$

with a dense domain $\mathcal{D}(\mathbf{L}_\Lambda) \subset \mathbb{L}_2(\omega)$ which is the closure of $\tilde{\mathcal{A}}$ in the corresponding graph norm. By the construction it has the following properties.

Theorem 5.4
Suppose the conditions of Lemma 5.3 are satisfied for $x = x_{\mathbf{j}}^a$, $\mathbf{j} \in \Lambda$, $a = 1,..,D$. Then the nonnegative selfadjoint operator $(\mathbf{L}_\Lambda, \mathcal{D}(\mathbf{L}_\Lambda))$ in $\mathbb{L}_2(\omega)$ is the generator of a finite volume Markov semigroup $P_t^\Lambda \equiv e^{t\mathbf{L}_\Lambda}$.
$\circ$

Proof: The operator $\mathbf{L}_\Lambda$ is defined as a finite sum of nonnegative selfadjoint operators $\mathbf{L_j}$ with a common essential domain $\mathcal{D}_0 \equiv \tilde{\mathcal{A}}$. Therefore it inherits the corresponding properties. We need only to show that it generates a Markov semigroup. For this, let us note that on its essential domain $\mathcal{D}_0$ we have

$$\mathbf{L}_\Lambda f = \lim_{T\to\infty} \mathbf{L}_{\Lambda,T} f, \qquad f \in \mathcal{D}_0 \tag{5.37}$$

where $\mathbf{L}_{\Lambda,T}$ is defined as a corresponding sum of bounded generators $\mathbf{L}_{\mathbf{j},T}$, $\mathbf{j} \in \Lambda$ given in (5.35). Clearly $\mathbf{L}_{\Lambda,T}$ is bounded on $\mathcal{A}$. Therefore it can be used to define a Markov semigroup $P_t^{\Lambda,T} \equiv e^{t\mathbf{L}_{\Lambda,T}}$ on $\mathcal{A}$. Let $\tilde{\mathbf{L}}_{\Lambda,T}$ be a selfadjoint nonnegative operator in $\mathbb{L}_2$ defined by the quadratic form

$$\mathcal{E}_{\Lambda,T}(f,g) \equiv \sum_{\mathbf{j}\in\Lambda} \mathcal{E}_{\mathbf{j},T}(f,g) \tag{5.38}$$

with

$$\mathcal{E}_{\mathbf{j},T}(f,g) \equiv \sum_{a=1,..,D} \mathcal{E}_{x_{\mathbf{j}}^a,T}(f,g) \tag{5.39}$$

Under the conditions of Lemma 5.3 it is now easy to see that the operator

$$\delta\mathbf{L}_{\Lambda,T} \equiv \tilde{\mathbf{L}}_{\Lambda,T} - \mathbf{L}_{\Lambda,T} \tag{5.40}$$

satisfies

$$\sup_{T\in(0,\infty)} ||\delta\mathbf{L}_{\Lambda,T}||_{\mathbb{L}_2\to\mathbb{L}_2} \le C|\Lambda| \tag{5.41}$$

with some positive constant C independent of Λ. Using this and observing that $P_t^{\Lambda,T} \equiv e^{t\mathbf{L}_{\Lambda,T}} = e^{t(\tilde{\mathbf{L}}_{\Lambda,T} - \delta\mathbf{L}_{\Lambda,T})}$, by an appropriate Duhamel expansion in $\mathbb{L}_2$, we arrive at the following stability estimate

$$||P_t^{\Lambda,T}||_{\mathbb{L}_2\to\mathbb{L}_2} \le e^{tC|\Lambda|} \tag{5.42}$$

This together with (5.37) implies (via Theorem 7.2, p. 44 in [**Go**]) that

$$P_t^\Lambda f = \mathbb{L}_2 - \lim_{T\to\infty} P_t^{\Lambda,T} f \tag{5.43}$$

for any $f \in \mathbb{L}_2$ and since every $P_t^{\Lambda,T}$ is positivity and unit preserving, so must be P_t^Λ. This ends the proof of Theorem 5.4.

$\diamondsuit$

Remark: *Since our proof relies on some $\mathbb{L}_2$ procedures, it does not tell us whether the semigroup P_t^Λ, $t \geq 0$, has a Feller property, although the approximating semigroups $P_t^{\Lambda,T}$ clearly have it.*

A global generator $\mathbf{L}$ is formally defined as follows

$$\mathbf{L} \equiv \sum_{\mathbf{j} \in \mathbb{Z}^d} \mathbf{L_j} \tag{5.44}$$

To give a rigorous meaning to this definition we will need to impose the following additional restriction called <u>Hyper Asymptotic Abelianess</u>:

Suppose $\tilde{\mathcal{A}} = \mathcal{A}_0$ and we have

(HAA) $$|| \, |\nabla_{s,\mathbf{j}}(f)| \, ||_{\mathbb{L}_2(\omega)} < \psi(s,f)$$

with some positive function $\psi(s,f)$ such that

$$\psi(s,f) \leq C(f)(1+|s|)^{-(d+1+\varepsilon)/2} \tag{5.45}$$

with some positive constants $C(f)$ and ε possibly dependent on the function f. $\circ$

The following result shows that definition of the global generator can make sense.

Theroem 5.5:

Suppose that the following <u>Finite Speed of Propagation</u> property for automorphism group α_s is true for any $f \in \mathcal{A}_\Lambda$, $\Lambda \in \mathcal{F}$,

$$|| \, |\nabla_{s,\mathbf{j}} f| \, || \leq D(f) e^{-\kappa(d(\mathbf{j},\Lambda) - vs)} \tag{5.46}$$

with some positive constants $D(f)$, κ and v possibly dependent on $f \in \mathcal{A}_0$. Then the global generator $\mathbf{L}$ is a well defined selfadjoint operator in $\mathbb{L}_2(\omega)$ with a dense domain $\mathcal{D} \supseteq \mathcal{A}_0$, provided the condition **HAA** *is satisfied. Moreover the corresponding semigroup $P_t \equiv e^{t\mathbf{L}}$ is Markov.* $\circ$

Remark: *The finite speed of propagation of information (5.46) for automorphism semigroups of quantum spin systems on a lattice has been proven long time ago in* [**LR**].

Proof: Let us consider the increasing sequence of nonnegative, symmetric and closed quadratic forms $\mathcal{E}_\Lambda(\cdot,\cdot)$, $\Lambda \in \mathcal{F}$, with a common dense essential domain $\tilde{\mathcal{A}}$. By general arguments, see e.g. [**Ka**] Theorem 3.13, p. 461, the quadratic form

$$\mathcal{E}(\cdot,\cdot) \equiv \lim_{\mathcal{F}_0} \mathcal{E}_\Lambda(\cdot,\cdot) \tag{5.47}$$

if well defined on a dense domain, is also closed, symmetric and nonnegative quadratic form. Thus in this case it defines a selfadjoint operator, denoted later on by $-\mathbf{L}$. Moreover we have by general arguments ([**Ka**] Theorem 3.13, p. 461), that the resolvent $\mathbf{R}(\lambda)$ of the operator $\mathbf{L}$ satisfies

$$\mathbf{R}(\lambda) = \lim_{\mathcal{F}_0} (\lambda - \mathbf{L}_\Lambda)^{-1} \tag{5.48}$$

Since by Theorem 5.4 the finite volume resolvents on the right hand side of (5.48) are positive for $\lambda \in \mathbb{R}^+$, so is $\mathbf{R}(\lambda)$. This implies that $\mathbf{L}$ is a Markov generator. Now to finish the proof it suffices to show that the quadratic form $\mathcal{E}(\cdot,\cdot)$ is well defined on $\tilde{\mathcal{A}}$. To do this we note that

$$0 \leq \mathcal{E}(f,f) \leq \sum_{\mathbf{j} \in \mathbb{Z}^d} \sum_{a=1,..,D} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dr ds \mathcal{K}_\lambda(r-s) ||\nabla_{\alpha_r \tau_{\mathbf{j}}(x^a)}(f)||_{\mathbb{L}_2} \cdot ||\nabla_{\alpha_s \tau_{\mathbf{j}}(x^a)}(f)||_{\mathbb{L}_2} \leq$$

$$\leq ||\hat{\mathcal{K}}_\lambda||_\infty \sum_{\mathbf{j}\in\mathbb{Z}^d} \sum_{a=1,..,D} \int_{-\infty}^{\infty} ds ||\nabla_{\alpha_s\tau_{\mathbf{j}}(x^a)}(f)||^2_{\mathbb{L}_2} \tag{5.49}$$

(where we have used property of the Fourier transform of a convolution and the Parseval's equality). Thus it is sufficient to show that for arbitrary $\Lambda_0 \in \mathcal{F}$ and every $f \in \mathcal{A}_{\Lambda_0}$, we have

$$\sum_{\mathbf{j}\in\mathbb{Z}^d} \int_{-\infty}^{\infty} ds ||\nabla_{\alpha_s(\tau_{\mathbf{j}}(x))} f||^2_{\mathbb{L}_2} < \infty \tag{5.50}$$

for any x in the chosen base of $\mathbf{M}_0$. To do that, first we represent the sum on the left hand side of (5.50) as follows

$$\sum_{N\in\mathbb{N}} \sum_{N-1\leq d(\mathbf{j},\Lambda_0)<N} \int_{-\infty}^{\infty} ds ||\nabla_{\alpha_s(\tau_{\mathbf{j}}(x))} f||^2_{\mathbb{L}_2} = \tag{5.51}$$

$$= \sum_{N\in\mathbb{N}} \sum_{N-1\leq d(\mathbf{j},\Lambda_0)<N} \left(\int_{|s|\leq\frac{N}{2v}} ds ||\nabla_{\alpha_s(\tau_{\mathbf{j}}(x))} f||^2_{\mathbb{L}_2} + \int_{|s|>\frac{N}{2v}} ds ||\nabla_{\alpha_s(\tau_{\mathbf{j}}(x))} f||^2_{\mathbb{L}_2} \right)$$

Now, using the finite speed of propagation (5.46), we get the following bound on the first part of the sum on the right hand side of (5.51).

$$\sum_{N\in\mathbb{N}} \sum_{N-1\leq d(\mathbf{j},\Lambda_0)<N} \int_{|s|\leq\frac{N}{2v}} ds ||\nabla_{\alpha_s(\tau_{\mathbf{j}}(x))} f||^2_{\mathbb{L}_2} \leq \sum_{N\in\mathbb{N}} \sum_{N-1\leq d(\mathbf{j},\Lambda_0)<N} \frac{N}{v} D(f) e^{-\kappa N} < \infty \tag{5.52}$$

To obtain an estimate on the second part of the sum on the right hand side of (5.51) we make use of our **HAA** assumption. We get

$$\sum_{N\in\mathbb{N}} \sum_{N-1\leq d(\mathbf{j},\Lambda_0)<N} \int_{|s|>\frac{N}{2v}} ds ||\nabla_{\alpha_s(\tau_{\mathbf{j}}(x))} f||^2_{\mathbb{L}_2} \leq \sum_{N\in\mathbb{N}} \sum_{N-1\leq d(\mathbf{j},\Lambda_0)<N} \int_{|s|>\frac{N}{2v}} ds\, \psi(s,f)^2 \leq$$

$$\leq \sum_{N\in\mathbb{N}} \sum_{N-1\leq d(\mathbf{j},\Lambda_0)<N} \int_{|s|>\frac{N}{2v}} ds\, C(f)(1+|s|)^{-(d+1+\varepsilon)} \leq \sum_{N\in\mathbb{N}} \sum_{N-1\leq d(\mathbf{j},\Lambda_0)<N} C_1 (N+1)^{-(d+\varepsilon)} \leq$$

$$\leq \sum_{N\in\mathbb{N}} C_2 N^{d-1} (N+1)^{-(d+\varepsilon)} < \infty \tag{5.53}$$

with some positive constants C_1 and C_2 dependent on f. Combining (5.52) and (5.53), we obtain the desired estimate (5.50). This ends the proof of Theorem 5.5

$\diamondsuit$

It does not follow from our construction whether the infinite volume Markov semigroup $P_t \equiv e^{t\mathbf{L}}$ can have a Feller property. This would be desirable in order to have a more interesting ergodicity theory. Therefore it would be useful to find some general conditions under which one could construct a Feller semigroup, i.e. a Markov semigroup mapping the algebra $\mathcal{A}$ into itself. One could have a hope that such result is possible if one would impose the following Ultrastrong Asymptotic Abelianess condition:
There are positive constants C and ε such that

$$\textbf{(UAA)} \qquad ||\nabla_{\alpha_s(x)} f|| \leq C(1+|s|)^{-d-1-\varepsilon}$$

for any $f \in \mathcal{A}_0$ and $x \in \mathbf{M}$.

Then of course one could mimic our arguments to show that the operator $\mathbf{L}$ from (5.44) is defined on the dense domain $\mathcal{A}_0$, which is in this case mapped into $\mathcal{A}$. Unfortunately such the appealing direction is wrong.

This is because already the modified **UAA** condition with the decay $(1+|s|)^{-B}$, with arbitrary $B>d$, implies for any $x \in \mathcal{A}_0$ the following estimate

$$|||\alpha_s(x)||| \leq C'(1+|s|)^{-(B-d)} \tag{5.54}$$

provided the Finite Speed of Propagation property is true; here the triple bar norm $|||\cdot|||$ is the same as in Section 4 (because of Lemma IV.1 in the Appendix IV). But if (5.54) holds, then for any two α_s-invariant states ω and $\tilde{\omega}$, and any selfajoint $f \in \mathcal{A}_0$ we get

$$|\omega(f)-\tilde{\omega}(f)| = |\omega(\alpha_s f)-\tilde{\omega}(\alpha_s f)| = |\omega\otimes\tilde{\omega}\,(\alpha_s f\otimes \mathbf{1} - \mathbf{1}\otimes\alpha_s f)| \leq 2|||\alpha_s(f)||| \longrightarrow 0 \tag{5.55}$$

when $s\to\infty$. This implies that there could be only one (α_t,β)-**KMS** state for all temperatures. Clearly such a situation is not very exciting and we should not follow in this direction. Let us note that actually this excludes also the possibility of introducing a strong version of **HAA** , with $\mathbb{L}_2$ norm replaced by the algebra norm, in case when $d=1$. We do not know at the moment whether or not the weak asymptotic abelianess with $\tilde{\mathcal{A}} = \mathcal{A}_0$ can hold with a faster decay than the strong one. It may be so that in one dimensional systems one can realize only a spin flip stochastic dynamics considered in previous sections.

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Appendix I:
In this Appendix we give a simple proof of the inequality

$$||f||_{\mathbb{L}_p(\omega_{\Lambda_0})} \leq ||f|| \tag{AI.0}$$

For this we will need the following lemma in which we set $A \equiv |\rho^{\frac{1}{2q}} f \rho^{\frac{1}{2q}}|$

Lemma AI.1
For any $k\in\mathbb{N}$, and $q \geq 2^k+1$ we have

$$||f||^{\frac{q+2^k-1}{2^k}}_{\mathbb{L}_q(\omega_{\Lambda_0})} \leq ||f||^{1+\frac{1}{2}+..+\frac{1}{2^{k-1}}} \left(\mathbf{Tr} A^{(q-1-2^k)}\rho^{\frac{1}{2q}} f^* \rho^{\frac{2^k}{q}} f \rho^{\frac{1}{2q}}\right)^{\frac{1}{2^k}} \tag{AI.1}$$

$\circ$

Proof: Let us consider first the case $k=1$. We have

$$||f||^q_{\mathbb{L}_q(\omega_{\Lambda_0})} = \mathbf{Tr} A^q = \mathbf{Tr}\left(A^{q-2}\rho^{\frac{1}{2q}} f^* \rho^{\frac{1}{q}} f\rho^{\frac{1}{2q}}\right) = \mathbf{Tr}\left(\left(A^{\frac{q-2}{2}-\frac{1}{2}}\rho^{\frac{1}{2q}} f^*\rho^{\frac{1}{q}}\right)\left(f\rho^{\frac{1}{2q}}A^{\frac{q-2}{2}+\frac{1}{2}}\right)\right) \tag{AI.2}$$

Applying to the right hand side of (AI.2) the Hölder inequality we get

$$||f||^q_{\mathbb{L}_q(\omega_{\Lambda_0})} \leq \left(\mathbf{Tr}\left(A^{\frac{q-2}{2}-\frac{1}{2}}\rho^{\frac{1}{2q}} f^* \rho^{\frac{2}{q}} f \rho^{\frac{1}{2q}} A^{\frac{q-2}{2}-\frac{1}{2}}\right)\right)^{\frac{1}{2}} \left(\mathbf{Tr}\left(A^{\frac{q-2}{2}+\frac{1}{2}}\rho^{\frac{1}{2q}} f^* f \rho^{\frac{1}{2q}} A^{\frac{q-2}{2}+\frac{1}{2}}\right)\right)^{\frac{1}{2}} =$$

$$\left(\mathbf{Tr}\left(A^{q-3}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2}{q}}f\rho^{\frac{1}{2q}}\right)\right)^{\frac{1}{2}}\left(\mathbf{Tr}\left(A^{\frac{q-2}{2}+\frac{1}{2}}\rho^{\frac{1}{2q}}f^*f\rho^{\frac{1}{2q}}A^{\frac{q-2}{2}+\frac{1}{2}}\right)\right)^{\frac{1}{2}} \qquad (AI.3)$$

The second factor on the right hand side of (AI.3) can be estimated as follows

$$\left(\mathbf{Tr}\left(A^{\frac{q-2}{2}+\frac{1}{2}}\rho^{\frac{1}{2q}}f^*f\rho^{\frac{1}{2q}}A^{\frac{q-2}{2}+\frac{1}{2}}\right)\right)^{\frac{1}{2}} \le ||f||\left(\mathbf{Tr}\left(A^{q-1}\rho^{\frac{1}{q}}\right)\right)^{\frac{1}{2}} \qquad (AI.4)$$

Since by our assumption $\mathbf{Tr}\rho=1$, by Hölder inequality for the trace, we estimate the second factor from the right hand side of (AI.4) as follows

$$\left(\mathbf{Tr}\left(A^{q-1}\rho^{\frac{1}{q}}\right)\right)^{\frac{1}{2}} \le \left(\mathbf{Tr}A^q\right)^{\frac{q-1}{2q}} \equiv ||f||^{\frac{q-1}{2}}_{\mathbb{L}_q(\omega_{\Lambda_0})} \qquad (AI.5)$$

From (AI.4)-(AI.5) we get

$$\left(\mathbf{Tr}\left(A^{\frac{q-2}{2}+\frac{1}{2}}\rho^{\frac{1}{2q}}f^*f\rho^{\frac{1}{2q}}A^{\frac{q-2}{2}+\frac{1}{2}}\right)\right)^{\frac{1}{2}} \le ||f||\cdot||f||^{\frac{q-1}{2}}_{\mathbb{L}_q(\omega_{\Lambda_0})} \qquad (AI.6)$$

Using this together with (AI.2)-(AI.3) we get

$$||f||^{\frac{q+1}{2}}_{\mathbb{L}_q(\omega_{\Lambda_0})} \le ||f||^1\left(\mathbf{Tr}A^{q-1-2^1}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2}{q}}f\rho^{\frac{1}{2q}}\right)^{\frac{1}{2}} \qquad (AI.7)$$

This ends the proof of the case $k=1$. Let us suppose now that (AI.1) is true for some $k-1\in\mathbb{N}$ such that $2^{k-1}+1<q$. We will show that it has to be true also for k. For this we note that the 2^{k-1} power of the second factor from the right hand side of (AI.1) with $k-1$ can be represented as follows

$$\mathbf{Tr}\left(A^{q-1-2^{k-1}}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2^{k-1}}{q}}f\rho^{\frac{1}{2q}}\right)=\mathbf{Tr}\left(\left(A^{\frac{q-1-2^{k-1}}{2}-\frac{2^{k-1}}{2}}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2^{k-1}}{q}}\right)\left(f\rho^{\frac{1}{2q}}A^{\frac{q-1-2^{k-1}}{2}+\frac{2^{k-1}}{2}}\right)\right) \qquad (AI.8)$$

Applying to the right hand side of (AI.8) the Hölder inequality we get

$$\mathbf{Tr}\left(A^{q-1-2^{k-1}}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2^{k-1}}{q}}f\rho^{\frac{1}{2q}}\right)\le \qquad (AI.9)$$

$$\le\left(\mathbf{Tr}\left(A^{q-1-2^k}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2^k}{q}}f\rho^{\frac{1}{2q}}\right)\right)^{\frac{1}{2}}\left(\mathbf{Tr}\left(A^{\frac{q-1-2^{k-1}}{2}+\frac{2^{k-1}}{2}}\rho^{\frac{1}{2q}}f^*f\rho^{\frac{1}{2q}}A^{\frac{q-1-2^{k-1}}{2}+\frac{2^{k-1}}{2}}\right)\right)^{\frac{1}{2}}$$

The first factor has the correct form. The second can be estimated, by similar arguments as in the case $k=1$, as follows

$$\left(\mathbf{Tr}\left(A^{\frac{q-1-2^{k-1}}{2}+\frac{2^{k-1}}{2}}\rho^{\frac{1}{2q}}f^*f\rho^{\frac{1}{2q}}A^{\frac{q-1-2^{k-1}}{2}+\frac{2^{k-1}}{2}}\right)\right)^{\frac{1}{2}}\le||f||\left(\mathbf{Tr}\left(A^{q-1}\rho^{\frac{1}{q}}\right)\right)^{\frac{1}{2}}\le||f||\cdot||f||^{\frac{q-1}{2}}_{\mathbb{L}(\omega_{\Lambda_0})} \qquad (AI.10)$$

Using the above considerations (AI.8)-(AI.10), we obtain the following bound

$$\left(\mathbf{Tr}\left(A^{q-1-2^{k-1}}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2^{k-1}}{q}}f\rho^{\frac{1}{2q}}\right)\right)^{\frac{1}{2^{k-1}}}\le||f||^{\frac{1}{2^{k-1}}}\cdot||f||^{\frac{q-1}{2^k}}_{\mathbb{L}(\omega_{\Lambda_0})}\left(\mathbf{Tr}\left(A^{q-1-2^k}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2^k}{q}}f\rho^{\frac{1}{2q}}\right)\right)^{\frac{1}{2^k}} \qquad (AI.11)$$

From this and (AI.1) for the case $k-1$, we get

$$||f||^{\frac{q+2^{k-1}-1}{2^{k-1}}}_{\mathbb{L}_q(\omega_{\Lambda_0})}\le \qquad (AI.12)$$

$$\le||f||^{1+\frac{1}{2}+..+\frac{1}{2^{k-1}}+\frac{1}{2^k}}\left(\mathbf{Tr}A^{q-1-2^k}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2^k}{q}}f\rho^{\frac{1}{2q}}\right)^{\frac{1}{2^k}}\cdot||f||^{\frac{q-1}{2^k}}_{\mathbb{L}(\omega_{\Lambda_0})}\left(\mathbf{Tr}\left(A^{q-1-2^k}\rho^{\frac{1}{2q}}f^*\rho^{\frac{2^k}{q}}f\rho^{\frac{1}{2q}}\right)\right)^{\frac{1}{2^k}}$$

Hence the case k follows. This ends the proof of the lemma.

$\diamondsuit$

In particular if $q = 2^k + 1$, Lemma A.1 gives us the following estimate

$$||f||^2_{\mathbb{L}_q(\omega_{\Lambda_0})} \leq ||f||^{2-\frac{1}{2^{k-1}}} \left(\mathbf{Tr} \left(\rho^{\frac{1}{2q}} f^* \rho^{\frac{2^k}{q}} f \rho^{\frac{1}{2q}} \right) \right)^{\frac{1}{2^k}} \tag{AI.13}$$

Now we note that the following lemma is true.

Lemma AI.2: For any $s \in [0,1]$ we have

$$\left(\mathbf{Tr} \left(\rho^s f^* \rho^{1-s} f \right) \right)^{\frac{1}{2}} \leq \left(\mathbf{Tr} \rho f^* f \right)^{\frac{s}{2}} \left(\mathbf{Tr} \rho f f^* \right)^{\frac{1-s}{2}} \tag{AI.14}$$

$\circ$

Use of Lemma AI.2 for $s = \frac{1}{q}$ together with (AI.13) ends the proof of the desired inequality (AI.0) for $p = 2^k + 1$. The general case now follows from the Hölder inequality on the left hand side of (2.32).

Proof of Lemma AI.2: The shortest proof one gets by applying the three lines theorem to the bounded analytic in the strip $Re\, z \in [0,1]$ function

$$\mathbf{Tr} \left(\rho^z f^* \rho^{1-z} f \right) \tag{AI.15}$$

For the case of interest to us with $s = \frac{1}{q}$, $q = 2^k + 1$, one can use also the following elementary induction. We apply the following elementary step 2^k times.

$$\mathbf{Tr} \left(\rho^{\frac{2^l}{q}} f^* \rho^{\frac{q-2^l}{q}} f \right) = \mathbf{Tr} \left(\rho^{\frac{2^l}{q}} f^* \rho^{\frac{q-2^{l+1}}{2q}} \right) \left(\rho^{\frac{q}{2q}} f \right) \leq \left(\mathbf{Tr} \left(\rho^{\frac{2^{l+1}}{q}} f^* \rho^{\frac{q-2^{l+1}}{q}} f \right) \right)^{\frac{1}{2}} \left(\mathbf{Tr} \rho f^* f \right)^{\frac{1}{2}} \tag{AI.16}$$

for $l = 0, .., k-1$. In this way we arrive at the following inequality

$$\left| \mathbf{Tr} \rho^{\frac{1}{q}} f^* \rho^{\frac{q-1}{q}} f \right| \leq \left(\mathbf{Tr} \rho f f^* \right)^{\frac{1}{2} + .. + \frac{1}{2^k}} \left(\mathbf{Tr} \rho^{\frac{1}{q}} f \rho^{\frac{q-1}{q}} f^* \right)^{\frac{1}{q-1}} \tag{AI.17}$$

The second term on the right hand side involves the similar expression as the starting one, with the roles of f and f^* exchanged. Therefore we can apply to it the same arguments. Using this, by induction we arrive at the inequality of interest to us.

$\diamondsuit$

Appendix II

Let us define the following function

$$\gamma_{X,\Lambda}(z) \equiv \rho_\Lambda^z (\mathbf{Tr}_X \rho_\Lambda)^{-z} = e^{-z\beta H_\Lambda} \left(\mathbf{Tr}_X e^{-\beta H_\Lambda} \right)^z \tag{AII.1}$$

As for every $\Lambda \in \mathcal{F}$ the symmetric operator H_Λ is bounded, it is clear that this is an operator analytic function on $\mathbb{C}$. Moreover the following useful fact is true.

Lemma AII.1:
Let β_0 be the radius of analyticity. Then there is a constant $C \in (0,\infty)$ such that for any $\Lambda \in \mathcal{F}$, $\beta \in (-\beta_0, \beta_0)$ and $z \in \mathbb{C}$, $|Rez| \leq 1$ we have

$$||\gamma_{X,\Lambda}(z)|| \leq C \tag{AII.2}$$

$\circ$

Proof: Since the function $\gamma_{X,\Lambda}(z)$ is analytic, applying the three lines theorem in the strip $0 \leq Rez \leq 1$, we have

$$||\gamma_{X,\Lambda}(z)|| \leq \sup_{t\in\mathbb{R}} ||\gamma_{X,\Lambda}(1+it)||^{Rez} \tag{AII.3}$$

Clearly from the definition of $\gamma_{X,\Lambda}(z)$, we have

$$||\gamma_{X,\Lambda}(1+it)|| \leq ||\gamma_{X,\Lambda}(1)|| \tag{AII.4}$$

Let us note now that

$$\gamma_{X,\Lambda}(1) \equiv e^{\beta H_\Lambda}\left(\mathbf{Tr}_X e^{-\beta H_\Lambda}\right)^{-1} = \xi_{X,\Lambda}(1)\left(\mathbf{Tr}_X \xi_{X,\Lambda}(1)\right)^{-1} \tag{AII.5}$$

where we have used a particular case of the following notation

$$\xi_{X,\Lambda}(s) \equiv e^{s\beta H_\Lambda} e^{-s\beta H_{\Lambda\setminus X}} \tag{AII.6}$$

Now we observe that, if the set X is sufficiently far from the boundary of Λ, we have

$$\frac{d}{ds}\xi_{X,\Lambda}(s) = \beta\alpha^\Lambda_{is\beta}(U_X)\cdot\xi_{X,\Lambda}(s) \tag{AII.7}$$

where $U_X \equiv \sum_{Y\cap X\neq\emptyset}\Phi_Y$. If $(1+\delta)\beta \in (-\beta_0, +\beta_0)$, with some $\delta \in (0,\infty)$, we have for $s \in [-1-\delta, 1+\delta]$, the following unique solution of the differential equation (AII.7) subjected to the initial condition $\xi_{X,\Lambda}(s=0) = 1$

$$\xi_{X,\Lambda}(s) = 1 + \sum_{n=1}^{\infty}\beta^n \int_0^s ds_1 ... \int_0^{s_{n-1}} ds_n\, \alpha^\Lambda_{is_1\beta}(U_X)...\alpha^\Lambda_{is_n\beta}(U_X) \tag{AII.8}$$

Hence we get

$$e^{-\beta_0 \sup_{s',\Lambda}||\alpha^\Lambda_{is'\beta}(U_X)||} \leq ||\xi_{X,\Lambda}(s)|| \leq e^{\beta_0 \sup_{s',\Lambda}||\alpha^\Lambda_{is'\beta}(U_X)||} \tag{AII.9}$$

Using this and (AII.5) we get that for $s_0 = \pm 1$ we have

$$||\gamma_{X,\Lambda}(s_0)|| \leq e^{2\beta_0 \sup_{s',\Lambda}||\alpha^\Lambda_{is'\beta}(U_X)||} \tag{AII.10}$$

On the other hand it is clear that

$$||\gamma_{X,\Lambda}(s=0)|| = 1 \tag{AII.11}$$

Since $\gamma_{X,\Lambda}(z)$ is analytic in the strip $|Rez| < 1+\delta$, (and obviously bounded for any fixed $\Lambda \in \mathcal{F}$), using the three lines theorem, we conclude that for any z with $0 \leq |Rez| \leq 1$ we have

$$||\gamma_{X,\Lambda}(z)|| \leq e^{2Re\, z\beta_0 \sup_{s',\Lambda}||\alpha^\Lambda_{is'\beta}(U_X)||} \tag{AII.12}$$

This ends the proof of Lemma AII.1.

$\diamondsuit$

Using the method of [**Ar**] one can also show the similar result for the spin systems with finite range interactions on one dimensional lattice at arbitrary temperature. From the uniform boundedness result for the sequence of operator valued analytic in the strip functions $\gamma_{X,\Lambda}(z)$, we see that one can choose a (weakly) convergent subsequence to an operator valued analytic function $\gamma_X(z)$. In general the limit point $\gamma_X(z)$ need not to be an element of the algebra $\mathcal{A}$.

Appendix III

In this Appendix we consider left and right multiplication operators in $\mathbb{L}_2(\omega,\lambda)$.

Lemma III.1

For any $f,g\in\mathbb{L}_2(\omega,\lambda)$ and an operator F such that Fg and gF are in $\mathbb{L}_2(\omega,\lambda)$ we have

$$|<f,Fg>_{\omega,\lambda}|\leq||\alpha_{i\lambda\beta/2}(F)||<f,f>_{\omega,\lambda}^{\frac{1}{2}}<g,g>_{\omega,\lambda}^{\frac{1}{2}} \qquad (AIII.1)$$

and

$$|<f,gF>_{\omega,\lambda}|\leq||\alpha_{i\lambda\beta/2}(F)\alpha_{i(1-\lambda/2)\beta}(F^*)||^{\frac{1}{2}}<f,f>_{\omega,\lambda}^{\frac{1}{2}}<g,g>_{\omega,\lambda}^{\frac{1}{2}} \qquad (AIII.2)$$

$\circ$

Proof: We have

$$<Fg,Fg>_{\omega,\lambda}=\omega((\alpha_{i\lambda\beta/2}(g))^*(\alpha_{i\lambda\beta/2}(F))^*(\alpha_{i\lambda\beta/2}(F))\alpha_{i\lambda\beta/2}(g))\leq||\alpha_{i\lambda\beta/2}(F)||^2<g,g>_{\omega,\lambda} \quad (AIII.3)$$

From this the inequality (AIII.1) follows. To get the inequality (AIII.2) we note that by definition of the scalar product and the **KMS** condition for the state ω we have

$$<gF,gF>_{\omega,\lambda}=\omega((\alpha_{i\lambda\beta/2}(F))^*(\alpha_{i\lambda\beta/2}(g))^*(\alpha_{i\lambda\beta/2}(g))\alpha_{i\lambda\beta/2}(F))= \qquad (AIII.4)$$

$$=\omega((\alpha_{i\lambda\beta/2}(g))^*(\alpha_{i\lambda\beta/2}(g))\alpha_{i\lambda\beta/2}(F)\alpha_{i\beta}(\alpha_{i\lambda\beta/2}(F)^*))$$

Hence by Schwartz inequality we obtain

$$<gF,gF>_{\omega,\lambda}\leq$$

$$\leq<g,g>_{\omega,\lambda}^{\frac{1}{2}}\cdot\left(\omega\left((\alpha_{i\lambda\beta/2}(F)\alpha_{i(1-\lambda/2)\beta}(F^*))^*(\alpha_{i\lambda\beta/2}(g))^*(\alpha_{i\lambda\beta/2}(g)(\alpha_{i\lambda\beta/2}(F)\alpha_{i(1-\lambda/2)\beta}(F^*)\right)\right)^{\frac{1}{2}} \qquad (AIII.5)$$

Iterating this procedure, in the limit we arrive at the following bound

$$<gF,gF>_{\omega,\lambda}\leq<g,g>_{\omega,\lambda}||(\alpha_{i\lambda\beta/2}(F)\alpha_{i(1-\lambda/2)\beta}(F^*)|| \qquad (AIII.6)$$

This clearly implies the inequality (AIII.2).

$\diamondsuit$

Appendix IV

Let $\{x^a_{\mathbf{j}}: a=1,..,D\}$ be a base of the single spin algebra $\mathbf{M_j}$, consisting of unitary operators. We define a seminorm $|||\cdot|||_\circ$ on $\mathcal{A}_0$ as follows

$$|||f|||_\circ\equiv\sum_{\mathbf{j}\in\mathbb{Z}^d}||\nabla_{x^a_{\mathbf{j}}}f|| \qquad (AIV.1)$$

Let

$$|||f||| \equiv \sum_{\mathbf{j}\in \mathbb{Z}^d} ||\partial_{\mathbf{j}} f|| \qquad (AIV.2)$$

where

$$\partial_{\mathbf{j}} \equiv f - \mathbf{Tr}_{\mathbf{j}} f \qquad (AIV.3)$$

Lemma IV.1

The triple bar seminorms introduced above are equivalent and one has with some $\lambda > 0$

$$\lambda^{-1}|||f||| \le |||f|||_\circ \le \lambda |||f||| \qquad (AIV.4)$$

$\circ$

Proof: We have

$$||\nabla_{x^a_{\mathbf{j}}} f|| = ||[x^a_{\mathbf{j}}, f]|| = ||[x^a_{\mathbf{j}}, f - \mathbf{Tr}_{\mathbf{j}} f]|| \le 2||f - \mathbf{Tr}_{\mathbf{j}} f|| = 2||\partial_{\mathbf{j}} f|| \qquad (AIV.5)$$

Summation over a's and $\mathbf{j}$'s yields the right hand side inequality in (AIV.4). To get the inequality on the left hand side we observe first that for any vector Φ and a positive operator f we have

$$||[x^a_{\mathbf{j}}, f]|| = ||x^{a*}_{\mathbf{j}} f x^a_{\mathbf{j}} - f|| \ge \pm(\Phi, (x^{a*}_{\mathbf{j}} f x^a_{\mathbf{j}} - f)\Phi) \qquad (AIV.6)$$

By an appropriate choice of the vector Φ and the base $x^a_{\mathbf{j}} : a = 1, .., D$, one can arrange that $x^{a*}_{\mathbf{j}}\Phi : a = 1, .., D'$, with $D' \le D$, is an O-N base in the corresponding finite dimensional Hilbert space associated to the point $\mathbf{j}$. Then summation over a's of (AIV.6) yields

$$\sum_{a=1,..,D'} ||[x^a_{\mathbf{j}}, f]|| \ge \pm(\Phi, (f - \mathbf{Tr}_{\mathbf{j}} f)\Phi) = \mp(\Phi, \partial_{\mathbf{j}} f\Phi) \qquad (AIV.7)$$

Hence taking the possible linear combinations with different Φ and the supremum over all possible choices, we arrive at the following inequality

$$\sum_{a=1,..,D'} ||[x^a_{\mathbf{j}}, f]|| \ge ||\partial_{\mathbf{j}} f|| \qquad (AIV.8)$$

Summing over $\mathbf{j}$'s we obtain the left hand side inequality $(AIV.4)$ for a positive operator f. From this the general case follows by an appropriate choice of the constant.

$\diamondsuit$

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