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Non-Associative Local Lie Groups

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Abstract. A general method for constructing local Lie groups which are not contained in any global Lie group is described. These examples fail to satisfy the global associativity axiom which, by a theorem of Mal'cev, is necessary and sufficient for globalizability. Furthermore, we prove that every local Lie group can be characterized by such a covering construction, thereby generalizing Cartan's global version of the Third Fundamental Theorem of Lie.

1. Introduction

In Sophus Lie's time, all Lie groups† were local. The group was identified with an open subset of Euclidean space, the group multiplication and inversion operations only being defined for elements sufficiently near the identity. The global theory had to await the proper abstract definition of manifold, and was finally codified in the work of Élie Cartan in the 1920's. The question then arose as to whether every local Lie group was contained in a global Lie group. This was answered affirmatively in the small by Cartan, [6], who showed that every local Lie group contains a neighborhood of the identity which is homeomorphic to a neighborhood of the identity of a global Lie group; see also [28; Theorem 84]. Cartan's result provides a global version Lie's Third Fundamental Theorem — every Lie algebra is the Lie algebra for a global Lie group. Indeed, the additional conditions that the Lie group be connected and simply connected are enough to uniquely specify it; any other connected global Lie group having the given Lie algebra is a quotient of this "maximal" Lie group by a discrete normal subgroup.

The global counterpart to Cartan's result is, however, not true. Not every local Lie group is contained in a global Lie group. The problem of globalizability of local topological groups was investigated in the 1930's by P.A. Smith, [33],

† See [4; pp. 410–429] for a short historical survey of Lie group theory. Hofmann, [11], argues that Lie's investigations were more along the lines of a local semigroup theory, which had its origins in a paper of Abel, [1]. In recent times, topological and Lie semigroups have become an important area of research in their own right; see, for example, the book [13] and the papers in [14], [15].

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[34], and by Mal'cev, [19], who pointed out the crucial connection between associativity and globalizability. Mal'cev proved that a necessary and sufficient condition for the existence of a global topological group containing a given local group is that it satisfy a certain "global" associativity hypothesis — see Definition 4 below. Their work was continued by, among others, Van Est, [35], Douady, [9], and, more recently, Plaut, [27], in connection with the study of local groups of isometries, metric convergence, and pinching, cf. [26]. However, these authors have tended to emphasize the infinite-dimensional version of Lie's Third Theorem, and all the examples of non-globalizable Lie groups appearing in their work are infinite-dimensional topological groups. Less well investigated, but also of interest is the finite-dimensional framework that forms the subject of this paper. A paper of Jacoby, [17], stated some globalizability theorems for finite-dimensional local groups, but his proofs are flawed. Indeed, the examples to be presented are readily seen to provide counterexamples to Jacoby's main globalizability theorem.

In this paper we describe a simple technique for constructing large families of examples of non-globalizable local Lie groups, and prove that every local Lie group is contained in one of these examples. The basic method is to start with a Lie group G, and let $S \subset G$ be an arbitrary closed subset not containing the identity element of G. It is not hard to see that any covering manifold $\pi: L \to G \setminus S$ can be endowed with the structure of a local Lie group which, in many instances, cannot be contained in any global Lie group. (A similar construction appears in [2; p. 388].) Indeed, if, as usually happens, the resulting local Lie group does not satisfy the global associativity axiom, Mal'cev's Theorem implies that it cannot be globalized. This paper includes a rather detailed analysis of the very simplest example of this general construction, when $G = \mathbb{R}^2$ is the two-dimensional abelian Lie group and S is a single point. However, the reader can readily see how these particular constructions can be generalized.

Thus, the results in this paper provide an interesting counterpoint to the infinite dimensional examples of Mal'cev, Van Est and Douady, leading to a wide class of finite dimensional local Lie groups, which, by the globalizability theorem of Cartan, are locally isomorphic to a neighborhood of the identity of a global Lie group, but which are not themselves embeddable into any global Lie group. Thus, in analogy with the tripartite division of the theory of Lie semigroups proposed in [12], the theory of finite-dimensional Lie groups also divides into three parts — the infinitesimal theory, the local theory, and the global theory. The present paper shows that the local theory is not a simple corollary of the global theory of Lie groups, but has its own set of interesting and delicate geometrical structures. Furthermore, our examples can be viewed as particular instances of non-associative local group actions on manifolds — we just view the group acting on itself by left multiplication — and can thus serve as useful test cases for the commonly misunderstood role played by associativity in the analysis of local transformation groups. See Mostow, [22], and Palais, [25], for results on globalizing local transformation group actions.

This work arose during the course of writing the book [24], when I was attempting to establish a global version of the local equivalence results found by Cartan's method of equivalence, [7], [10]. The equivalence method, which (in the cases relevant to our constructions) is based on the Frobenius Theorem, cf. [24; Chapter 14], [36], governing the existence of solutions to certain involutive systems of partial differential equations, can, in fact be employed to prove that these examples are the most general possible: every local Lie group is a generalized covering, in the sense of Definition 16, of an open subset of a global Lie group, leading to a complete characterization of the most general possible local Lie group. See Theorem 21 for a precise statement of this result. In the final section, we revisit Mal'cev's Theorem, adapting and fleshing out the proof to show that a local Lie group can be globalized if and only if it satisfies the global associativity condition.

2. Local and Global Lie Groups

We begin by formalizing the definitions of local and global Lie groups. For simplicity we work in the smooth category, although results of Lie, [18; p. 366], and Schur, [32], could be used to significantly relax our differentiability hypotheses. To avoid constant repetition, we assume the blanket hypothesis that all groups and manifolds are assumed to be connected. We begin with the standard definition of a Lie group.

Definition 1. A *global Lie group* is a group G which also carries the structure of a smooth manifold so that the operations of group multiplication

$$\mu: G \times G \longrightarrow G, \qquad \mu(g,h) = g \cdot h, \qquad g,h \in G$$

and inversion

$$\iota: G \longrightarrow G, \qquad \iota(g) = g^{-1}, \qquad g \in G$$

are smooth, globally defined maps.

While the definition of a global Lie group is standard, the precise definition of a local Lie group varies from author to author. The following one is slightly adapted from that appearing in Pontryagin, [28; p. 83]; see also [4; §III.1.10], [23; Definition 1.20]. The group multiplication is defined locally, meaning one of the multiplicands must be sufficiently close to the identity element. The only difference between our definition and Pontryagin's is that we require the left and right inverses of a group element to both be defined, and the same, whenever one or the other exists. This requirement could clearly be relaxed, without appreciable complication.

Definition 2. A smooth manifold L is called a *local Lie group* if there exists a) a distinguished element $e \in L$, the identity element, b) a smooth product map $\mu: \mathcal{U} \to L$ defined on an open subset $(\{e\} \times L) \cup (L \times \{e\}) \subset \mathcal{U} \subset L \times L$, and c) a smooth inversion map $\iota: \mathcal{V} \to L$ defined on an open subset $e \in \mathcal{V} \subset L$ such that $\mathcal{V} \times \iota(\mathcal{V}) \subset \mathcal{U}$, and $\iota(\mathcal{V}) \times \mathcal{V} \subset \mathcal{U}$, all satisfying the following properties:

(i) *Identity:* $\mu(e,x) = x = \mu(x,e)$ for all $x \in L$.

(ii) *Inverse:* $\mu(\iota(x),x) = e = \mu(x,\iota(x))$ for all $x \in \mathcal{V}$.

(*iii*) *Associativity:* If (x,y), (y,z), $(\mu(x,y),z)$, and $(x,\mu(y,z))$ all belong to $\mathcal{U}$, then

$$\mu(x,\mu(y,z)) = \mu(\mu(x,y),z). \tag{1}$$

In the classical texts, L is assumed to be an open subset of Euclidean space, so that the multiplication map $\mu(x,y)$ and the inversion map $\iota(x)$ are expressed in terms of coordinates. However, allowing more general manifolds does not lead to any appreciable change in either the methods or the results to be presented here.

Example 3. The most basic example of a local Lie group is provided by any neighborhood $e \in N \subset G$ of the identity element in a global Lie group. Indeed, we set $\mathcal{U}$ to be any open subset of $N \times N$ such that $(\{e\} \times N) \cup (N \times \{e\}) \subset \mathcal{U} \subset (N \times N) \cap \mu^{-1}(N)$, and $\mathcal{V}$ to be any open subset of N such that $\{e\} \subset \mathcal{V} \subset N \cap \iota^{-1}(N)$ and $(\mathcal{V} \times \iota(\mathcal{V})) \cup (\iota(\mathcal{V}) \times \mathcal{V}) \subset \mathcal{U}$. The group multiplication μ and inversion ι on G then clearly restrict to define local group multiplication and inversion maps on N.

Note that we can restrict the domains of definition of the local group multiplication and inversion maps, leading to a family of "restricted" local group structures on L. Two local group structures on the manifold L are *locally homeomorphic* if they have a common restriction. In other words, the two local group structures on the same manifold L, defined by $(\mu,\, \mathcal{U}, \iota, \mathcal{V})$, and $(\widetilde{\mu},\, \widetilde{\mathcal{U}}, \widetilde{\iota}, \widetilde{\mathcal{V}})$, as per Definition 2, are locally homeomorphic provided there exists open sets $(\{e\} \times L) \cup (L \times \{e\}) \subset \widehat{\mathcal{U}} \subset \mathcal{U} \cap \widetilde{\mathcal{U}}$, and $e \in \widehat{\mathcal{V}} \subset \mathcal{V} \cap \widetilde{\mathcal{V}}$, on which the two multiplications and inversions agree: $\mu | \widehat{\mathcal{U}} = \widetilde{\mu} \,|\, \widehat{\mathcal{U}}$, $\iota | \widehat{\mathcal{V}} = \widetilde{\iota} | \widehat{\mathcal{V}}$. Continuity then implies that the two local group structure agree on the connected components $(\mathcal{U} \cap \widetilde{\mathcal{U}})_0$, $(\mathcal{V} \cap \widetilde{\mathcal{V}})_0$ of the intersections of their domains. However, the two local group structures need not agree on the entire intersections $\mathcal{U} \cap \widetilde{\mathcal{U}}$, $\mathcal{V} \cap \widetilde{\mathcal{V}}$, which precludes the existence of a "maximal" local group structure among the locally homeomorphic local group structures on a given manifold L. By abuse of terminology, though, we shall refer to "the local Lie group L" with the understanding that the local group structure may need to be appropriately restricted if necessary.

There are several inequivalent ways of imposing the associativity requirement on a local Lie group. The least restrictive is to require that it hold locally, meaning for triples of group elements where two are sufficiently close to the identity; in other words, we require (1) hold for $(x,y,z) \in \mathcal{W}$, where

$$(\{e\} \times \{e\} \times L) \cup (\{e\} \times L \times \{e\}) \cup (L \times \{e\} \times \{e\}) \subset \mathcal{W} \subset \mathcal{Z} \subset L \times L \times L,$$

is an open subset contained in the set

$$\mathcal{Z} = \left[(\mathbb{1} \times \mu)^{-1}\, \mathcal{U} \cap (L \times \mathcal{U})\right] \cap \left[(\mathbb{1} \times \mu)^{-1}\, \mathcal{U} \cap (L \times \mathcal{U})\right] \subset L \times L \times L$$

of all $(x,y,z) \in L \times L \times L$ for which both sides of equation (1) are defined. We shall call such local groups *locally associative*. A similar condition appears in [17], [13; p. 542], where one assumes the existence of an open set $e \in \mathcal{Y} \subset \mathcal{V}$ such that $\mathcal{Y} \times \mathcal{Y} \subset \mathcal{U}$, and requires (1) to hold whenever $x, y, z, \mu(x,y), \mu(y,z) \in \mathcal{Y}$;

this amounts to requiring local associativity with $\mathcal{W} = (\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y}) \cap (\mathcal{Y} \times \mu^{-1}(\mathcal{Y})) \cap (\mu^{-1}(\mathcal{Y}) \times \mathcal{Y})$.

More restrictively, one can require that the associativity law hold for all triples of group elements for which the required group products are defined. This means that $\mathcal{W} = \mathcal{Z}$, and this is the version that I have chosen in Definition 2. Such local groups will be called *associative* for short.

There is a further *generalized associative law* that requires that associativity holds for all possible iterated products. From now on, if S is any set, we let $S^{\times n} = S \times \cdots \times S$ denote the n-fold Cartesian product of S with itself. Given an *ordered* n-tuple $(x_1, \ldots, x_n) \in L^{\times n}$ of local group elements, we can, provided they are defined, form a variety of n*-fold products*. For example, the two sides of the associativity law (1) define the two possible three-fold products. There are, potentially, five different products associated with an ordered four-tuple $(x_1, x_2, x_3, x_4) \in L^{\times 4}$, namely

$$\mu(x_1, \mu(x_2, \mu(x_3, x_4))), \qquad \mu(x_1, \mu(\mu(x_2, x_3), x_4)), \qquad \mu(\mu(x_1, x_2), \mu(x_3, x_4)),$$
$$\mu(\mu(x_1, \mu(x_2, x_3)), x_4), \qquad \mu(\mu(\mu(x_1, x_2), x_3), x_4).$$

Each possible n-fold product† of a given n-tuple is determined by an n-fold parenthesis system, cf. [8; p. 53], or, equivalently, a complete, planar, rooted binary tree on whose leaves are labeled by the x_i's, cf. [29; p. 4]. The precise number of possible n-fold products of an ordered n-tuple of group elements $(x_1, \ldots, x_n) \in L^{\times n}$, that is the number of possible ways of introducing parentheses into the list, is therefore given by the Catalan number $C_n = \dfrac{1}{n}\dbinom{2n-2}{n-1}$.

Definition 4. A local Lie group is *associative to order* n if, for every $3 \le m \le n$, and every ordered m-tuple of group elements $(x_1, \ldots, x_m) \in L^{\times m}$, all corresponding well-defined m-fold products are equal. A local group is called *globally associative* if it is associative to every order $n \ge 3$.

Note that, in particular, the associativity assumption (*iii*) in Definition 2 requires that any local group be associative to order 3 (at least). Of course, for a global group, associativity to order 3 automatically implies global associativity. The remarkable fact, which is explicitly borne out by our examples, is that this is not the case for local groups. Mal'cev's theorem, [19], demonstrates that the condition of globally associativity is both necessary and sufficient condition for a globally inversional local topological group to be globalizable. Moreover, Mal'cev exhibits infinite-dimensional examples of globally inversional topological groups which are associative to some fixed order n, but which fail to satisfy the global associativity requirement, and hence cannot be contained in any global group. Later, we shall see how to produce similar examples of local Lie groups that are associative, but not globally associative.

Remark. In an attempt to generalize the Gleason–Montgomery–Zippin Theo-

† Of course, not all of these products may be well defined.

rem, [21], on topological groups and thereby solve Hilbert's Fifth Problem ‡ for local Lie groups, Jacoby, [17], claimed to prove that any locally Euclidean local group is contained in a global Lie group. However, as pointed out by Plaut, [27], Jacoby did not appreciate the significance of the global associativity assumption. Indeed, Jacoby's Theorem 8, which is stated without proof, essentially makes the false claim that local associativity implies global associativity. Despite this initial flaw, it is likely that some of Jacoby's results can be recovered in a local or germ sense, or, alternatively, subject to an appropriate global associativity assumption. Thus, it is probably correct to state that every locally Euclidean local group is locally isomorphic to a neighborhood of the identity in a global Lie group. Unfortunately, since the rest of Jacoby's paper relies heavily on his incorrect Theorem 8, the re-establishment of his results (in an appropriately amended form) would require a significant effort. Such a project would not be without motivation — for instance, results of Brown and Houston, [5], and Hofmann and Weiss, [16], that provide a positive solution to Hilbert's Fifth Problem for semigroups on manifolds, rely heavily on Jacoby's results.

A map between local groups is a local group homomorphism provided it respects the two multiplication and inversions where they are defined. Specifically:

Definition 5. Let $(L, \mu, \mathcal{U}, \iota, \mathcal{V})$, and $(\widetilde{L}, \widetilde{\mu}, \widetilde{\mathcal{U}}, \widetilde{\iota}, \widetilde{\mathcal{V}})$, be local Lie groups. A smooth map $\Phi: L \to \widetilde{L}$ is called a *local group homomorphism* if

(*i*) $\Phi \times \Phi(\mathcal{U}) \subset \widetilde{\mathcal{U}}, \quad \Phi(\mathcal{V}) \subset \widetilde{\mathcal{V}}, \quad \Phi(e) = \tilde{e}$,

(*ii*) $\Phi(\mu(g,h)) = \widetilde{\mu}(\Phi(g), \Phi(h))$ for $(g,h) \in \mathcal{U}$, and

(*iii*) $\Phi(\iota(g)) = \widetilde{\iota}(\Phi(g))$ for $g \in \mathcal{V}$.

A local group homomorphism is called a *homeomorphism* if it is one-to-one, onto, with smooth inverse.

Note that the image of a local group homeomorphism $\Phi: L \to \widetilde{L}$ effectively identifies L as a restriction of the local group structure on $\widetilde{L}$. We could further relax the requirement for a local group homomorphism by only requiring that Φ respect the group multiplication and inversion maps on suitably small open subsets. However, this completely local definition can be subsumed in the present definition by utilizing a suitable restriction of the local group structure on L.

The main focus of the present paper is to understand to what extent local groups can be globalized, thereby identifying them as a neighborhood of the identity element in a global Lie group of the same dimension. This will be formalized in the following definition.

Definition 6. A local Lie group L is called *globalizable* if there exists a local group homeomorphism $\Phi: L \to N$ mapping L onto a neighborhood $e \in N \subset G$ of the identity of a global Lie group G.

‡ Hilbert's fifth problem concerns the role of analyticity in general transformation groups, and seeks to generalize the result of Lie, [18; p. 366], and Schur, [32]. The Gleason–Montgomery–Zippin result only addresses the special case when a global Lie group acts on itself by right or left multiplication.

Example 7. Before proceeding further, it is useful to illustrate these concepts by a simple example of a one-dimensional local Lie group, cf. [23; pp. 19–20]. Let $M = \mathbb{R}$. The identity element will be $e = 0$. The formulas for the multiplication and inversion maps are given by

$$\mu(x,y) = \frac{2xy - x - y}{xy - 1}, \qquad \iota(x) = \frac{x}{2x-1}. \tag{2}$$

First, we restrict the multiplication and inversion maps by defining $L = \{|x| < \frac{1}{2}\}$, and using

$$\mathcal{U} = \left\{(x,y) \;\middle|\; |x| < \tfrac{1}{2}, |y| < \tfrac{1}{2}, (3x-2)(3y-2) > 1, (5x-2)(5y-2) < 9\right\},$$
$$\mathcal{V} = \{-\tfrac{1}{2} < x < \tfrac{1}{4}\},$$

as the domains of definition of the multiplication and inversion maps. The reader can straightforwardly verify that (2) does indeed define a globally associative, abelian, local Lie group. Indeed, the local Lie group L can be globalized via an embedding into the unique global, simply connected, one-dimensional Lie group $G = \mathbb{R}$. The map $\Phi: L \to G$ given by $\Phi(x) = x/(x-1)$ satisfies $\Phi(\mu(x,y)) = \Phi(x) + \Phi(y)$, and $\Phi(\iota(x)) = -\Phi(x)$ where defined. Therefore Φ provides the desired local group homeomorphism mapping L to the open interval $N = \{-1 < x < \frac{1}{3}\} \subset G$.

In fact, one can expand the domains of definition of the multiplication and inversion maps in this example, but this leads to a curious phenomenon. The largest possible domains that will still satisfy the local group axioms are $\widehat{\mathcal{U}} = \{(x,y) \,|\, |xy| \neq 1\} \subset M \times M$ and $\widehat{\mathcal{V}} = \{x \,|\, x \neq \frac{1}{2}, x \neq 1\} \subset M$. However, in this situation, the group element $1 \in M$ plays a strange role. We find $\mu(x,1) = \mu(1,x) = 1$ for all $x \neq 1$, so that 1 defines an "infinite group element". Moreover, its "inverse" $\iota(1) = 1$ can even be defined, although the product of the two, $\mu(1, \iota(1))$, is not defined, and hence not equal to the identity. On the other hand, the infinite group element is "inaccessible" since $\mu(x,y) = 1$ if and only if either $x = 1$ or $y = 1$. Note that we can complete M and G by adjoining a point at infinity, realizing both as subsets of the projective line: $M, G \subset \mathbb{RP}^1$. The "group homomorphism" Φ then extends to a global projective map of $\mathbb{RP}^1$, thereby endowing the projective line with the structure of a local Lie group with an infinite group element, containing a global Lie group as a dense open subset.

Remark. The latter construction in Example 7 is a special instance of a general compactification process that can be applied to any noncompact, but locally compact group; see [31] for details.

To avoid the pathologies caused by such infinite local group elements, we must impose an additional invertibility assumption on the local group. The most common way to do this, advocated in [19], [33], is to require that every group element have an inverse.

Definition 8. A local Lie group L is called *globally inversional* if the inversion map ι is defined everywhere, so that $\mathcal{V} = L$.

For example, a local group $\{e\} \subset N \subset G$ given by a neighborhood of the identity element in a global Lie group will be globally inversional if and only if it satisfies the symmetry restriction $N = \iota(N)$. Bourbaki's Lie group germs[†], [4; §III.1.10], are the same as globally inversional local Lie groups.

For our purposes, though, it is useful to weaken the rather severe global inversionality assumption, without permitting the pathology of infinite group elements. One way to do this is to require that the group multiplication map does not degenerate. More specifically, define the left and right multiplication maps λ_x, ρ_x so that

$$\lambda_x(y) = \mu(x,y), \qquad \rho_x(y) = \mu(y,x). \tag{3}$$

Definition 9. A local Lie group L is called *regular* if, for each $x \in L$, the maps λ_x, ρ_x defining left and right multiplication, (3), are diffeomorphisms on their respective domains of definition.

Thus, in the local Lie group of Example 7, in order to retain regularity we must throw away the infinite group element 1, since its left and right multiplication maps are constant. A slightly more restrictive approach is to require that the group elements always be expressible as products of invertible ones.

Definition 10. An element x of a local Lie group L is called *inversional* if there exist invertible elements $x_1, \ldots, x_n \in \mathcal{V}$ such that x equals an n-fold product of the x_i's. A local Lie group L is called *inversional* if every $x \in L$ is an inversional element.

The following result is straightforward; its converse, though, is not necessarily valid, since the group might include additional components containing regular, but non-invertible elements.

Proposition 11. *Every inversional local Lie group is regular.*

The property of being inversional is related to the notion of connectivity of a local Lie group. A global Lie group is connected if it forms a connected manifold. Although a first glance this appears a reasonable definition in the case of a local Lie group, we have seen that it permits inaccessible, non-regular group elements. Moreover, one of the principal means of characterizing connected global Lie groups, which lies at the foundations of the infinitesimal approach to Lie group analysis, [23], is no longer equivalent to the mere connectivity of the manifold in the case of local groups.

Definition 12. A subset $U \subset L$ of a local Lie group L is said to *generate* L if $L = \bigcup_{n=1}^{\infty} U^{(n)}$. Here $U^{(n)} \subset L$ denotes the subset consisting of all group elements $x \in L$ which can be written as a well-defined n-fold product of elements $x_1, \ldots, x_n \in U$.

For example, Definition 10 just says that a local Lie group is inversional if and only if the domain $\mathcal{V}$ of the inversion map generates L. A key result in the theory of global Lie groups states that any open neighborhood $e \in U$ of

[†] The original French version [3; §III.1.10] uses the wonderful term "groupuscle" for a local group or group germ.

the identity of a connected global Lie group G generates the entire Lie group G, cf. [36; Proposition 3.18]. In particular, choosing U to be contained in the image of the exponential map $\exp: \mathfrak{g} \to G$, we deduce that every element of a connected Lie group can be written as a finite product of exponentials: $g = \exp(\mathbf{v}_1) \cdot \ldots \cdot \exp(\mathbf{v}_n)$ for $\mathbf{v}_i \in \mathfrak{g}$. Example 7 shows that, without additional assumptions, this result is not necessarily valid for local Lie groups. We are therefore led to propose a more restrictive notion of connectedness for a local Lie group.

Definition 13. A local Lie group L is *connected* if

(*i*) L is a connected manifold,

(*ii*) the domains of definition of the multiplication and inversion maps are also connected,

(*iii*) if $U \subset L$ is any neighborhood of the identity, then U generates L.

Note that a connected global Lie group satisfies the revised Definition 13. In particular, we can choose the generating neighborhood to be $U = \mathcal{V}$, and thus deduce that connected local Lie groups are automatically inversional:

Proposition 14. *Any connected local Lie group is inversional, and hence regular.*

¿From now on, in accordance with our blanket hypothesis, all local Lie groups are be assumed to be connected in the sense of Definition 13. Thus we replace a more general local Lie group L, by the connected component L_0 containing the identity, which is defined the largest open, connected, local Lie subgroup $L_0 \subset L$.

3. The Simplest Examples

In order to understand our general construction of non-associative local Lie groups, it is helpful to discuss the very simplest examples in some detail. The first example will be an associative, regular, local Lie group which fails to be globally associative. This preliminary example however, does not satisfy the global inversional property, and so we describe a straightforward modification which is, in addition, globally inversional. The failure of the global associativity axiom implies that neither example can be contained in any global Lie group.

Let $G = \mathbb{R}^2$ denote the real, simply connected two-dimensional abelian Lie group. It is convenient to identify G with the complex plane $\mathbb{C} \simeq \mathbb{R}^2$, and use complex notation for the elements of G, even though we are considering G as a real Lie group. Let $M = G \setminus \{-1\}$ denote the punctured plane. (Other than the fact that it is not the identity, there is nothing special about choosing -1 as the distinguished point to omit.) Let $L \simeq \mathbb{R}^2$ denote the simply connected covering space of M, and let $\pi: L \to M$ denote the covering map. Note that L can be identified with the Riemann surface associated with the function $\log(z+1)$. An element $\hat{z} \in L$ is described by its projection $z = \pi(\hat{z})$ and an integer $n \in \mathbb{Z}$ indicating which sheet in the covering space $\hat{z} = (z, n)$ lies in. In the sequel, we shall consistently use the notation z, w, etc. for the projections of local

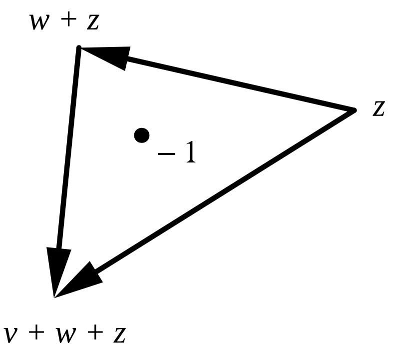

Figure 1. Non–associativity.

group elements $\hat{z}, \hat{w}$, etc. If we introduce polar coordinates (relative to the point -1) then we can identify L with the half plane $L \simeq \{(r,\theta) \,|\, r > 0\}$. The projection is explicitly given by $\pi(r,\theta) = re^{i\theta} - 1$. Thus, the point (r,θ) corresponds to $\hat{z} = (z,n)$, where $z = re^{i\theta} - 1$, and n is the integer such that $(2n-1)\pi < \theta \leq (2n+1)\pi$. Suppose $C \subset M$ is any (connected) curve not passing through -1, and let $z \in C$. Given a point $\hat{z} \in \pi^{-1}\{z\}$, we can find a unique lifted connected curve $\widehat{C} \subset L$ passing through $\hat{z}$ and which projects back down to C. In particular, if C is a closed curve, then $\widehat{C}$ is also closed if and only if the winding number of C with respect to -1 is zero.

We shall endow L with the structure of a connected, regular local Lie group which does not satisfy the global associativity property, and hence cannot be contained in any global Lie group. The group multiplication law on L will, locally, agree with vector addition on $\mathbb{R}^2$. The identity element will be the element $\hat{e} = (1,0)$ lying in the zero$^{\text{th}}$ sheet of L which projects to $e = 0$, the identity element of the abelian Lie group G. The multiplication $\mu(\hat{z},\hat{w})$ between two elements $\hat{z},\hat{w} \in L$ will be defined provided one or the other is sufficiently close to the identity. When defined, the product $\hat{v} = \mu(\hat{z},\hat{w})$ will have projection $v = z + w$ in M. Moreover, the product $\mu(\hat{z},\hat{w})$ itself is defined either by lifting the line segment connecting z to $z+w$ to form a curve connecting $\hat{z}$ to $\mu(\hat{z},\hat{w})$ in L, or, alternatively, by lifting the segment connecting w to $z+w$ to connect $\hat{w}$ to $\mu(\hat{z},\hat{w})$. Note that the lifted line segment will be unambiguously defined as long as the line segment in question does not pass through the forbidden point -1.

Our goal is to suitably prescribe the domain of definition of the product μ so that all three local group axioms are valid, (and the group is even locally commutative!), but then prove that the group fails to satisfy the globally associativity property. The key technical complication is to appropriately restrict the domain of definition of μ so that the ordinary associativity law (1) holds. Note that if we allow too large a domain of definition, then the resulting local group will only be locally associative. Indeed, let $\hat{v}, \hat{w}, \hat{z} \in L$ be group elements whose projections v, w, z, have the property that the triangle with vertices z, $w+z$,

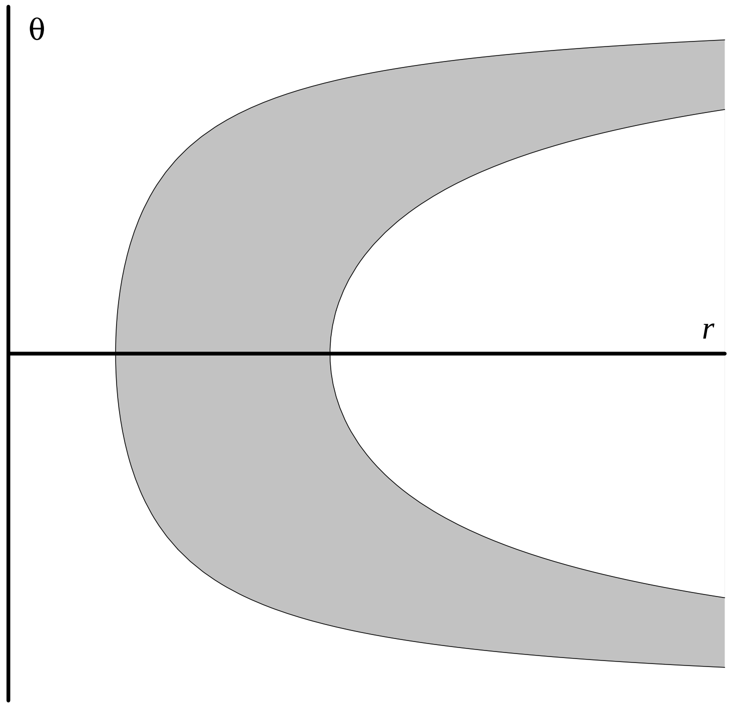

Figure 2. The domain $L_0 \subset L$.

$v+w+z$, contains the singular point -1 in its interior (and hence the triangle has nonzero winding number); see Figure 1. Then the product $\mu(\hat{v}, \mu(\hat{w}, \hat{z}))$ cannot equal $\mu(\mu(\hat{v}, \hat{w}), \hat{z})$ since the final points would have to lie on different sheets of the covering space L. In this way, we see that associativity of an n-fold product is related to the winding number of a certain polygonal path with respect to the distinguished point -1.

The domain of definition of the product on L will now be fixed as follows. Let

$$L_0 = \left\{ (r, \theta) \;\middle|\; \tfrac{1}{2} \sec \theta < r < \tfrac{3}{2} \sec \theta, -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \right\} \tag{4}$$

denote that part of the zero$^{\text{th}}$ sheet of L (i.e., the sheet containing the identity $\hat{e}$) which lies above the strip $M_0 = \{ -\frac{1}{2} < \operatorname{Re} z < \frac{1}{2} \}$; see Figure 2. Note that the projection $\pi: L_0 \to M_0$ is one-to-one when restricted to L_0. Further let

$$L_1 = \left\{ (r, \theta) \;\middle|\; -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \right\} \tag{5}$$

denote that part of the zero$^{\text{th}}$ sheet of L which projects diffeomorphically onto the half plane $M_1 = \{ \operatorname{Re} z > -1 \}$. A trivial, but important observation is that if $z, w \in M_0$, then $z + w \in M_1$.

Given $z, w \in M$, we let $-\pi < \alpha(z, w) \leq \pi$ denote the angle from z to w with respect to the singular point -1; see Figure 3. Given $z \in M$, define

$$H_z = \left\{ \hat{w} \in L_0 \;\middle|\; -\tfrac{1}{2}\pi < \alpha(z, z + w) < \tfrac{1}{2}\pi \right\}. \tag{6}$$

The domain of definition of the group operation is then taken to be

$$\mathcal{U} = \{ (\hat{z}, \hat{w}) \in L \times L \mid \hat{z} \in H_w \quad \text{or} \quad \hat{w} \in H_z \}. \tag{7}$$

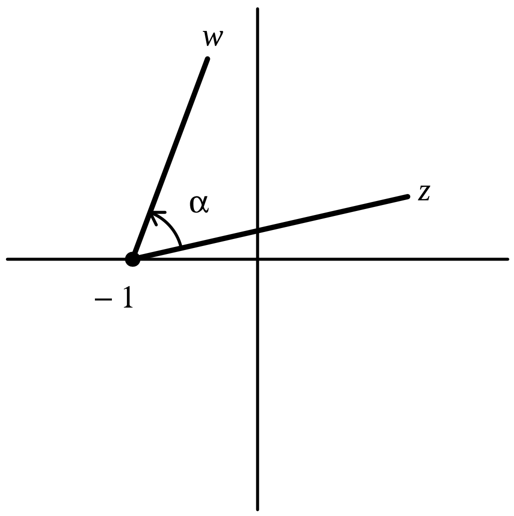

Figure 3. The angle $\alpha = \alpha(z, w)$**.**

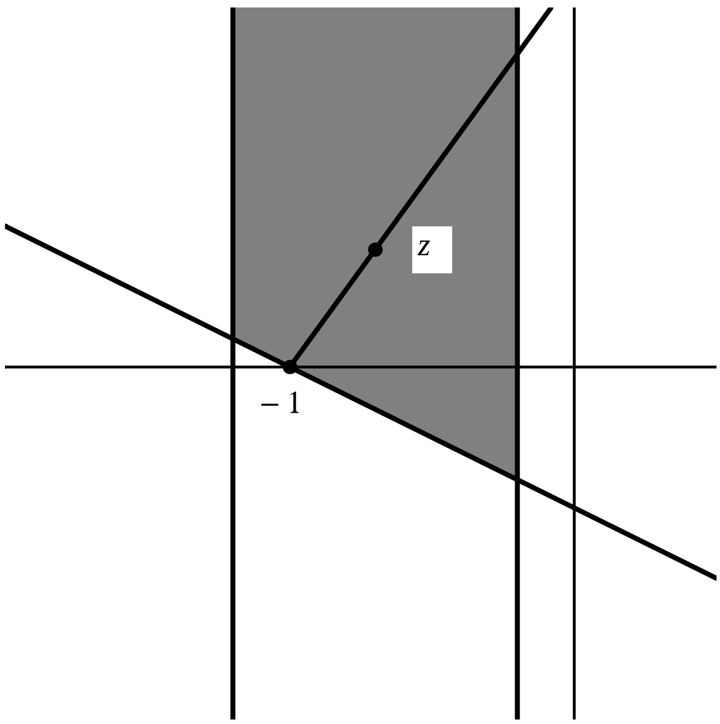

Figure 4. The domain of definition of the product.

In other words, the product $\mu(\hat{z}, \hat{w})$ is defined provided either $\hat{w} \in H_z$ or $\hat{z} \in H_w$. In the former case, $\hat{w} \in H_z$, and the product is defined by lifting the line segment connecting z to $z + w$ to form a connected curve from $\hat{z}$ to $\mu(\hat{z}, \hat{w})$ in L; in the latter case, $\hat{z} \in H_w$, we lift the segment connecting w to $z + w$ to connect $\hat{w}$ to $\mu(\hat{z}, \hat{w})$. Note that the restriction on the domain implies that the line segment in question does not pass through the forbidden point -1, and hence the lifted line segment is unambiguously defined. Given a point $\hat{z}$ with projection z, the shaded region in Figure 4 represents the projection of the set of possible

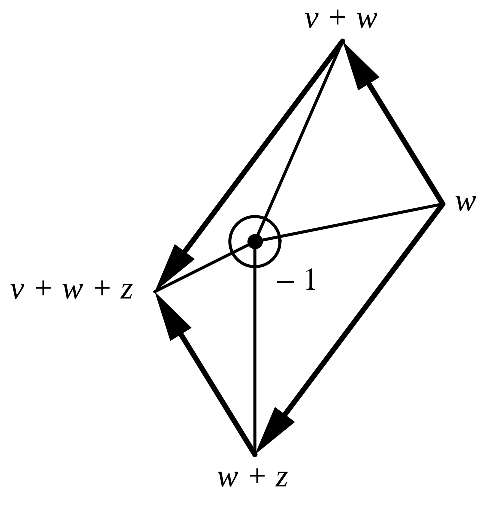

Figure 5. Fundamental parallelogram.

products $\mu(\hat{z}, \hat{w})$ for all $\hat{w} \in H_z$; the domain H_z is obtained by translating the figure so as to place z at the origin. Further, we define the domain of definition of the inversion map ι to be $\mathcal{V} = L_0$, as given in (4). Given $\hat{z} \in L_0$, the inverse $\iota(\hat{z})$ is the unique element in L_0 projecting to $-z$.

Theorem 15. *Under the above constructions, the product $\mu: \mathcal{U} \to L$ and inversion $\iota: \mathcal{V} \to L$ endow L with the structure of a regular, connected, associative, local Lie group which is not globally associative.*

Proof. The identity and inverse requirements are immediate. Local associativity of L is also not difficult. To prove that L is associative (to order 3) we need to show that

$$\mu(\hat{v}, \mu(\hat{w}, \hat{z})) = \mu(\mu(\hat{v}, \hat{w}), \hat{z}), \tag{8}$$

whenever both sides of the equation are defined. The left-hand side of (8) is defined provided either *a*) $\hat{z} \in H_w$ or *b*) $\hat{w} \in H_z$, and either *a*) $\hat{v} \in H_{w+z}$ or *b*) $\mu(\hat{w}, \hat{z}) \in H_v$, leading to four different possibilities. Similarly, there are four different ways that the right hand side could be defined: *a*) $\hat{v} \in H_w$ or *b*) $\hat{w} \in H_v$, and *a*) $\hat{z} \in H_{v+w}$ or *b*) $\mu(\hat{v}, \hat{w}) \in H_z$. Therefore, there is a grand total of sixteen different ways in which (8) could make sense! We label the different possibilities by four letters, each either *a* or *b*, indicating which one of the preceding different possibilities; for example, Case *abba* would require $\hat{z} \in H_w$, $\mu(\hat{w}, \hat{z}) \in H_v$, $\hat{w} \in H_v$, and $\hat{z} \in H_{v+w}$. The sixteen cases subdivide into four classes, each of which is handled by a slightly different proof.

The first class consists solely of Case *aaaa*, which means $\hat{z} \in H_w$, $\hat{v} \in H_{w+z}$, $\hat{v} \in H_w$, $\hat{z} \in H_{v+w}$. Consider the *fundamental parallelogram*, defined as the parallelogram contained in M with vertices w, $v+w$, $w+z$ and $v+w+z$ — see Figure 5. Clearly, the associative law (8) holds in this case if and only if the point -1 does not lie in the interior of this fundamental parallelogram. In Case *aaaa*, the four angles

$$\alpha(w, w+z), \quad \alpha(w+z, v+w+z), \quad \alpha(w, v+w), \quad \alpha(v+w, v+w+z),$$

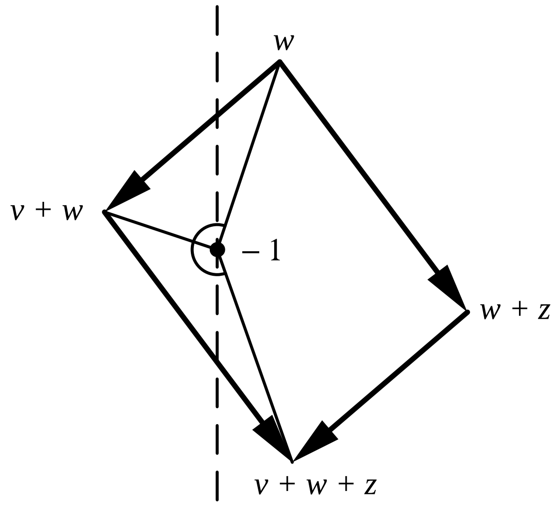

Figure 6. **Case** *abaa***.**

are all strictly less than $\frac{1}{2}\pi$, which, as illustrated in Figure 5 immediately implies that the fundamental parallelogram cannot contain -1.

The second class is exemplified by Case *abaa*, which requires $\hat{z} \in H_w$, $\mu(\hat{w}, \hat{z}) \in H_v$, $\hat{v} \in H_w$, $\hat{z} \in H_{v+w}$. In this case, $v, z, w + z \in M_0$, and hence $w = (w+z) - z$ and $v + w + z$ lie in M_1. This means that the only vertex of the fundamental parallelogram which can lie to the left of the line $\{\mathrm{Re}\ z = -1\}$ is $v+w$. On the other hand, the two angles $\alpha(w, v+w)$ and $\alpha(v+w, v+w+z)$ must be less than $\frac{1}{2}\pi$, which is impossible unless $v + w \in M_1$ also — see Figure 6. Thus -1 is not contained in the fundamental parallelogram, and associativity holds here also. Similar arguments apply to Cases *aaab*, *aaba*, *aabb*, *baaa*, and *baba*.

The third class is indicated by Case *baab*, which requires $\hat{w} \in H_z$, $\hat{v} \in H_{w+z}$, $\hat{v} \in H_w$, $\mu(\hat{v}, \hat{w}) \in H_z$. In this case, consider the *fundamental triangle* with vertices at z, $w + z$ and $v + w + z$, as illustrated in Figure 7. The three angles

$$\alpha(z, v + w + z), \qquad \alpha(z, w + z), \qquad \alpha(w + z, v + w + z),$$

are all less than $\frac{1}{2}\pi$, hence the triangle cannot contain -1 in its interior. Therefore, the corresponding three points lie on the same sheet of L, and associativity holds. Similar arguments apply to Cases *abba*, *babb*, and *bbba*.

The final, and easiest class is exemplified by Case *abab*. Here $\hat{z} \in H_w$, $\mu(\hat{w}, \hat{z}) \in H_v$, $\hat{v} \in H_w$, $\mu(\hat{v}, \hat{w}) \in H_z$. Thus $v, z, v+w, w+z \in M_0$, and hence w, $v + w + z \in M_1$. Thus every vertex of the fundamental parallelogram lies to the right of -1, so -1 is not contained in its interior, and hence associativity holds. The same argument applies to Cases *abbb*, *bbaa*, *bbab*, and *bbbb*. We have now analyzed every one of the sixteen different possibilities, and hence associativity of our local Lie group is proved.

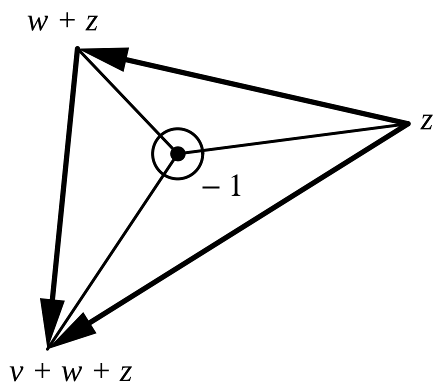

Figure 7. Fundamental triangle.

To prove global non-associativity, it suffices to produce one counter-example to the next simplest associative identity:

$$\mu(\hat{u},\mu(\hat{v},\mu(\hat{w},\hat{z}))) = \mu(\mu(\mu(\hat{u},\hat{v}),\hat{w}),\hat{z}). \tag{9}$$

This will demonstrate that L is not associative of order 4, and, hence, not globally associative. Let $\omega = \exp \frac{1}{4}\pi i$ be the primitive eighth root of unity. Let

$$\begin{aligned} u &= \frac{2}{3}(\overline{\omega}-1) = -\frac{2-\sqrt{2}}{3} - i\,\frac{\sqrt{2}}{3}, & w &= \frac{2}{3}(i-\omega) = -\frac{\sqrt{2}}{3} + i\,\frac{2-\sqrt{2}}{3}, \\ v &= \frac{2}{3}(-i-\overline{\omega}) = -\frac{\sqrt{2}}{3} - i\,\frac{2-\sqrt{2}}{3}, & z &= \frac{2}{3}(\omega-1) = \frac{2-\sqrt{2}}{3} + i\,\frac{\sqrt{2}}{3}. \end{aligned} \tag{10}$$

Finally, let $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{z}$ be the corresponding points in L_0 which project to u,v,w,z. Note that, as illustrated in Figure 8, the points 0, u, $u+v$, $u+v+w$, $u+v+w+z$, $v+w+z$, $w+z$, and z form the vertices of a regular octagon centered at the point $-\frac{2}{3}$ having radius $\frac{2}{3}$. In particular, this implies that the singular point -1 lies *inside* the octagon, and hence any lift of the octagon does not form a closed curve in the covering space L. Therefore, if we start at 0, and go to $u+v+w+z = -\frac{4}{3}$ along the two different polygonal paths, we will end up at different points back up on the covering space L, which implies that the associativity condition (9) does *not* hold! We leave it to the reader to check that both four-fold products are well defined — in particular that the relevant angles are all less than $\frac{1}{2}\pi$. This produces the required example.

The only remaining point is to show that L satisfies the connectivity conditions in Definition 13, but this is fairly straightforward. First, L itself is clearly pathwise connected, as is $\mathcal{V} = L_0$. The domain (7) is also pathwise connected. Indeed, consider a point $(\hat{z},\hat{w}) \in \mathcal{U}$ such that $\hat{w} \in H_z$. We can connect $(\hat{z},\hat{w})$ to $(\hat{z},0)$ by a curve $\{\hat{z}\}\times S$, where $S \subset H_z$ is the lift of the straight line segment that connects w to 0. Further, since L is pathwise connected, we can then connect $(\hat{z},0)$ to $(0,0)$ via a curve $C\times\{0\}\subset L\times\{0\}$. The alternative case $\hat{z}\in H_w$ is handled analogously, thus proving that $\mathcal{U}$ is

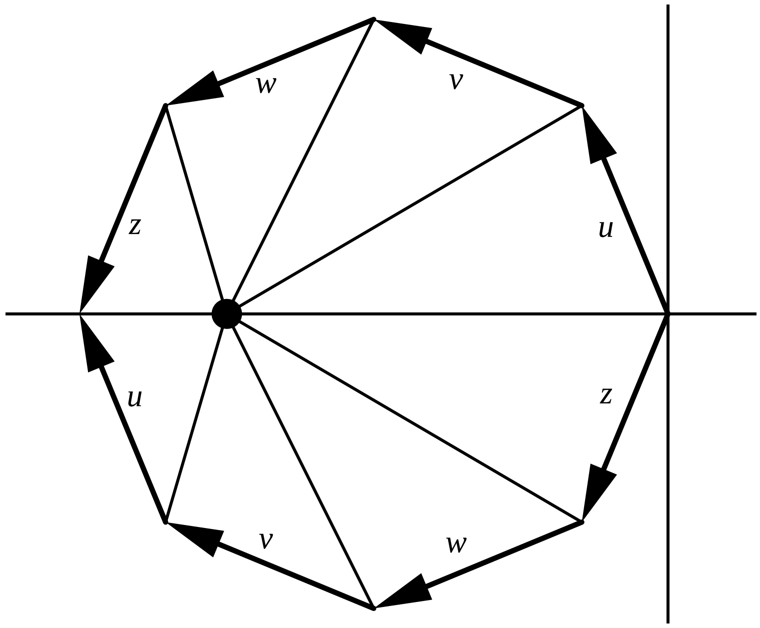

Figure 8. Non–associative octagon.

pathwise connected. Finally, note that the open set $U = \{\hat{z} \in L_0 \,|\, |z| < \frac{3}{4}\}$ generates L. Indeed, repeatedly multiplying the primitive eighth root elements $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{z} \in U$ constructed above, cf. (10), allows us to circle the singular point -1 as many times as we like in order to reach any desired sheet of L. Once on a given sheet, it is straightforward to continue to multiply by appropriate elements of U to generate any group element lying in that sheet. Moreover, any open subneighborhood $e \in \widetilde{U} \subset U \subset L$ will clearly generate U, thereby verifying the last connectivity requirement. This at last completes the proof of the theorem.■

Finally, we remark that one can suitably shrink the domain of definition of the product on L, enabling one to construct a local Lie group that is associative to order n, but which is not globally associative. The precise definition of the n^{th} order domain, though, is rather tricky, owing to the many possible n-fold products that may be defined!

The examples modeled on the covering space of $M = \mathbb{R}^2 \setminus \{-1\}$ are not globally inversional, since the inverse was only defined for group elements $\hat{z} \in \mathcal{V} = L_0$. A modification of this basic construction can, though, be used to produce a globally inversional, but not globally associative local Lie group. We briefly indicate how this can be accomplished. The key is that the group domain needs to be symmetric with respect to the origin, while still maintaining non-associativity. Consider the twice punctured plane $\widetilde{M} = \mathbb{R}^2 \setminus \{+1, -1\}$, and let $\pi\colon \widetilde{M}^* \to \widetilde{M}$ denote the simply connected covering space of $\widetilde{M}$; we can identify $\widetilde{M}^*$ with the Riemann surface corresponding to the complex function $\log(z^2 - 1)$. Fix a point $e \in \pi^{-1}\{0\}$ as the identity. Let $\widetilde{L} \subset \widetilde{M}^*$ be a domain which contains e, which is four-fold symmetric under the analytic continuation of the coordinate axis reflections $z \mapsto \bar{z}$, $z \mapsto -\bar{z}$, and whose projection overlaps itself on the outer real axes $\{(x,0) \,|\, |x| > 1\}$ just once. See Figure 9 for a sketch of the projection of such a domain. Clearly, by symmetry, we can arrange $\widetilde{L}$ to

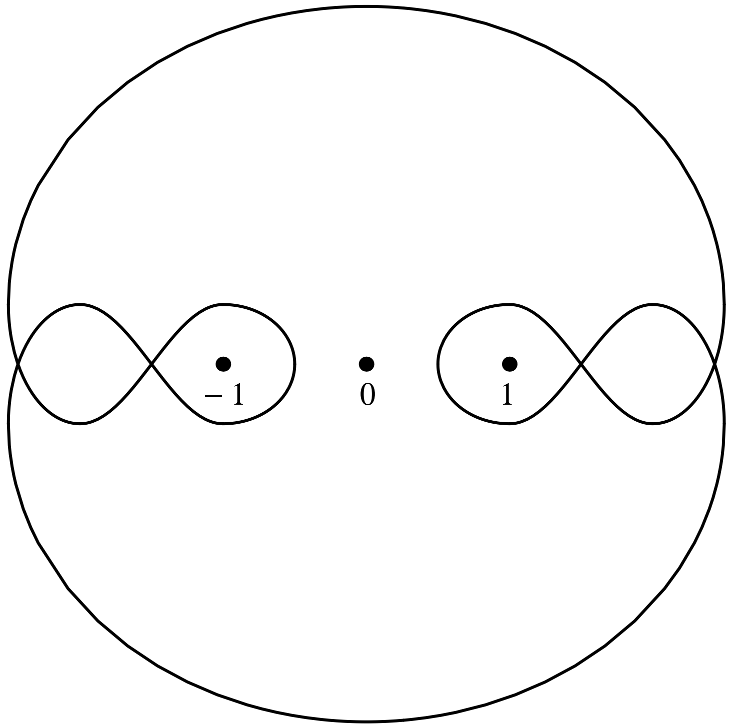

Figure 9. Globally inversional non–associative local Lie group.

be globally inversional by analytic continuation of the inversion map $z \mapsto -z$ lifted to $\widetilde{M}^*$. Thus, the inverse of an element $\hat{z}$ that lies in the upper sheet of $\widetilde{L}$ that is to the right of $+1$ will end up in the lower sheet of $\widetilde{L}$ that is to the left of -1. It is also clear, that by suitably restricting the domain of definition of the product, which will be the lift of the usual vector addition as in our previous example, one can endow $\widetilde{L}$ with the structure of a globally inversional, abelian, local Lie group. Nevertheless, due to the overlap of the projection of $\widetilde{L}$ back down to $\widetilde{M}$, the resulting local Lie group is clearly not globally associative. The precise definition of the domain of definition of the product on $\widetilde{L}$ so as to make it associative to order, say, 3, but not to order 4, is left as a (challenging) exercise for the reader.

4. A General Characterization of Local Lie Groups

The basic construction in the previous section can be readily generalized to apply to covering spaces of open subsets of arbitrary Lie groups. The main technical complication is the determination of a suitably small domain of definition of the multiplication map so as to maintain local associativity. The existence of such a domain is straightforward — one merely ensures that the corresponding "polygonal paths" do not encircle any of the components of the complement of the open subset. However, an explicit construction in particular examples can get complicated.

We now investigate in what sense these are the most general examples.

The main result states that this is the case "up to covering". Here, though, our definition of covering is more general than usual.

Definition 16. Let M and N be m-dimensional manifolds. A local diffeomorphism $\Phi: M \to N$ which maps M onto N is called a *generalized covering map*.

Thus, according to the Inverse Function Theorem, a map $\Phi: M \to N$ is a generalized covering map if and only if $\Phi(M) = N$ and $d\Phi: TM|_x \to TN|_{\Phi(x)}$ is an invertible linear map for each $x \in M$. Our use of the term "covering map" is more inclusive than the usual terminology, since M does not necessarily cover N uniformly; indeed the inverse image $\Phi^{-1}(y)$ of a point $y \in N$ is a discrete subset of M, but its cardinality may vary from point to point. For instance, according to our definition, the restriction of any generalized covering map to any open subset $\widetilde{M} \subset M$ such that $\Phi(\widetilde{M}) = N$ remains a generalized covering map. For example, the map $F(t) = (\cos t, \sin t)$ provides an ordinary covering map from the real line $M = \mathbb{R}$ to the circle $N = S^1$, which remains a generalized covering map when restricted to any open interval of length at least 2π. By abuse of terminology, we will say M *covers* N if there is a generalized covering map from M to N.

If L and $\overline{L}$ are local Lie groups, we will say that $\overline{L}$ is a *covering group* of L if there is a generalized covering map $\Phi: \overline{L} \to L$ which is a local group homomorphism. A simple example is provided in Section 3, where the non-associative local Lie group L is realized as a covering group of the globally associative local Lie group $M \subset \mathbb{R}^2$. A more exotic example of a local Lie group is found by starting with the local Lie group L described in Section 3, and choosing a point $\hat{e} \neq \hat{z}_0 \in L$. Then the simply connected covering space $\overline{L} \to L \setminus \{\hat{z}_0\}$ is also a (non-uniform) local covering group of M.

We next recalling some standard facts concerning frames and coframes on manifolds and (local) Lie groups.

Definition 17. Let M be a smooth m-dimensional manifold. A *frame* on M is an ordered set of vector fields $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ having the property that they form a basis for the tangent space $TM|_x$ at each point $x \in M$. Dually, a *coframe* on M is an ordered set of one-forms $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ which form a basis for the cotangent space $T^*M|_x$ at each point $x \in M$.

A manifold admits a (global) frame if and only if its tangent bundle is topologically trivial: $TM \simeq M \times \mathbb{R}^m$. The *structure equations* associated with a frame are obtained by expressing the Lie brackets in terms of the frame vector fields themselves:

$$[\mathbf{v}_i, \mathbf{v}_j] = \sum_{k=1}^{m} C_{ij}^k \mathbf{v}_k, \qquad i, j = 1, \ldots, m. \tag{11}$$

The coefficient functions C_{ij}^k are called the *structure functions* of the given frame. The frame is said to have *rank* 0 if the structure coefficients C_{ij}^k are all constant. In this case, the C_{ij}^k form the structure constants of the Lie algebra $\mathfrak{g}$ spanned by the frame vector fields. We will show that we can endow M itself with the structure of a local Lie group such that the frame vector fields form a basis for the Lie algebra $\mathfrak{g}$ of right-invariant† vector fields thereon.

Theorem 18. *If L is a regular, locally associative, local Lie group, then it admits a right invariant frame of rank 0. Conversely, if M is a manifold that admits a rank 0 frame, then M can be endowed with the structure of a regular, locally associative local Lie group having the given frame as right-invariant Lie algebra elements.*

Proof. Let L be the m-dimensional local Lie group in question. We first construct the Lie algebra $\mathfrak{g}$ consisting of all right-invariant vector fields on L by standard methods, cf. [28]. In other words, given a tangent vector $\mathbf{v}|_e \in TL|_e$ to L at the identity, we define its right-invariant counterpart to be the vector field with value $\mathbf{v}|_x = d\rho_x(\mathbf{v}|_e)$ at any $x \in L$, where ρ_x denotes the standard right multiplication map, cf. (3). Associativity implies the right-invariance of $\mathbf{v}$, so $d\rho_y(\mathbf{v}|_x) = \mathbf{v}|_{\mu(x,y)}$ for any $(x,y) \in \mathcal{U}$. The regularity hypothesis implies that $\mathfrak{g}$ defines a global frame on L, i.e., we can choose a basis $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_m$ which span the tangent space $TL|_x$ at each point $x \in L$, satisfy the standard Lie algebra commutation relations (11) for structure constants C_{ij}^k relative to the chosen basis.

Remark. If the group does not satisfy our regularity assumption, then the construction of the right-invariant frame leads to degeneracies at the "infinite elements". For instance, in the irregular Example 7, the right-invariant vector field $\widehat{\mathbf{v}} = (x-1)^2 \partial_x$ vanishes at the infinite element $x = 1$.

To prove the converse, let G denote the global Lie group whose Lie algebra has the given structure constants determined by the structure equations (11) of the rank 0 frame. According to the basic theorem on transformation groups, [28; Theorem 88], [23; Theorem 1.57], the vector fields $\mathbf{v}_i$ form the infinitesimal generators for a local action of G on M. In particular, we can choose any point $e \in M$ to serve as the identity element for the local group structure on M, and the map $g \mapsto g \cdot e$ serves to define a group isomorphism between a sufficiently small neighborhood of $e \in M$ with a neighborhood of the identity in G. In this way, we can identify the transformation $x \mapsto g \cdot x$ by a group element $g \in G$ sufficiently close to the identity with the product $g \cdot x = \mu(y,x)$ by the corresponding element $y = g \cdot e \in M$. Thus, by suitably restricting the domain of definition of the transformation group action, we endow M with the structure of a local Lie group. The construction of the associated inverse map and verification of the local group axioms is straightforward. ∎

Any manifold admitting a global frame $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ also admits a global coframe provided by the dual one-forms $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$, and conversely. These are constructed so that $\langle\, \theta^i \,;\, \mathbf{v}_j \,\rangle = \delta^i_j$ at each $x \in M$, where δ^i_j is the usual Kronecker symbol. The dual structure equations for the coframe are

$$d\theta^k = -\sum_{1 \le i < j \le m} C_{ij}^k\, \theta^i \wedge \theta^j, \tag{12}$$

† One can equally well use left-invariant vector fields if one prefers.

and have the same (up to sign) structure functions C^k_{ij}. In particular, the dual coframe to a right-invariant frame on a local Lie group is known as the *Maurer–Cartan coframe*, and (12) are the fundamental Maurer–Cartan structure equations, with the C^k_{ij} the structure constants for the Lie algebra. Note that Theorem 18 immediately implies that any manifold admitting a rank 0 coframe can be given the structure of a local Lie group.

Theorem 19. *If L is a regular, locally associative local Lie group, then it admits a right invariant Maurer–Cartan coframe satisfying the rank 0 structure equations (12). Conversely, if M is a manifold with a coframe of rank 0, then M can be endowed with the structure of a regular, locally associative local Lie group having the given coframe as Maurer–Cartan coframe.*

Remark. The preceding considerations imply that any smooth, connected local Lie group comes equipped with a Cartan connection $\boldsymbol{\theta}: TL \to \mathfrak{g}$ with vanishing curvature $\Omega = d\boldsymbol{\theta} + [\boldsymbol{\theta}, \boldsymbol{\theta}] = 0$. See [20], [30], for the theory of "almost Lie groups" in which one relaxes this condition by allowing manifolds with a Cartan connection having small curvature. Under certain conditions, such manifolds are necessarily local Lie groups.

Finally, we state the basic infinitesimal characterization of local Lie group homeomorphisms.

Theorem 20. *Suppose L and M are connected m-dimensional local Lie groups, and let $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ denote their respective right-invariant Maurer–Cartan coframes. If a map $\Phi: L \to M$ satisfies $\Phi^*(\boldsymbol{\eta}) = \boldsymbol{\theta}$ and $\Phi(e) = \tilde{e}$, then Φ defines a local group homeomorphism from L onto its image, which forms an open local subgroup of M.*

Proof. The pull-back conditions imply that the corresponding dual right-invariant vector fields are mapped to each other by the push forward map: $d\Phi(\mathbf{v}_i) = \mathbf{w}_i$, $i = 1, \ldots, m$. Therefore Φ is equivariant under the associated flows or one-parameter subgroup actions: $\Phi(\exp(t\mathbf{v})y) = \exp(t\mathbf{w})\Phi(y)$, where $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m \in \mathfrak{g}$ and t is sufficiently small. Therefore, the homeomorphism property

$$\Phi(\mu(x, y)) = \widetilde{\mu}(\Phi(x), \Phi(y)) \tag{13}$$

holds provided $x = \exp(t\mathbf{v})$ for $\mathbf{v} \in \mathfrak{g}$ and t sufficiently small. Now, the exponential map maps an open neighborhood of $0 \in \mathfrak{g}$ onto a neighborhood of $e \in L$, and hence (13) holds whenever $x \in L$ is sufficiently close to the identity. Since L is connected, we can use continuity of Φ to assert that (13) holds for all $(x, y) \in \mathcal{U}$. The corresponding assertion for the inversion is proved similarly, completing the proof. ∎

Our main result will completely characterize local Lie groups up to (generalized) covering. The proof relies on the Cartan equivalence method, [7], [10], applied to coframes of rank zero, which in turns rests on an application of Frobenius' Theorem to construct the covering map via Cartan's technique of the graph. See [24] for additional details.

Theorem 21. *Let L be a connected local Lie group. Then there exists a local covering group $\overline{L} \to L$ which is also a local covering group $\overline{L} \to M$ of an open subset $M \subset G$ of a global Lie group G.*

Proof. Let L be a local group and let $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ be its Maurer–Cartan coframe. The structure equations (12) serve to define the structure constants of an m-dimensional Lie algebra $\mathfrak{g}$. Let G denote the connected, simply connected, m-dimensional global Lie group having the same Lie algebra $\mathfrak{g}$. Let $\boldsymbol{\alpha} = \{\alpha^1, \ldots, \alpha^m\}$ be the corresponding Maurer–Cartan coframe of globally right-invariant one-forms on G with respect to the same basis of $\mathfrak{g}$. The Maurer–Cartan structure equations of $\boldsymbol{\alpha}$ are the same as those of $\boldsymbol{\theta}$, cf. (12). Thus, according to the Cartan equivalence method, [7], [10], [24], if $x \in L$ and $g \in G$ are any two elements, then there is a unique local equivalence map $\Phi: L \to G$, defined in a neighborhood of x, such that $\Phi(x) = g$ and Φ pulls-back the Maurer–Cartan coframe on G to that on L, so $\Phi^*(\boldsymbol{\alpha}) = \boldsymbol{\theta}$. In particular, if we take $x = e$ to be the identity element of L, and $g = e$ the identity element of G, then Theorem 20 implies that Φ is a local group homeomorphism from a neighborhood of the identity in L to a neighborhood of $e \in G$. The map Φ serves to globalize a neighborhood of the identity in L, thereby reproducing Cartan's global version, [6], of Lie's Third Fundamental Theorem. (Of course, this is cheating, since we have assumed the existence of the appropriate global Lie group G in advance!)

To see how Φ can itself be globalized, we need to understand its construction in more detail. The proof of the local equivalence result is based on Cartan's technique of the graph. One determines the local equivalence map $\Phi: L \to G$ by constructing its graph $\Gamma_\Phi = \{(x, \Phi(x)\}$, which is a $r = \dim G$ dimensional submanifold of the Cartesian product space $L \times G$. The graph is realized as an integral submanifold of a suitable differential system $\mathcal{I}$, namely the one generated by the differences $\vartheta^i = \pi_G^*\alpha^i - \pi_L^*\theta^i$, $i = 1, \ldots, m$, of the pull-backs of the two coframes under the standard projections $\pi_L: L \times G \to L$ and $\pi_G: L \times G \to G$. The fact that the two coframes have identical constant structure coefficients implies that this differential system is in involution, and hence Frobenius' Theorem, cf. [24], [36], guarantees the existence of a unique maximal integral submanifold N, of dimension m, passing through any point $(x, g) \in L \times G$. The integral submanifold N coincides, locally, with the graph of the required equivalence map Φ. The fact that N satisfies the required transversality conditions at (x, g) is immediate, and so the Implicit Function Theorem guarantees that it defines a local diffeomorphism.

In order to complete the proof, we need to show that the full maximal integral submanifold covers all of L. Let $N \subset L \times G$ denote the maximal integral submanifold corresponding to the differential system described above passing through the point $(e, e) \in L \times G$. We know that, locally, N is the graph of an equivalence map. Indeed, the restrictions of the two projections $\pi_L: N \to L$, $\pi_G: N \to G$, determine local diffeomorphisms. Moreover, equivalence implies that the given coframe $\boldsymbol{\theta}$ on L and the Maurer–Cartan coframe $\boldsymbol{\alpha}$ on G both pull back to the same rank 0 coframe, $\boldsymbol{\xi} = \pi_L^*\boldsymbol{\theta} = \pi_G^*\boldsymbol{\alpha}$ on N. According to Theorem 19, the existence of a rank zero coframe on N allows us to identify it as a local Lie group with the point (e, e) serving as the identity element. Moreover,

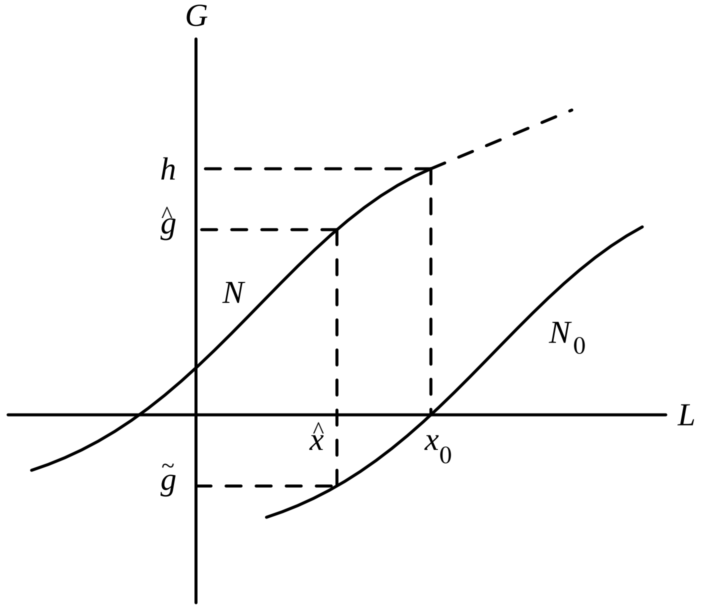

Figure 10. Integral submanifolds.

Theorem 20 implies that the two (generalized) covering maps will then be local group homomorphisms. In fact, since the one-forms in $\boldsymbol{\xi}$ form a subset of the Maurer–Cartan coframe on the local product group $L \times G$, consisting of the one-forms $\pi_L^*\boldsymbol{\theta}, \pi_G^*\boldsymbol{\alpha}$, the local group structure on N coincides with the restriction of the Cartesian product local group structure on $L \times G$, and so we can identify N as an m-dimensional local Lie subgroup of $L \times G$.

I claim that the restricted projection $\pi_L: N \to L$ is, in fact, a generalized covering map, which requires us to prove that $\pi_L(N) = L$. Assuming this, the theorem follows directly, since N will then be the required covering group of L, and also cover an open subset of G under π_G, thereby fulfilling the conditions of Definition 16. To prove the claim, note that if N is any integral submanifold of the differential system generated by $\boldsymbol{\vartheta}$, then, because of the right-invariance of the Maurer–Cartan forms, any right translate $\rho_h[N] = \{(x, g \cdot h) \mid (x,g) \in N\}$ by a group element $h \in G$ is also an integral submanifold. Suppose $\pi_L(N) \neq L$, and let $x_0 \in L \setminus \pi_L(N)$ be a point in the closure of $\pi_L(N)$. According to Frobenius' Theorem, we can find a local equivalence $\Phi_0: U \to G$, where U is a neighborhood of x_0 in L, mapping x_0 to the identity element $e = \Phi_0(x_0)$; see Figure 10. Let N_0 denote the graph of Φ_0, so that N_0 is also an integral submanifold of our differential system passing through (x_0, e). Choose any point $\hat{x} \in U \cap \pi_L(N)$, so $\hat{x} = \pi_L(\hat{x}, \hat{g})$ for some point $(\hat{x}, \hat{g}) \in N$ in the original integral submanifold. Let $\tilde{g} = \Phi_0(\hat{x})$, and define $h = \tilde{g}^{-1} \cdot \hat{g}$. Then, by the previous remark, $R_h[N_0]$ is an integral submanifold of the differential system; moreover, the point $(\hat{x}, \hat{g}) = (\hat{x}, \tilde{g} \cdot h)$ is contained both in $\rho_h[N_0]$ and in our original integral submanifold N. Therefore, by uniqueness and maximality of N, the integral submanifold $\rho_h[N_0]$ must be an open submanifold of N. But the point $(x_0, h) = \rho_h(x_0, e)$ lies in $\rho_h[N_0]$, and hence in N. This contradicts our original assumption that x_0 was not in the projection $\pi_L(N)$, and so proves the claim. ∎

Example 22. Consider the coframe

$$\theta^1 = \cos\varphi\, dr - r\sin\varphi\, d\varphi, \qquad \theta^2 = \sin\varphi\, dr + r\cos\varphi\, d\varphi, \tag{14}$$

defined on the half plane $L = \{(r,\varphi) \mid r > 0\}$. Clearly, in terms of polar coordinates, this coframe is locally diffeomorphic to the Maurer–Cartan coframe $\alpha^1 = dx$, $\alpha^2 = dy$ for the two-dimensional abelian Lie group $\mathbb{R}^2$. Indeed, L can be identified with the simply connected covering space for the punctured plane $M = \mathbb{R}^2 \setminus \{(-1,0)\}$, and so is isomorphic to our earlier example of a non-associative local Lie group. Consequently, this coframe is the Maurer–Cartan coframe for the local Lie group L and is therefore not globally equivalent to the Maurer–Cartan coframe on any open subset of the Lie group $\mathbb{R}^2$.

This example includes yet one additional interesting pathology. The dual frame vector fields to (14) are

$$\mathbf{v}_1 = \cos\varphi\,\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\,\frac{\partial}{\partial\varphi}, \qquad \mathbf{v}_2 = \sin\varphi\,\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\,\frac{\partial}{\partial\varphi}. \tag{15}$$

They are mapped to the coordinate vector fields ∂_x, ∂_y under the polar coordinate map, and hence commute:

$$[\mathbf{v}_1, \mathbf{v}_2] = 0. \tag{16}$$

Nevertheless, their flows do not commute! The reader can explicitly verify that, for any $r_0 > 0$, if we set $s = t = \sqrt{2}\, r_0$, then

$$\exp(s\mathbf{v}_1)\exp(t\mathbf{v}_2)\big(r_0, \tfrac{5}{4}\pi\big) = \exp(s\mathbf{v}_1)\big(r_0, \tfrac{3}{4}\pi\big) = \big(r_0, \tfrac{1}{4}\pi\big),$$

whereas

$$\exp(t\mathbf{v}_2)\exp(s\mathbf{v}_1)\big(r_0, \tfrac{5}{4}\pi\big) = \exp(t\mathbf{v}_2)\big(r_0, \tfrac{5}{4}\pi\big) = \big(r_0, \tfrac{5}{4}\pi\big).$$

(This is easy to do by reverting to rectangular coordinates.) Thus we have an explicit, elementary counterexample to the commutativity theorem in [23; Theorem 1.34], which states that two vector fields on a manifold M satisfy (16) if and only if their flows satisfy

$$\exp(s\mathbf{v}_1)\exp(t\mathbf{v}_2)x_0 = \exp(t\mathbf{v}_2)\exp(s\mathbf{v}_1)x_0, \tag{17}$$

for all $x_0 \in M$ and all $(s,t) \in V$ where V denotes the subset of the s,t plane where both sides of (17) are defined! This theorem, that commuting vector fields induce commuting flows, *is* true locally. A correct version should state that (17) holds for all (s,t) lying in the connected component of V containing the origin. In the above example, for $x_0 = \big(r_0, \frac{5}{4}\pi\big)$, the left hand side of (17) is defined for all $(s,t) \in V_1 = \mathbb{R}^2 \setminus \{s = \frac{1}{\sqrt{2}} r_0, t \geq \frac{1}{\sqrt{2}} r_0\}$. The right hand side of (17) is defined for all $(s,t) \in V_2 = \mathbb{R}^2 \setminus \{t = \frac{1}{\sqrt{2}} r_0, s \geq \frac{1}{\sqrt{2}} r_0\}$. The set $V = V_1 \cap V_2$ on which both sides of (17) are defined, then, consists of two connected components, $V = V_0 \cup V_1$. If $(s,t) \in V_0 = \{s < \frac{1}{\sqrt{2}} r_0 \text{ or } t < \frac{1}{\sqrt{2}} r_0\}$, which is the component containing the origin, the commutativity equation (17) holds. On the other hand, when $(s,t) \in V_1 = \{s > \frac{1}{\sqrt{2}} r_0 \text{ and } t > \frac{1}{\sqrt{2}} r_0\}$, commutativity equation (17) does *not* hold.

Remark. Using a version of the Hopf–Rinow Theorem, Gardner, [10; p. 72], shows that a rank zero coframe $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^m\}$ on a simply connected manifold M which is metrically complete with respect to the Riemannian metric $\sum_i (\theta^i)^2$ induced by the coframe is globally equivalent to a Maurer–Cartan coframe on a global Lie group. Thus, metric completeness is, in some subtle way, related to associativity.

Remark. The covering map $\pi_L: N \to L$ identifies (in a neighborhood of each point) the local group structure on N with some restriction of the local group structure on L. An open question is whether this too can be globalized: Can we construct a covering local group $\pi: N \to L$ such that every well-defined product $\mu(x, y)$, and inverse $\iota(x)$ has its counterpart in N. In other words, if $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ denote the local group domains for L, can we ensure that $(\pi \times \pi)\,\widetilde{\mathcal{U}} \to \mathcal{U}$ and $\pi(\widetilde{\mathcal{V}}) \to \mathcal{V}$ are also covering maps? This appears to be more difficult, in light of the global incompatibility of locally homeomorphic group structures on a given local Lie group.

5. Globalization

Finally, we prove a version of Mal'cev's Theorem, [19], that the condition of global associativity is both necessary and sufficient for a local Lie group to be contained in a global Lie group. Here we adapt Mal'cev's method of proof, and generalize his result to connected (and hence inversional) local Lie groups. Mal'cev argument extremely brief, and is not entirely convincing at first sight, so we need to fill in some of the missing details. Moreover, Mal'cev assumes that the group is globally inversional, so we need to include additional arguments to cover mere inversionality. The fundamental result relies on the basic Definition 6 of a globalizable local Lie group.

Theorem 23. *A connected local Lie group L is globalizable if and only if it is globally associative.*

Proof. The necessity of global associativity is clear. To prove its sufficiency, we must construct a global Lie group G containing the given connected local Lie group L. Let

$$\mathcal{W} = \mathcal{W}(L) = \bigcup_{n=1}^{\infty} L^{\times n}$$

denote the set of words based on the set L, i.e., ordered n-tuples $(x_1, \ldots, x_n) \in L^{\times n}$ for any $n \geq 1$. We define an equivalence relation on $\mathcal{W}$ as follows: If x_k, x_{k+1} are adjacent elements in an n-tuple $X = (x_1, \ldots, x_n) \in \mathcal{M}$ such that $(x_k, x_{k+1}) \in \mathcal{U} \subset L \times L$, then X will be equivalent to the $(n-1)$-tuple $Y = (x_1, \ldots, x_{k-1}, y, x_{k+2}, \ldots, x_n) \in L^{\times(n-1)}$ obtained by replacing them by the product $y = \mu(x_k, x_{k+1}) \in L$. Vice versa, if an element $x_k = \mu(y_1, y_2)$ in X is written as a product of local group elements, then X will be equivalent to the $(n+1)$-tuple $Z = (x_1, \ldots, x_{k-1}, y_1, y_2, x_{k+1}, \ldots, x_n) \in L^{\times(n+1)}$ obtained by replacing x_k by y_1, y_2. The first type of equivalence relation will be called a *contraction* and the second an *expansion*. Thus two words $X, Y \in \mathcal{W}$ are equivalent if and only if there is a finite chain of basic equivalences — expansions and contractions — starting with X and ending with Y. Finally, we define $G = \mathcal{W}/\sim$ to be the set of equivalence classes under the full equivalence relation. The claim is that G can be endowed with the structure of a global, connected Lie group such that the map $\sigma: L \to G$ that identifies each $x \in L$ with the corresponding 1-tuple $\sigma(x) = x \in L^{\times 1} \subset \mathcal{W}$ is an injective local group homomorphism.

An associative product on $\mathcal{W}$ is given by juxtaposition of words. This product clearly respects the equivalence relation, and hence provides a globally defined, globally associative product on G. The inverse of an m-tuple $Y = (y_1, \ldots, y_m) \in \mathcal{V}^{\times m} \subset \mathcal{W}$ consisting of invertible elements $y_i \in \mathcal{V} \subset L$ is given by inverting the individual group elements and reversing the order, so $Y^{-1} = (y_m^{-1}, \ldots, y_1^{-1})$. Clearly $Y \cdot Y^{-1}$ and $Y^{-1} \cdot Y$ are both equivalent to the identity element in G. More generally, thanks to the regularity of L and Proposition 14, given any $(x_1, \ldots, x_n) \in \mathcal{W}$, we can write each $x_i = y_{i1} \cdot y_{i2} \cdot \ldots \cdot y_{ik_i}$ as a product of invertible elements $y_{i\nu} \in \mathcal{V}$, whereby X is equivalent to the element

$$X = (x_1, \ldots, x_n) \sim Y = (y_{11}, \ldots, y_{1k_1}, y_{21}, \ldots, y_{2k_2}, \ldots, y_{n1}, \ldots y_{nk_n}).$$

The inverse of X in G can then be identified with the inverse of Y.

The global associativity of L ensures that this construction is well-defined on G, making G into a group. At this point in his proof of [19; Theorem 1], Mal'cev merely says "…on peut démontrer …" that the global associativity assumption implies that the map $\sigma: L \to G$ is an injective local group homomorphism. A detailed argument can be constructed as follows: The fact that σ is a local group homomorphism is immediate from the definition of the product on G. The difficulty is in the equivalence procedure, which could conceivably prevent the injectivity of σ. This would mean that we might be able to find two different local group elements $x \neq z$ which admit a chain of equivalences consisting of expansions and contractions. We must show that this cannot happen due to our global associativity hypothesis. Now, if the chain of equivalences starting at x and ending at z consists of a sequence of n expansions, increasing the word length by 1 at each step, followed by a sequence of n contractions ending at z, which we refer to as a *simple equivalence chain*, then global associativity is applicable and we may conclude that $x = z$. The desired conclusion is less clear if the chain of equivalences is not simple, mixing expansions and contractions. However, this will reduce to the simple case if we can show that all the intermediate contractions can be replaced by equivalent expansions, so that the given chain of equivalences can be replaced by one of simple form.

Consider a contraction that replaces an adjacent pair x_k, x_{k+1} of elements in a word X by their product $y = \mu(x_k, x_{k+1})$. If $x_k \in \mathcal{V}$ is an invertible element sufficiently close to the identity, then $x_{k+1} = \mu(x_k^{-1}, z)$, and hence the singleton $y \in L$ can be replaced by the equivalent triple $(x_k, x_k^{-1}, y) \in L^{\times 3}$, which we can also view as an *expansion* of the pair (x_k, x_{k+1}). More generally, using the connectivity of L, we expand x_k into an equivalent sequence $x_{k1}, \ldots, x_{kn}$ of invertible elements $x_{kj} \in \mathcal{V}$ such that $y = \mu(x_{k1}, \mu(x_{k1}, \ldots, \mu(x_{kn}, x_{k+1}) \ldots))$

and $x_{k+1} = \mu(x_{kn}^{-1}, \mu(x_{k,n-1}^{-1}, \ldots, \mu(x_{k1}^{-1}, y) \ldots))$. We then replace y by the equivalent word $(x_{k1}, \ldots, x_{kn}, x_{kn}^{-1}, \ldots, x_{k1}^{-1}, y)$, which can also be regarded as a sequence of successive expansions of the original pair (x_k, x_{k+1}). Alternatively, we can replace y by the word $(y, x_{k1}, \ldots, x_{kn}, x_{kn}^{-1}, \ldots, x_{k1}^{-1})$, since, by global associativity,

$$(y, x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}) \sim y \sim (x_1, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}, x),$$

for any $y \in L$, and $x_1, \ldots, x_n \in \mathcal{V}$. Thus, we can always ensure that adjacent group elements in the original words can always be arranged to be adjacent in the new expanded words, and thus we can continue to replace each contraction in the original sequence of equivalences by an equivalent expansion. Proceeding in this manner, we finally arrive at an ultimate word that is composed of a large number of subwords of the form $(y_1, \ldots, y_k, y_k^{-1}, \ldots, y_1^{-1})$ juxtaposed with the final singleton z at some position. We can now follow our sequence of expansions by a sequence of elementary contractions, each one replacing a pair $y_j y_j^{-1}$ by e and then canceling out the e. Thus we have produced the desired simple chain of equivalences connecting x to z. This completes our demonstration that the map σ is injective.

Finally we demonstrate that G is a manifold and hence forms a global Lie group. The second countability axiom follows immediately from the fact that $\mathcal{W}$ itself is second countable. We introduce local coordinates on G as follows: Given an n-tuple $(x_1, \ldots, x_n) \in L^{\times n}$, we define its k^{th} coordinate neighborhood to consist of all n-tuples of the form $(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots x_n)$, where y lies in a sufficiently small local coordinate neighborhood of $x_k \in L$. The smoothness of the overlap maps relies on global associativity. For instance, the overlap map between the k^{th} and $(k+1)^{\text{st}}$ coordinate neighborhoods determined by a fixed n-tuple arises by assuming that y is sufficiently close to x_k, which, by regularity, implies that we can write $y = \mu(\hat{y}, x_k)$ for some $\hat{y}$ close to e. Hence, by global associativity, we can identify the words

$$\begin{aligned}(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots x_n) &\sim (x_1, \ldots, x_{k-1}, \hat{y}, x_k, x_{k+1}, \ldots x_n) \\ &\sim (x_1, \ldots, x_{k-2}, z, x_k, x_{k+1}, \ldots x_n)\end{aligned}$$

where $z = \mu(x_{k-1}, \hat{y})$. This requires that $(x_{k-1}, \hat{y}) \in \mathcal{U}$, which again holds for $\hat{y}$ sufficiently close to e. The overlap map takes y to z, and is clearly smooth. Iterating this construction produces the overlap maps for the different (small) neighborhoods of a given n-tuple. The overlap maps between neighborhoods of two nearby elements comes from a similar construction: for example, if $(y_0, x_2, \ldots, x_n) \sim (z_0, w_2, \ldots, w_n)$ are two equivalent words lying in a first coordinate neighborhood of $(x_1, \ldots, x_n)$ and $(w_1, \ldots, w_n)$ respectively, then we can write the nearby elements in the form

$$(y, x_2, \ldots, x_n) \sim (\hat{y}, y_0, x_2, \ldots, x_n) \sim (\hat{y}, z_0, w_2, \ldots, w_n) \sim (z, w_2, \ldots, w_n),$$

where $y = \mu(\hat{y}, y_0)$ and $z = \mu(\hat{y}, z_0)$. Smoothness of the overlap map taking y to z is now clear. The smoothness of the product and inverse maps is proved similarly — the remaining details are left to the reader. ■

Remark. The local group homeomorphism $\sigma: L \to G$ constructed in the proof of Theorem 23 is defined on the entire multiplication and inversion domains — we do not need to introduce a restriction of the local group structure on L for this to work.

Theorem 23 has some interesting, non-trivial consequences, whose full import remains to be explored. For example, consider the rotation group $\mathrm{SO}(3)$, and let $S = \{R_\theta \mid 0 \le \theta < 2\pi\}$ be a one-parameter subgroup consisting of the rotations around an axis in $\mathbb{R}^3$. Let $M \subset \mathrm{SO}(3)$ be a small open neighborhood of the circular subgroup S, which is thus topologically equivalent to a solid three-dimensional torus. Let $L \to M$ denote the simply connected covering manifold of M. The local group structure on M induces a local group structure on its covering space L. However, L cannot be globalized, since it would have to be contained in the simply connected Lie group with Lie algebra $\mathfrak{so}(3)$, but this is the special unitary group $\mathrm{SU}(2)$, which is only a double cover of $\mathrm{SO}(3)$. Therefore, according to Theorem 23, L cannot be globally associative. In other words, there exists an sequence of small rotations $(R_1, \ldots, R_n) \in U^{\times n}$, where $U \subset M$ is a neighborhood of the identity, and two different n-fold products which are well-defined in M, but whose associated polygonal paths (as in the example in Section 3) define different homotopy classes in M, and hence are no longer equal when lifted up to the covering local group L. It would be interesting to produce, and understand, explicit examples exhibiting such phenomena.

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References

[1] Abel, N.H. *Untersuchung der Funktionen zweier unabhängig veränderlicher Größen x und y, wie $f(x, y)$, welche die Eigenschaft haben, daß $f(z, f(x, y))$ eine symmetrische Funktion von z, x, und y ist*, J. Reine Angew. Math. **1** (1826) 11–15.

[2] Alekseevsky, D.V., and P. W. Michor, *Differential geometry of $\mathfrak{g}$-manifolds*, Diff. Geom. Appl. **5** (1995), 371–403.

[3] Bourbaki, N., "Groupes et Algèbres de Lie," Chapitres 2–3, Diffusion C.C.L.S., Paris, 1972.

[4] —, "Lie groups and Lie algebras," Chapters 1–3, Springer-Verlag, New York etc., 1989.

[5] Brown, D., and R. S. Houston, *Cancellative semigroups on manifolds*, Semigroup Forum **35** (1987), 279–302.

[6] Cartan, É., "La Topologie des Groupes de Lie," Exposés de Géométrie No. 8, Hermann, Paris, 1936.

[7] —, *Les problèmes d'équivalence* Oeuvres Complètes; part. II, vol. 2, Gauthiers–Villars, Paris, 1952, pp. 1311–1334.

[8] Comtet, L., "Advanced Combinatorics," D. Reidel Publ., Dordrecht, 1974.

[9] Douady, A., *Un espace de Banach dont le groupe linéaire n'est pas connexe*, Proc. Konink. Nederl. Akad. Wetensch. Ser. A **68** (1965), 787–789.

[10] Gardner, R. B., "The Method of Equivalence and Its Applications," Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1989.

[11] Hofmann, K. H., *Zur Geschichte des Halbgruppenbegriffs*, Historia Math. **90** (1992), 40–59.

[12] —, *Lie groups and semigroups*, in [14] below, 3–26.

[13] Hilgert, J., K. H. Hofmann,and J. D. Lawson, "Lie Groups, Semigroups, and Convex Cones," Oxford University Press, Oxford, 1989.

[14] Hofmann, K. H., J. D. Lawson, and J. S. Pym, Eds., "The Analytical and Topological Theory of Semigroups," Exp. in Math. **1**, De Gruyter, Berlin etc., 1990.

[15] Hofmann, K. H., J. D. Lawson, and E. B. Vinberg, Eds., "Semigroups in Algebra, Geometry and Analysis," Exp. in Math. **20**, De Gruyter, Berlin etc., 1995.

[16] Hofmann, K. H., and W. Weiss, *More on cancellative semigroups on manifolds*, Semigroup Forum **37** (1988), 93–111.

[17] Jacoby, R., *Some theorems on the structure of locally compact local groups*, Ann. Math. **66** (1957), 36–69.

[18] Lie, S., "Theorie der Transformationsgruppen," Vol. 3, B. G. Teubner, Leipzig, 1893.

[19] Malcev, A. I. *Sur les groupes topologiques locaux et complets*, Comptes Rendus Acad. Sci. URSS **32** (1941), 606–608.

[20] Min–oo, and E. A. Ruh, *Comparison theorems for compact symmetric spaces*, Ann. Sci. École Norm. Sup. **12** (1979), 335–353.

[21] Montgomery, D., and L. Zippin, "Topological Transformation Groups," Instersc. Publ. New York, 1955.

[22] Mostow, G. D. *The extensibility of local Lie groups of transformations and groups on surfaces*, Ann. Math. **52** (1950), 606–636.

[23] Olver, P. J., "Applications of Lie Groups to Differential Equations," Second Edition, Graduate Texts in Mathematics, vol. **107**, Springer–Verlag, New York etc., 1993.

[24] —, "Equivalence, Invariants, and Symmetry," Cambridge University Press, 1995.

[25] Palais, R. S., "A Global Formulation of the Lie Theory of Transformation Groups," Memoirs of the Amer. Math. Soc. **22**, Providence, R. I., 1957.

[26] Plaut, C., *Metric curvature, convergence, and topological finiteness*, Duke Math. J. **66** (1992), 43–57.

[27] —, "Associativity and the local version of Hilbert's fifth problem," Notes, University of Tennesee, 1993.

[28] Pontryagin, L. S., "Topological Groups," Princeton University Press, Princeton, N. J., 1939; Gordon and Breach, New York etc., 1966.

[29] Reutenauer, C., "Free Lie Algebras," London Math. Soc. Monographs, New Series 7, Clarendon Press, Oxford, 1993.

[30] Ruh, E. A., *Almost Lie groups*, in: A. M. Gleason, Ed., Proceedings of the International Congress of Mathematicians, Berkeley, 1986; vol. 1, Amer. Math. Soc., Providence, R.I., 1987, pp. 561–564.

[31] Ruppert, W. A. F., "The Structure of Semitopological Semigroups: An Intrinsic Theory," Lecture Notes in Math. **1079**, Springer-Verlag, New York etc., 1984.

[32] Schur, F., *Über den analytischen Charakter der eine endliche continuirliche Transformationsgruppe darstellenden Functionen*, Math. Ann. **41** (1893), 509–538.

[33] Smith, P. A., *Topological foundations in the theory of continuous transformation groups*, Duke Math. J. **2** (1936), 246–279.

[34] —, Topological groups; Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950; vol. 2, L.M. Graves, E. Hille, P.A. Smith, O. Zariski, eds., Amer. Math. Soc., Providence, R.I., 1952, 436–441.

[35] Van Est, W. T.,*Local and global groups I*, Proc. Kon. Neder. Akad. v. Wetensch. **65** (1962) 391–425.

[36] Warner, F. W., "Foundations of Differentiable Manifolds and Lie Groups," Springer–Verlag, New York etc., 1983.

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