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The Weyl group as fixed point set of smooth involutions

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Abstract. We show that the Weyl group $W = M'/M$ of a noncompact semisimple Lie group is obtained by taking fixed point sets of smooth involutions in K/M. More precisely, one considers first the fixed point set X of the involutions defined on K/M by the elements of order 2 in $\exp i\mathfrak{a}$. The Weyl group is either X, or the fixed point set of the involutions defined on X by special elements of order 4 in $\exp i\mathfrak{a}$.

1. Introduction.

The primary motivation for studying the problem at hand comes from the observation ([2]) that if the Weyl group W can be obtained from K_0/M_0 by successively taking fixed point sets of smooth involutions preserving Hessenberg manifolds, then one can also reverse Morse inequalities for real Hessenberg manifolds using Floyd's theorem [4] (see Section 1 for notations, and the second remark at the end of Section 2 for the Hessenberg–preserving property). The result proved in this paper, however, is of some independent interest, and we think that it might be useful in a variety of contexts. If one removes the smoothness assumption, it is easy to obtain W as the fixed point set of a single discontinuous involution: just view K_0/M_0 as the adjoint K_0–orbit of a suitable regular element in $\mathfrak{a}_0$ and "flip" across $\mathfrak{a}_0$.

In order to explain our ideas, we now briefly discuss an example. Let the group $M_0 = \{\mathrm{diag}\,(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mid \varepsilon_j = \pm 1, \prod \varepsilon_j = 1\}$ act by conjugation on $K_0 = SO(3,\mathbb{R})$. Clearly, each $m \in M_0$ induces a smooth involution on the flag manifold K_0/M_0. The points in K_0/M_0 simultaneously fixed by all three non–trivial involutions are in correspondence to those $k \in K_0$ for which the following property holds: for all $m \in M_0$ there exists $m' \in M_0$ such that $mkm = km'$. It is immediate to see that such a k normalizes – by conjugation – the set $\mathfrak{a}_0$ of trace zero diagonal matrices. Indeed, if $H \in \mathfrak{a}_0$ and $Y = kHk^{-1}$, then Y is left fixed by all $m \in M_0$. On the other hand, this last condition is expressed by the equalities $Y_{ij} = \varepsilon_i \varepsilon_j Y_{ij}$, so that Y is itself diagonal. Thus, modulo M_0, k is a permutation, i.e. an element of the Weyl group of $SL(3,\mathbb{R})$.

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If we attempt to obtain the same result for $SL(2,\mathbb{R})$ using as set of involutions the centralizer of $\mathfrak{a}_0$ in $K_0 = SO(2,\mathbb{R})$, that is $M_0 = \{\pm \mathrm{id}\}$, we get as fixed point set in the projective space K_0/M_0 the identity coset alone. But if we use the involution defined by $\mathrm{diag}\,(i,-i)$, which still centralizes $\mathfrak{a}_0$, we achieve the target. These considerations suggest on the one hand that the right involutions should centralize $\mathfrak{a}_0$, and on the other hand that the problem should be analyzed inside some "complexification" of the semisimple Lie group G_0, that is, inside the adjoint group G of its complexified Lie algebra.
The appropriate set of involutions turns out to be $F_2 = \{f \in \exp i\mathfrak{a}_0 \mid f^2 = e\}$, where $\mathfrak{a}_0$ denotes as usual a maximal abelian subspace of the symmetric part of the Lie algebra $\mathfrak{g}_0$ of G_0. When acting as group of smooth maps on $X_0 := K_0/M_0$, however, F_2 singles out the Weyl group W as fixed point set only if the (restricted) root system associated with $(\mathfrak{g}_0, \mathfrak{a}_0)$ is reduced. Otherwise $X_1 := \mathrm{Fix}(F_2, X_0)$ contains properly W and one has to consider a special set F_4 of elements of order 4 in $\exp i\mathfrak{a}_0$. At this stage one gets equality, namely $\mathrm{Fix}(F_4, X_1) = W$. The elements of F_4 take into precise account the non–reduced roots, in a sense that will be made clear in Section 4. The nature of $\mathrm{Fix}(F_4, X_1)$, in particular the fact that the action of F_4 on X_1 is well–defined, is a slightly delicate matter, and is best understood via the Bruhat decomposition. The key step (Theorem 10) is proved by using basic properties of the Bruhat decomposition (Crollary 5.3) and $SU(2,1)$–reduction (Lemma 5.6).

2. Preliminaries and notation.

Let G_0 be a semisimple, connected, non–compact Lie group with finite center, $\mathfrak{g}_0$ its Lie algebra, and $\mathfrak{g} = \mathfrak{g}_0^c$ its complexification viewed as real Lie algebra. Thus $\mathfrak{g}$ is semisimple. Denote by σ the automorphism of $\mathfrak{g}$ corresponding to conjugation with respect to $\mathfrak{g}_0$, i.e. $\sigma : X + iY \mapsto X - iY$ for $X, Y \in \mathfrak{g}_0$.
Let ad denote the adjoint representation of $\mathfrak{g}$. We then have Lie algebra inclusions $\mathrm{ad}\,\mathfrak{g}_0 \subset \mathrm{ad}\,\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$. Let $G = \mathrm{Int}(\mathfrak{g})$ be the adjoint group of $\mathfrak{g}$, i.e. the connected Lie subgroup of $GL(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})$ correspoding to $\mathrm{ad}\,\mathfrak{g}$. If G_* denotes the connected Lie subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to $\mathrm{ad}\,\mathfrak{g}_0$, then G_* is a closed Lie subgroup of G diffeomorphic to $\mathrm{Int}(\mathfrak{g}_0)$, the adjoint group of $\mathfrak{g}_0$ ([6], Lemma 6.2, Ch. III, p.181). The adjoint representation of G_0 maps G_0 onto G_* with kernel Z_0, the center of G_0. Thus $G_* \simeq G_0/Z_0$.
Let now θ be a Cartan involution of $\mathfrak{g}_0$ and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ the resulting Cartan decomposition. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$ of dimension say l. Call a linear functional $\alpha \in \mathfrak{a}_0^*$ a *restricted* root if

$$\mathfrak{g}_{0\alpha} = \{X \in \mathfrak{g}_0 \mid [H, X] = \alpha(H)X \ \forall\, H \in \mathfrak{a}_0\} \neq 0.$$

The set of non–zero restricted roots (resp. positive, simple) will be denoted by Σ (resp. Σ^+, Δ). The simultaneous diagonalization of all the $X \mapsto [H, X] \in \mathrm{End}(\mathfrak{g}_0)$ with $H \in \mathfrak{a}_0$ leads to the root–space decomposition of $\mathfrak{g}_0$

$$\mathfrak{g}_0 = \mathfrak{g}_{00} + \sum_{\alpha\in\Sigma} \mathfrak{g}_{0\alpha}.$$

In turn,

$$\mathfrak{g}_{00} = \mathfrak{a}_0 + \mathfrak{m}_0,$$

where $\mathfrak{m}_0 = \{X \in \mathfrak{k}_0 \mid [X, H] = 0 \, \forall H \in \mathfrak{a}_0\}$ is the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}$. Put

$$\mathfrak{n}_0 = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{0\alpha},$$

so that the Iwasawa decomposition of $\mathfrak{g}_0$ reads:

$$\mathfrak{g}_0 = \mathfrak{n}_0 + \mathfrak{a}_0 + \mathfrak{k}_0.$$

Let now N_0, A_0 and K_0 denote the Lie subgroup of G_0 corresponding to $\mathfrak{k}_0$, $\mathfrak{a}_0$ and $\mathfrak{n}_0$. Thus $G_0 = N_0 A_0 K_0$ is the Iwasawa decomposition of G_0. Here K_0 is a maximal compact subgroup of G_0, A_0 is abelian and N_0 is nilpotent. Moreover, there exists an involutive automorphism Θ of G_0 with $d\Theta = \theta$, such that K_0 is the set of points fixed by Θ. Let M_0 and M_0' denote respectively the centralizer and normalizer of $\mathfrak{a}_0$ in K_0, i.e. $M_0 = \{m \in K_0 \mid \operatorname{Ad} m(H) = H, \, \forall H \in \mathfrak{a}_0\}$ and $M_0' = \{m \in K_0 \mid \operatorname{Ad} m(H) \in \mathfrak{a}_0, \, \forall H \in \mathfrak{a}_0\}$. Here Ad, stands for the adjoint representation of G_0. The Lie algebras of M_0 and M_0' coincide and are equal to $\mathfrak{m}_0$. The finite group $W = M_0'/M_0$ is the Weyl group of G_0 associated to the previous data. Clearly, W sits inside the boundary K_0/M_0.

3. The case of reduced root systems.

The Cartan involution θ extends in a unique fashion to an involution of $\mathfrak{g}$, also denoted by θ ([7], Ch. III, p. 368), and there exists an involutive automorphism Θ of G such that $d\Theta = \theta$. Let K_Θ denote the Lie subgroup of G of fixed points of Θ and by K its identity component. Put

$$F_2 = \left\{ f \in \exp i\mathfrak{a}_0 \mid f^2 = e \right\} \subset G.$$

It is easy to give an explicit description of F_2 ([7], ex. 7 p. 384). Indeed, if $\{H_1, \ldots, H_l\}$ is the basis of $\mathfrak{a}_0$ dual to Δ, then:

$$F_2 = \left\{ \exp i\pi \sum_{j=1}^{l} \nu_j H_j \mid \nu_j = 0, 1 \right\}.$$

Thus card $F_2 = 2^l$. Observe also that if $f = \exp iA \in F_2$, then

$$\Theta f = \Theta \exp iA = \exp(\theta iA) = \exp(-iA) = f^{-1} = f,$$

so that $F_2 \subset K_\Theta$. More precisely, F_2 is the group of components of K_Θ, i.e. ([8])

$$K_\Theta = K \cdot F_2.$$

The first step of our construction consists in showing that F_2 acts on $X_0 = K_0/M_0$ by smooth involutions. The fixed point set $X_1 = \operatorname{Fix}(F_2, X_0)$ is in many cases W, but not always. If the root system Σ is not reduced, then X_1 is an "intermediate" manifold between K_0/M_0 and W.
¿From now until the end of this section, we will use the subscript $*$ to indicate images under the adjoint representation Ad of G_0.

Proposition 3.1. *If $f \in F_2$, , then $fK_*f = K_*$.*

Proof. Let $fK_*f = K_f$, and let $\mathfrak{k}_f$ be its Lie algebra in $\mathfrak{g}$. It is clear that $\mathfrak{k}_f = \mathrm{Ad}_G f\mathfrak{k}_0$. Therefore, $\mathfrak{k}_f$ is θ–invariant. If $f = \exp iA$, and $T_0 \in \mathfrak{k}_0$, then:

$$\begin{aligned}\sigma(\mathrm{Ad}_G fT_0) &= \sigma(\mathrm{Ad}_G(\exp iA)T_0) \\ &= \mathrm{Ad}_G(\exp\sigma(iA))(\sigma T_0) \\ &= \mathrm{Ad}_G(\exp -iA)T_0 \\ &= \mathrm{Ad}_G fT_0.\end{aligned}$$

Thus $\mathrm{Ad}_G fT_0 \in \mathfrak{g}_0 \cap \mathfrak{k} = \mathfrak{k}_0$. This shows $\mathfrak{k}_f = \mathfrak{k}_0$, thereby proving the Proposition, since K_* is the connected subgroup of G corresponding to the Lie subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}$. ■

Proposition 3.2. *F_2 acts on K_0/M_0 by:*

$$f \cdot \langle k \rangle = \langle \mathrm{Ad}^{-1}(fk_*f) \rangle, \tag{1}$$

$\langle\cdot\rangle$ denoting the class mod M_0 and Ad *the adjoint representation of G_0. Moreover, as maps, all the f's commute with each other and $f^2 = \mathrm{id}$.*

Proof. First of all notice that $\mathrm{Ad}^{-1} e = Z_0$, the center of G_0. But Z_0 is contained in M_0, so that all the elements in $\mathrm{Ad}^{-1} x$ belong to the same M_0 coset. Since F_2 centralizes M_*, for $k \in K_0$ and $m \in M_0$, we have:

$$\begin{aligned}f(km)_*f &= fk_*m_*f \\ &= (fk_*f)(fm_*f) \\ &= (k')_*m_* \\ &= (k'm)_*,\end{aligned}$$

where $k'_* = fk_*f \in K_*$ because of the previous Proposition. Thus:

$$\mathrm{Ad}^{-1}(f(km)_*f) = k'mZ_0$$

and $\langle k'mZ_0 \rangle = \langle k' \rangle$ depends only on $\langle k \rangle$. The remaining assertions are clear, since F_2 is abelian in G. ■

Denote by $\mathrm{Fix}(F_2, X_0)$ the set of points in $X_0 = K_0/M_0$ which are simultaneously fixed by all $f \in F_2$. It is clear that $\mathrm{Fix}(F_2, X_0)$ is a smooth manifold. Indeed, if $F_2 = \{f_1, \ldots, f_N\}$, $N = 2^l$, let $X_0^1 = \mathrm{Fix}(f_1, X_0)$ and for $1 \leq j \leq N$ put $X_0^j = \mathrm{Fix}(f_j, X_0^{j-1})$, (here $X_0^0 = X_0$). Then X_0^1 is a smooth manifold because it is the image under $\pi : K_0 \to K_0/M_0$ of $K_0^1 = \{k \in K_0 \mid k^{-1}f_1kf_1 \in M_0\}$. Similarly, X_0^j is a smooth manifold because it is the image under $\pi : K_0 \to K_0/M_0$ of $K_0^j = \{k \in K_0^{j-1} \mid k^{-1}f_jkf_j \in M_0\}$. But $X_0^N = \mathrm{Fix}(F_2, X_0)$ because all the f's commute as maps.

Theorem 3.3. *If Σ is reduced, then $\mathrm{Fix}(F_2, X_0) = W$.*

Proof. Let $\overline{H} \in \mathfrak{a}_0$ be a regular element, and let $\langle k \rangle \in \mathrm{Fix}(F_2, X_0)$. This means that given $f \in F_2$ there exists $m \in M_0$ such that:

$$fk_* f = k_* m_*.$$

Then $Y := \mathrm{Ad}\, k_* \overline{H} = \mathrm{Ad}\, k\overline{H} \in \mathfrak{p}_0$ depends only on $\langle k \rangle$ and we may write:

$$Y = Y_0 + \sum_{\alpha \in \Sigma} Y_\alpha,$$

where $Y_0 \in \mathfrak{m}_0$ and $Y_\alpha \in \mathfrak{g}_\alpha$. Since f centralizes $\mathfrak{a}_0$,

$$\mathrm{Ad}\, fY = \mathrm{Ad}\, fk_* f\overline{H} = \mathrm{Ad}\, k_* m_* \overline{H} = \mathrm{Ad}\, k_* \overline{H} = Y,$$

so that Y is fixed by all $f \in F_2$. Therefore, if $f = \exp iA_0$,

$$Y_\alpha = (\mathrm{Ad} \exp iA_0 Y)_\alpha = (e^{\mathrm{ad}\, iA_0} Y)_\alpha = e^{i\alpha(A_0)} Y_\alpha.$$

Fix now α and write

$$\alpha = \sum_{\delta \in \Delta} \nu_\delta(\alpha)\delta.$$

If Σ is reduced, there are no roots α for which $\nu_\delta(\alpha)$ is even for all δ ([1]). Thus $\nu_{\overline{\delta}}(\alpha)$ is odd for at least one simple restricted root $\overline{\delta}$. Select $A_0 = \pi H_{\overline{\delta}}$, so that $\alpha(A_0)$ is an odd multiple of π. It follows that $Y_\alpha = -Y_\alpha = 0$. This shows that $Y = Y_0 \in \mathfrak{a}_0$, namely that $k \in M_0'$. Thus $\langle k \rangle \in W$ and $\mathrm{Fix}(F_2, X_0) \subset W$. The reverse inclusion is obvious. ■

Remarks i) It is clear from the proof of 3.3 that if the root system Σ is not reduced, $\langle k \rangle \in \mathrm{Fix}(F_2, X_0)$ and $\overline{H} \in \mathfrak{a}_0$ is a regular element, then $Y = \mathrm{Ad}\, k\overline{H} = Y_0 + \sum_{\alpha \in E} Y_\alpha$, where E is the set of *even* roots, namely those roots $\alpha = \sum_{\delta \in \Delta} \nu_\delta(\alpha)\delta$ for which $\nu_\delta(\alpha) \in 2\mathbb{Z}$ for all $\delta \in \Delta$.

ii) The involutions defined by F_2 have the additional property of preserving Hessenberg manifolds. We recall ([2], [3]) that for a fixed regular element $\overline{H} \in \mathfrak{a}_0$, a (real) Hessenberg manifold $\mathrm{Hess}_{\mathcal{R}}(\overline{H})$ is defined for any subset $\mathcal{R}$ of the set Σ^- of negative roots having the following *Hessenberg property*:

$$\alpha \in \mathcal{R}, \quad \beta \in \Sigma^+, \quad \alpha + \beta \in \Sigma^- \Longrightarrow \alpha + \beta \in \mathcal{R}.$$

One then defines the Hessenberg subspaces $\mathfrak{p}_0(\mathcal{R})$ of $\mathfrak{p}_0$ by summing vectors of the form $X_\alpha - \theta X_\alpha$ with $\alpha \in \mathcal{R}$. More precisely, set

$$\mathfrak{p}_0(\mathcal{R}) = \mathfrak{a}_0 + \sum_{\alpha \in \mathcal{R}} ((\mathfrak{g}_{0,\alpha} + \mathfrak{g}_{0,-\alpha}) \cap \mathfrak{p}_0).$$

Finally,

$$\operatorname*{Hess}_{\mathcal{R}}(\overline{H}) = \left\{ \langle k \rangle \in K_0/M_0 \;\mid\; \mathrm{Ad}\, k^{-1}\overline{H} \in \mathfrak{p}_0(\mathcal{R}) \right\}.$$

It is clear that since F_2 centralizes $\mathfrak{a}_0$, it maps $\mathrm{Hess}_{\mathcal{R}}(\overline{H})$ into itself. A similar situation occurs for the action of F_4 which will be presented in Section 5.

We will simplify the notation thinking of all the subgroups of G_0 as subgroups of G under the adjoint representation. In this process, the center becomes trivial. Thus we suppress the subscripts $*$ and write, for example, K_0 in place of K_* whenever no confusion arises.

4. The Weyl group of SU(2,1).

As we will see in the next section, the general case is handled by reducing the problem to an $SU(2,1)$ computation, via the "Bruhat decomposition" of K_0. We therefore analyze this group in full detail.
Recall that $G_0 = SU(2,1)$ consists of those elements in $GL(3,\mathbb{C})$ having determinant equal to one and satisfying $g^* I_{2,1} g = I_{2,1}$, where

$$I_{2,1} = \begin{bmatrix} -I_2 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Lie algebra $\mathfrak{g}_0$ of G_0 is therefore

$$\begin{aligned} \mathfrak{su}(2,1) &= \{X \in \mathfrak{gl}(3,\mathbb{C}) \mid X^* I_{2,1} + I_{2,1} X = 0,\ \mathrm{tr}\ X = 0\} \\ &= \left\{ \begin{bmatrix} A & B \\ B^* & -\mathrm{tr}\ A \end{bmatrix} \mid A \in \mathfrak{u}(2),\ B \in M_{2,1}(\mathbb{C}) \right\}, \end{aligned}$$

where evidently $\mathfrak{u}(2)$ is the Lie algebra of 2×2 skew–hermitian matrices. Let θ and Θ denote the Cartan involutions on $\mathfrak{g}_0$ and G_0 respectively, so that $\theta = d\Theta$. Then:

$$\theta X = I_{2,1} X I_{2,1}, \qquad \Theta g = I_{2,1} g I_{2,1}.$$

Consequently,

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & -\mathrm{tr}\ A \end{bmatrix} \mid A \in \mathfrak{u}(2) \right\},$$

$$\mathfrak{p}_0 = \left\{ \begin{bmatrix} 0 & {}^t\zeta \\ \overline{\zeta} & 0 \end{bmatrix} \mid \zeta \in \mathbb{C}^2 \right\}.$$

A maximal abelian suspace of $\mathfrak{p}_0$ is:

$$\mathfrak{a}_0 = \left\{ tH_0 \mid H_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ t \in \mathbb{R} \right\}.$$

Next, a maximal compact subgroup of G_0 corresponding to $\mathfrak{k}_0$ is:

$$K_0 = S(U_2 \times U_1) = \left\{ \begin{bmatrix} A & 0 \\ 0 & e^{it} \end{bmatrix} \mid A \in U(2),\ \det A = e^{-it},\ t \in \mathbb{R} \right\}.$$

The centralizer M_0 of $\mathfrak{a}_0$ in K_0 is

$$M_0 = \left\{ \begin{bmatrix} e^{is} & 0 & 0 \\ 0 & e^{-2is} & 0 \\ 0 & 0 & e^{is} \end{bmatrix} \mid s \in \mathbb{R} \right\},$$

whereas the normalizer M_0' of $\mathfrak{a}_0$ in K_0 is

$$M_0' = \left\{ \begin{bmatrix} \varepsilon e^{is} & 0 & 0 \\ 0 & \varepsilon e^{-2is} & 0 \\ 0 & 0 & e^{is} \end{bmatrix} \mid s \in \mathbb{R},\ \varepsilon = \pm 1 \right\}.$$

It follows that the Weyl group $W = M_0'/M_0$ is (isomorphic to) the two–element group.
The root–space structure of $SU(2,1)$ is easily written. Indeed, $\mathfrak{g}_{00} = \mathfrak{a}_0 + \mathfrak{m}_0$ with

$$\mathfrak{m}_0 = \left\{ tT_0 \mid T_0 = \begin{bmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{bmatrix},\ t \in \mathbb{R} \right\},$$

and if $\alpha : \mathfrak{a}_0 \to \mathbb{R}$ is the linear functional defined by $tH_0 \mapsto t$, then:

$$\mathfrak{g}_{0,\alpha} = \left\{ \begin{bmatrix} 0 & z & 0 \\ -\overline{z} & 0 & \overline{z} \\ 0 & z & 0 \end{bmatrix},\ z \in \mathbb{C} \right\}, \quad \mathfrak{g}_{0,-\alpha} = \left\{ \begin{bmatrix} 0 & z & 0 \\ -\overline{z} & 0 & -\overline{z} \\ 0 & -z & 0 \end{bmatrix},\ z \in \mathbb{C} \right\},$$

$$\mathfrak{g}_{0,2\alpha} = \left\{ \begin{bmatrix} it & 0 & -it \\ 0 & 0 & 0 \\ it & 0 & -it \end{bmatrix},\ t \in \mathbb{R} \right\}, \quad \mathfrak{g}_{0,-2\alpha} = \left\{ \begin{bmatrix} it & 0 & it \\ 0 & 0 & 0 \\ -it & 0 & -it \end{bmatrix},\ t \in \mathbb{R} \right\}.$$

Taking exponentials in $GL(3,\mathbb{C})$, we have

$$\exp i\mathfrak{a}_0 = \left\{ \begin{bmatrix} \cos t & 0 & i\sin t \\ 0 & 1 & 0 \\ i\sin t & 0 & \cos t \end{bmatrix} \mid t \in \mathbb{R} \right\},$$

so that

$$F_2 = \left\{ f \in \exp i\mathfrak{a}_0 \mid f^2 = 1 \right\} = \left\{ \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \mid \varepsilon = \pm 1 \right\}.$$

Observe that:

$$\exp i\pi H_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We will also need the following set of elements of order 4 in $\exp i\mathfrak{a}_0$: $F_4 = \{f \in \exp i\mathfrak{a}_0 \mid f^2 = \exp i\pi H_0\}$. Clearly:

$$F_4 = \left\{ \begin{bmatrix} 0 & 0 & i\varepsilon \\ 0 & 1 & 0 \\ i\varepsilon & 0 & 0 \end{bmatrix} \mid \varepsilon = \pm 1 \right\}.$$

Next we analyze $X_1 := \mathrm{Fix}(F_2, K_0/M_0)$. To this end, let $\langle k \rangle \in X_1$. If $k = \begin{bmatrix} A & 0 \\ 0 & e^{it} \end{bmatrix}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(2)$, and $f = f_\varepsilon \in F_2$, then there exists $m = m_s(\varepsilon) \in M_0$ such that $fkf = km$. Since

$$fkf = \begin{bmatrix} a & \varepsilon b & 0 \\ \varepsilon c & d & 0 \\ 0 & 0 & e^{it} \end{bmatrix}, \quad \text{and} \quad km = \begin{bmatrix} ae^{is} & be^{-2is} & 0 \\ ce^{is} & de^{-2is} & 0 \\ 0 & 0 & e^{it}e^{is} \end{bmatrix},$$

we immediately obtain $e^{is} = 1$, i.e. $m_s(\varepsilon) = \mathrm{id} = e$ independently of ε. Choosing $\varepsilon = -1$, we see that A must be diagonal, i.e.,

$$k = \begin{bmatrix} e^{iu} & 0 & 0 \\ 0 & e^{iv} & 0 \\ 0 & 0 & e^{it} \end{bmatrix} \qquad u + v + t \in 2\pi\mathbb{Z}.$$

We stress that we have proved that if $\langle k \rangle \in X_1$, then $fkf = k$ for all $f \in F_2$, a very special situation which will be used in the proof of 5.6. Nonetheless, $X_1 \neq W$. In order to obtain W as the fixed point set of (smooth) involutions, we need to consider the action of F_4 on $\mathrm{Fix}(F_2, K_0/M_0)$. This fact illustrates a general situation. Observe that if $\langle k \rangle \in X_1$ and $f \in F_4$, then:

$$fkf^{-1} = \begin{bmatrix} 0 & 0 & i\varepsilon \\ 0 & 1 & 0 \\ i\varepsilon & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{iu} & 0 & 0 \\ 0 & e^{iv} & 0 \\ 0 & 0 & e^{it} \end{bmatrix} \begin{bmatrix} 0 & 0 & -i\varepsilon \\ 0 & 1 & 0 \\ -i\varepsilon & 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & e^{iv} & 0 \\ 0 & 0 & e^{iu} \end{bmatrix},$$

so that F_4 sends fixed points into fixed points. Now, the requirement that $\langle k \rangle \in \mathrm{Fix}(F_4, X_1)$ is equivalent to asking that, given ε, there exists $m \in M_0$ such that $fkf^{-1} = km$. Since

$$km = \begin{bmatrix} e^{i(u+s)} & 0 & 0 \\ 0 & e^{i(v-2s)} & 0 \\ 0 & 0 & e^{i(t+s)} \end{bmatrix},$$

we obtain the system

$$\begin{cases} u + s = t + 2n_1\pi \\ v - 2s = v + 2n_2\pi \\ t + s = u + 2n_3\pi \end{cases}$$

for some integers n_1, n_2 and n_3, that is:

$$s = n\pi, \qquad u = t + n'\pi,$$

for some integers n and n'. If n is odd, then $m = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, and $fkf = km$ implies $e^{it} = -e^{iu}$, whence $v = (2n'' + 1)\pi - 2u$, and so:

$$k = \begin{bmatrix} -e^{it} & 0 & 0 \\ 0 & -e^{-2it} & 0 \\ 0 & 0 & e^{it} \end{bmatrix} \in M_0'.$$

Finally, if n is even, then $m = \mathrm{id}$ and $fkf = km$ forces $e^{it} = e^{iu}$, whence $e^{iv} = -e^{-2it}$, i.e.

$$k = \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & e^{-2it} & 0 \\ 0 & 0 & e^{it} \end{bmatrix} \in M_0 \subset M_0'.$$

This shows that $\mathrm{Fix}(F_4, X_1) = W$.

For the reader's convenience, we conclude this section by briefly recalling the general structure of a root system of type $(BC)_l$ because it will be used in Section 5. The root system associated to $SU(2,1)$ is of course of type $(BC)_1$. Let e_j denote the j^{th} standard basis element in $\mathbb{R}^l$. The rank l root system of type $(BC)_l$ is (isomorphic to):

$$(BC)_l = \{\pm e_i \pm e_j,\ 1 \leq i < j \leq l,\ \pm e_j,\ 1 \leq j \leq l,\ \pm 2e_j,\ 1 \leq j \leq l\}.$$

Slightly modifying the standard notation ([6], Theorem 3.25, Ch. X, p. 475), a basis $\Delta = \{\delta_0, \delta_1, \ldots \delta_{l-1}\}$ of simple roots is given by:

$$\delta_0 = e_l, \qquad \delta_j = e_j - e_{j+1}, \quad 1 \leq j \leq l-1.$$

Finally we list the expression of the positive roots in terms of the basis elements; here $1 \leq i < j \leq l$, and if $j = l$ the sum $\delta_j + \ldots + \delta_{l-1}$ is zero:

$$\begin{aligned}
&e_i - e_j = \delta_i + \ldots + \delta_{j-1};\\
&e_j = \delta_0 + (\delta_j + \ldots + \delta_{l-1});\\
&2e_j = 2\delta_0 + 2(\delta_j + \ldots + \delta_{l-1});\\
&e_i + e_j = (\delta_i + \ldots + \delta_{j-1}) + 2\delta_0 + 2(\delta_j + \ldots + \delta_{l-1}).
\end{aligned}$$

Using the terminology introduced in the first remark following Theorem 3, the only even roots in $(BC)_l$ are $2e_j$, for $j = 1, \ldots, l$.

5. Bruhat decomposition and $SU(2,1)$ reduction. The general case.

We recall that if $B_0 = N_0 A_0 M_0$, then the Bruhat decomposition of G_0 is the disjoint union:

$$G_0 = \coprod_{w \in W} B_0 w B_0.$$

A better parametrization of the double cosets $B_0 w B_0$ may be achieved as follows (see [5]). For $w \in W$, let $\Sigma_w^+ = \{\alpha \in \Sigma^+ \mid -w\alpha \in \Sigma^+\}$, and set

$$\overline{\mathfrak{n}}_w = \sum_{\alpha \in \Sigma_w^+} \mathfrak{g}_{w\alpha}, \quad N_w^- = \exp \overline{\mathfrak{n}}_w.$$

Then the Bruhat decomposition may be rewritten as

$$G_0 = \coprod_{w \in W} (N_0 A_0) N_w^- w M_0,$$

giving rise to a "Bruhat decomposition" of K_0:

$$K_0 = \coprod_{w \in W} k(N_w^-) w M_0,$$

where $k(\cdot)$ refers to the K_0–coordinate function in the Iwasawa decomposition $G_0 = N_0A_0K_0$. When we write $k = k(\overline{n})wm$, we think of w as a fixed representative in M_0'.
We will write $\mathcal{B}(w) = B_0wB_0 = (N_0A_0)N_w^- wM_0$ and $\mathcal{C}(w) = k(N_w^-)wM_0$, and refer to $\mathcal{B}(w)$ and $\mathcal{C}(w)$ as the Bruhat *cells* of G_0 and K_0, respectively. Observe that:

$$\mathcal{C} = k(\mathcal{B}).$$

Next, we recall ([1]) a few crucial properties enjoied by the cells $\mathcal{B}(w)$, and show that they hold for the cells $\mathcal{C}(w)$ as well.
Let S be a set of generators of W, with $e \notin S$. Let $s \in S$ and $w \in W$, then

$$\mathcal{B}(s)\mathcal{B}(w) = \begin{cases} \mathcal{B}(sw) & \text{if} \quad \mathcal{B}(w) \not\subset \mathcal{B}(s)\mathcal{B}(w) \\ \mathcal{B}(w) \cup \mathcal{B}(sw) & \text{if} \quad \mathcal{B}(w) \subset \mathcal{B}(s)\mathcal{B}(w) \end{cases} \tag{2}$$

Let $w = s_1 \cdot \ldots \cdot s_p$ be a reduced expression in terms of generators, and let $v \in W$; then

$$\mathcal{B}(s_1 \cdot \ldots \cdot s_p)\mathcal{B}(v) \subset \coprod_{1 \leq i_1 \leq \ldots \leq i_t \leq p} \mathcal{B}(s_{i_1} \cdot \ldots \cdot s_{i_t} v), \tag{3}$$

where $(i_1, \ldots, i_t)$ ranges over all – possibly empty – t-uples of increasing integers in the interval $[1, p]$. Finally, let $l_S(\cdot)$ denote the length function on W with respect to the set S of generators of W. Let $w_1, \ldots, w_p \in W$ and suppose that $l_S(w) = l_S(w_1) + \ldots + l_S(w_p)$, then $\mathcal{B}(w) = \mathcal{B}(w_1) \cdot \ldots \cdot \mathcal{B}(w_p)$. In particular, if $w = s_1 \cdot \ldots \cdot s_p$ is a reduced expression of w in terms of generators, then:

$$\mathcal{B}(w) = \mathcal{B}(s_1) \cdot \ldots \cdot \mathcal{B}(s_p). \tag{4}$$

Our first concern will be to show that 2, 3 and 4 hold for $\mathcal{C}(w)$ as well.

Lemma 5.1. *Let $u, w \in W$. Then $k(\mathcal{B}(u)) \cdot k(\mathcal{B}(w)) = k(\mathcal{B}(u) \cdot \mathcal{B}(w))$.*

Proof. Let $x \in \mathcal{B}(u)$ and $y \in \mathcal{B}(w)$. Write $x = b_1ub_2$, $b_i \in B_0$, and $y = na \cdot k(y)$. Then $xy = b_1ub_2na \cdot k(y) = (b_1ub_2') \cdot k(y)$. Since $b_1ub_2' \in \mathcal{B}(u)$, $b_1ub_2' = n'a'k'$ with $k' \in k(\mathcal{B}(u))$. Thus $xy = n'a'k' \cdot k(y)$, so that $k(xy) = k' \cdot k(y) \in k(\mathcal{B}(u)) \cdot k(\mathcal{B}(w))$, thereby showing $k(\mathcal{B}(u) \cdot \mathcal{B}(w)) \subset k(\mathcal{B}(u)) \cdot k(\mathcal{B}(w))$. On the other hand, $k(\mathcal{B}(u)) \cdot k(\mathcal{B}(w)) = k(\mathcal{B}(u) \cdot k(\mathcal{B}(w))) \subset k(\mathcal{B}(u) \cdot \mathcal{B}(w))$, and the lemma is proved. ∎

Lemma 5.2. *$k(\mathcal{B}) \subset \mathcal{B}$ and $N_0A_0 \cdot k(\mathcal{B}) = \mathcal{B}$.*

Proof. As for the first statement, let $nak \in \mathcal{B} = B_0wB_0$ and put $b = (na)^{-1} \in N_0A_0M_0 = B_0$. Then $k = b(nak) \in B_0\mathcal{B} = \mathcal{B}$. As for the second, $N_0A_0 \cdot k(\mathcal{B}) \subset B_0 \cdot k(\mathcal{B}) \subset B_0\mathcal{B} = \mathcal{B}$, and $\mathcal{B} \subset n(\mathcal{B}) \cdot a(\mathcal{B}) \cdot k(\mathcal{B}) \subset N_0A_0 \cdot k(\mathcal{B})$. ∎

Corollary 5.3. *Formulae (2), (3) and (4) hold for the Bruhat cells in K_0.*

Proof. Suppose $\mathcal{C}(w) \subset \mathcal{C}(s)\mathcal{C}(w)$, i.e. $k(\mathcal{B}(w)) \subset k(\mathcal{B}(s))k(\mathcal{B}(w))$. Then $N_0A_0 \cdot k(\mathcal{B}(w)) \subset N_0A_0 \cdot k(\mathcal{B}(s))N_0A_0 \cdot k(\mathcal{B}(w))$, that is, by Lemma 5, $\mathcal{B}(w) \subset \mathcal{B}(s)\mathcal{B}(w)$. Thus (2) gives $\mathcal{B}(s)\mathcal{B}(w) = \mathcal{B}(w) \cup \mathcal{B}(sw)$, so that, applying Lemma 4 and observing that $k(A \cup B) = k(A) \cup k(B)$, we obtain $\mathcal{C}(s)\mathcal{C}(w) = \mathcal{C}(w) \cup \mathcal{C}(sw)$. If instead $\mathcal{C}(w) \not\subset \mathcal{C}(s)\mathcal{C}(w)$, then necessarily $\mathcal{B}(w) \not\subset \mathcal{B}(s)\mathcal{B}(w)$, otherwise taking $k(\cdot)$ and applying Lemma 5.1 we would obtain the inclusion which negates our assumption. But then, by (2), $\mathcal{B}(s)\mathcal{B}(w) = \mathcal{B}(sw)$, which yields $\mathcal{C}(s)\mathcal{C}(w) = \mathcal{C}(sw)$. This proves that 2 holds for the cells in K_0. As for 3 and 4, simply apply Lemma 5.1. ∎

Next we analyze the action of F_2 on the Bruhat cells $\mathcal{C}(w)$. We simply write fkf in place of (1), and keep in mind the convention established at the end of Section 2.

Lemma 5.4. *F_2 leaves each Bruhat cell $\mathcal{C}(w)$ invariant. In particular, if $f \in F_2$ and $k = k(\overline{n}_w)wm \in \mathcal{C}(w)$:*

$$fkf = k(f\overline{n}_wf)w^fm,$$

where $f\overline{n}_wf \in N_w^-$, $w^f = w \cdot m(f,w) \in M_0'$, and $m(f,w) \in M_0$.

Proof. Let $\overline{n}_w = nau$ be the Iwasawa decomposition of $\overline{n}_w$, so that $u = k(\overline{n}_w)$. Then $f\overline{n}_wf = (fnf)(faf)(fuf) = n'a(fuf)$, where clearly

$$fnf = \exp \sum_{\alpha\in\Sigma^+} \mathrm{Ad}\, fX_\alpha = \sum_{\alpha\in\Sigma^+} e^{i\pi\alpha(A)}X_\alpha \in N_0.$$

Therefore $k(f\overline{n}_wf) = fk(\overline{n}_w)f$.
Moreover, if $f = \exp i\pi A$, and $\overline{n}_w = \exp \sum_{\alpha\in\Sigma_w^+} X_{w\alpha}$, then

$$f\overline{n}_wf = \exp \sum_{\alpha\in\Sigma_w^+} \mathrm{Ad}\, fX_{w\alpha} = \exp \sum_{\alpha\in\Sigma_w^+} e^{i\pi w\alpha(A)}X_\alpha \in N_w^-.$$

so that $f\overline{n}_wf \in N_w^-$. Next, it is clear that since f centralizes M_0, $w^f = fwf$ normalizes $\mathfrak{a}_0$:

$$\mathrm{Ad}\, w^fA = \mathrm{Ad}\, fwfA = \mathrm{Ad}\, fwA = \mathrm{Ad}\, wA \in \mathfrak{a}_0,$$

which also shows that $w^f \in M_0'$ coincides with w modulo M_0, i.e. $w^f = wm(f,w)$. The result follows, since $fkf = (fk(\overline{n}_w)f)(fwf)(fmf)$ and $fmf = m$. ∎

Let now $S = \{s_1, \ldots, s_l\}$ be the set of generators of W consisting of all the reflections associated with the simple roots $\Delta = \{\delta_1, \ldots, \delta_l\}$. Recall that $\{H_1, \ldots, H_l\}$ is the dual basis of Δ. Thus to each $s \in S$ there corresponds a unique H of the basis, and viceversa.

Lemma 5.5. *Let $s \neq s_0$, with $s, s_0 \in S$. Let $f_0 = \exp i\pi H_0 \in F_2$, where H_0 is the element corresponding to s_0. Then f_0 leaves $\mathcal{C}(s)$ pointwise fixed.*

Proof. Let $k \in \mathcal{C}(s)$. Thanks to Lemma (5.4), if $k = k(\overline{n}_s)sm$, then $f_0 k f_0 = k(f_0 \overline{n}_s f_0) s^{f_0} m$. Now, Σ_s^+ consists of the integral multiples of δ in Σ^+. Indeed, s permutes the positive roots which are not multiples of δ, and sends δ (resp. 2δ) to $-\delta$ (resp. -2δ). It follows that $\overline{n}_s = \exp(X_{-\delta} + X_{-2\delta})$, where we agree that $X_{-2\delta} = 0$ if 2δ is not a root. Therefore:

$$\begin{aligned}
f_0 \overline{n}_s f_0 &= f_0 \exp(X_{-\delta} + X_{-2\delta}) f_0 \\
&= \exp \operatorname{Ad} f_0 (X_{-\delta} + X_{-2\delta}) \\
&= \exp \operatorname{Ad}(\exp i\pi H_0)(X_{-\delta} + X_{-2\delta}) \\
&= \exp e^{\operatorname{ad} i\pi H_0}(X_{-\delta} + X_{-2\delta}) \\
&= \exp(e^{-i\pi\delta(H_0)} X_{-\delta} + e^{-i\pi 2\delta(H_0)} X_{-2\delta}) \\
&= \exp(X_{-\delta} + X_{-2\delta}). \\
&= \overline{n}_s
\end{aligned}$$

Moreover, $\operatorname{Ad} sH_0 = \sum_{j=1}^{l} \nu_j H_j$ for some coefficients $\nu_1, \ldots, \nu_l$. But if δ corresponds to s, from the fact that $\delta(H_0) = 0$ it follows:

$$\nu_j = \delta_j(\operatorname{Ad} sH_0) = s \cdot \delta_j(H_0) = (\delta_j - c_{\delta_j,\delta}\delta)H_0 = \delta_j(H_0),$$

namely $\operatorname{Ad} sH_0 = H_0$. Thus

$$s f_0 s = \exp i\pi(\operatorname{Ad} sH_0) = \exp i\pi H_0 = f_0,$$

that is $s^{f_0} = s$. ∎

Let now $\Sigma = \cup_j \Sigma^j$ be the decomposition of Σ into irreducible root systems, and assume Σ non–reduced. Then at least one of the Σ^j is non–reduced, hence of type $(BC)_l$ ([6], Theorem 3.25, Ch. X, p. 475). Let $\Sigma_0 = \cup_j \Sigma_0^j$ denote the collection of all such subsystems, let δ_0^j be the only simple root in Σ_0^j such that $2\delta_0^j$ is a root, and let H_0^j be associated with δ_0^j. These data enable us to select special elements in F_2, namely

$$f_0^j = \exp i\pi H_0^j.$$

Denote by F_2^0 the set of all such elements and define a new set of elements of order 4 :

$$F_4 = \left\{ f \in \exp i\mathfrak{a}_0 \ \mid \ f^2 \in F_2^0 \right\}.$$

For simplicity, we may assume that Σ_0 consists of a single system, so that F_2^0 reduces to $\{f_0\}$. All the results that follow hold in full generality, but in order to avoid cumbersome notation, they will be stated and proved under this simplifying assumption. The necessary modifications are obvious.

Lemma 5.6. *Let $k \in \mathcal{C}(s_0)$ be such that $f_0 k f_0 = km$ for some $m \in M_0$. Then $m = e$.*

Proof. Let $k = k(\overline{n}_0)s_0\mu$. Then $\overline{n}_0 = \exp(X_0^1 + X_0^2)$, with X_0^1 (resp. X_0^2) in the root space corresponding to $-\delta_0$ (resp. $-2\delta_0$). Select now two non-zero vectors in these spaces, coinciding with X_0^1 and X_0^2 if neither one vanishes, and denote them again X_0^1 and X_0^2. Then since δ_0 is indivisible, the Lie algebra $\mathfrak{g}_0^*$ generated by X_0^1, X_0^2, θX_0^1 and θX_0^2 is isomorphic to $\mathfrak{su}(2,1)$ ([6], Theorem 3.1, Ch. IX, p. 409). Give now to all general Lie algebra concepts connected with $\mathfrak{g}_0^*$ the superscript *. In particular, let G_0^*, K_0^* A_0^*, N_0^* and $\overline{N}_0^*$ be the analytic subgroups corresponding to $\mathfrak{g}_0^*$, $\mathfrak{k}_0^*$, $\mathfrak{a}_0^*$, $\mathfrak{n}_0^*$ and $\overline{\mathfrak{n}}_0^*$. In particular $\overline{n_0} \in \exp \overline{\mathfrak{n}}_0^*$. Next, consider the elements $Y \in \mathfrak{p}_0^*$ and $Z \in \mathfrak{k}_0^*$ corresponding to

$$\overline{Y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \in \mathfrak{su}(2,1), \qquad \overline{Z} = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{su}(2,1),$$

respectively. It is immediate to check that for all $H \in \mathfrak{a}_0^*$:

$$[H,Y] = \delta_0(H)Z, \qquad [H,Z] = \delta_0(H)Y.$$

Thus, ([6], Lemma 2.4, Ch. VII, p. 286), s_0 may be realized as

$$s_0 = \exp(\frac{\pi}{\langle\delta_0,\delta_0\rangle^{1/2}} Z) \in G^* \cap K_0 = K_0^*.$$

Observe also that $k(\overline{n}_0) \in K_0^*$. It follows that the equality $f_0(k(\overline{n}_0)s_0\mu)f_0 = (k(\overline{n}_0)s_0\mu)m$ in K_0 yields the equality:

$$f_0(k(\overline{n}_0)s_0)f_0 = (k(\overline{n}_0)s_0)\mu m \mu^{-1}$$

in K_0^*, that is in $S(U_2 \times U_1) \subset SU(2,1)$. This gives $\mu m \mu^{-1} = e$, namely $m = e$. ■

Remark. The previous lemma should be thought of as an "$SU(2,1)$–reduction". It also explains the first reason for choosing δ_0 as we did; the second reason will become clear in the proof of Theorem 12.

Theorem 5.7. *Let k be such that $\langle k \rangle \in \mathrm{Fix}(F_2, K_0/M_0)$. Then $f_0 k f_0 = k$.*

Proof. Let $k \in \mathcal{C}(w)$ be as in the statement. We now show by induction on $l_S(w)$ that $f_0 k f_0 = k$.
If $l_S(w) = 1$ the results follows from Lemma 5.5 and Lemma 5.6.
Suppose the statement true for $l_S(w) \le p$ and let $w = ss_1\cdot, \ldots \cdot s_p$ be a reduced expression. Two cases arise: either $s \ne s_0$ or $s = s_0$.
Suppose first that $s \ne s_0$. By hypothesis, $f_0 k f_0 = km$ for a suitable $m \in M_0$. Because of (4), $\mathcal{C}(w) = \mathcal{C}(s)\mathcal{C}(s_1 \cdot \ldots \cdot s_p)$ and writing $\sigma = s_1\cdot, \ldots \cdot s_p$, we have $k = k_s k_\sigma$, where evidently $k_s \in \mathcal{C}(s)$ and $k_\sigma \in \mathcal{C}(\sigma)$. But $s \ne s_0$, so that Lemma (5.5) gives $f_0 k_s f_0 = k_s$. Therefore

$$f_0(k_s k_\sigma)f_0 = (f_0 k_s f_0)(f_0 k_\sigma f_0) = k_s(f_0 k_\sigma f_0),$$

and this is equal to $k_s k_\sigma m$. It follows that $f_0 k_\sigma f_0 = k_\sigma m$. By induction, $m = e$.
Suppose next that $s = s_0$. Put again $\sigma = s_1\cdot, \ldots \cdot s_p$ and $k = k_s k_\sigma$. From $(f_0 k_s f_0)(f_0 k_\sigma f_0) = k_s k_\sigma m$ we obtain

$$f_0 k_s f_0 = k_s(k_\sigma m f_o k_\sigma^{-1} f_0).$$

Now $k_s \in \mathcal{C}(s)$, and let $k_\sigma m f_o k_\sigma^{-1} f_0 \in \mathcal{C}(u)$, so that:

$$f_0 k_s f_0 \in \mathcal{C}(s) \cap \mathcal{C}(s)\mathcal{C}(u).$$

Using now (2), we infer that

a) either $f_0 k_s f_0 \in \mathcal{C}(s) \cap \mathcal{C}(su)$.

b) or $f_0 k_s f_0 \in \mathcal{C}(s) \cap \mathcal{C}(u)$.

In case a), u must be equal to e, which yields $k_\sigma m f_o k_\sigma^{-1} f_0 \in \mathcal{C}(e) = M_0$, that is $f_0 k_s f_0 = k_s \mu$ for suitable $\mu \in M_0$ which is forced to be e by induction. Thus $k_\sigma m = f_0 k_\sigma f_0$ and again by induction $m = e$.
Next, we see that case b) cannot occur. Indeed, in this circumstance $u = s$. Let us be more precise about u. The defining condition is that

$$\rho := (k_\sigma m)(f_0 k_\sigma^{-1} f_0) \in \mathcal{C}(u).$$

But $k_\sigma m \in \mathcal{C}(\sigma)$ and $f_0 k_\sigma^{-1} f_0 \in \mathcal{C}(\sigma^{-1})$. By (3)

$$\mathcal{C}(s_1 \cdot \ldots \cdot s_p)\mathcal{C}(\sigma^{-1}) \subset \coprod_{1 \leq i_1 \leq \ldots \leq i_t \leq p} \mathcal{C}(s_{i_1} \cdot \ldots \cdot s_{i_t} \sigma^{-1}),$$

so that $\rho \in \mathcal{C}(s_{i_1} \cdot \ldots \cdot s_{i_t} \sigma^{-1})$ for a suitable choice of indices. On the other hand, $s_{i_1} \cdot \ldots \cdot s_{i_t} \sigma^{-1} = u = s$, i.e. $s_{i_1} \cdot \ldots \cdot s_{i_t} = s\sigma = w$. This is a contradiction because $l_S(s_{i_1} \cdot \ldots \cdot s_{i_t}) = t \leq p < p + 1 = l_S(w)$. ∎

Recall that the set F_4 of order 4 elements is now assumed for simplicity to be:

$$F_4 = \left\{ f \in \exp i\mathfrak{a}_0 \mid f^2 = f_0 \right\}.$$

Proposition 5.8. *F_4 acts on $X_1 = \mathrm{Fix}(F_2, K_0/M_0)$.*

Proof. With the notation of Section 2, the action of F_4 on X_1 is given by $f \cdot \langle k \rangle = \langle \mathrm{Ad}^{-1}(f k_* f^{-1}) \rangle$. As agreed earlier, however, whenever appropriate we will think of all the supgroups of G_0 as subgroups of G under Ad (whereby the center has become trivial), and remove the subscript $*$.
Let $K_1 = \{k \in K_0 \mid f_0 k f_0 = k\}$, a closed, hence compact, Lie subgroup of K_0. Let $\mathfrak{k}_1$ be its Lie algebra. Our first concern is to show that:

$$\mathrm{Ad}\, f_0 T = T, \qquad \forall T \in \mathfrak{k}_1 \tag{5}$$

Indeed, for all $t \in \mathbb{R}$:

$$\exp(t\, \mathrm{Ad}\, f_0 T) = f_0 (\exp tT) f_0 = (\exp tT) \in K_1,$$

thereby showing $\mathrm{Ad}\, f_0 T \in \mathfrak{k}_1$. Next, since $\mathrm{Ad}\, f_0$ is an involution on $\mathfrak{k}_1$, we have a vector space decomposition $\mathfrak{k}_1 = \mathfrak{k}_1^+ + \mathfrak{k}_1^-$ corresponding to the ± 1 eigenvalues of $\mathrm{Ad}\, f_0$. Thus, if $T = T^+ + T^-$ is the resulting decomposition of T, then for all $t \in \mathbb{R}$:

$$\exp(t(T^+ - T^-)) = \exp(t\, \mathrm{Ad}\, f_0 T) = f_0 (\exp tT) f_0 = \exp tT = \exp(t(T^+ + T^-)).$$

For t sufficiently small, this yields $T^- = 0$, which is equivalent to saying $\mathfrak{k}_1^- = 0$. This proves (5).
Now we prove that if $k \in K_1$, then fkf^{-1} is fixed by Θ. In fact since $f^2 = f_0$, we have

$$f = f^{-1} f_0 = f_0 f^{-1}, \qquad f^{-1} = f f_0 = f_0 f.$$

On the other hand, since $\Theta(f) = \Theta(\exp iA) = \exp(i\theta A) = \exp(-iA) = f^{-1}$ we have

$$\Theta(fkf^{-1}) = f^{-1}kf = ff_0 k f_0 f^{-1} = fkf^{-1}.$$

Suppose now $k = \exp T \in K_1$. We may take $T \in \mathfrak{k}_1$. Then $fkf = \exp(\mathrm{Ad}\, fT)$ and if σ denotes conjugation in $\mathfrak{g}$ with respect to $\mathfrak{g}_0$, then by (5)

$$\begin{aligned}
\sigma(\mathrm{Ad}\, fT) &= \sigma(\mathrm{Ad}(\exp iA)T) \\
&= \mathrm{Ad}(\exp \sigma(iA))(\sigma(T)) \\
&= \mathrm{Ad}(\exp -iA)T \\
&= \mathrm{Ad}\, f^{-1}T \\
&= \mathrm{Ad}\, ff_0 T \\
&= \mathrm{Ad}\, fT.
\end{aligned}$$

Thus, $\mathrm{Ad}\, fT \in \mathfrak{k}_0$, i.e. $fkf^{-1} \in K_0$.
Finally, let $\langle k \rangle \in X_1$, so that, by Theorem 5.7, $k \in K_1$ and by the above argument $fkf^{-1} \in K_0$. In order to see that F_4 sends fixed points of F_2 into fixed points of F_2, we must show that for all $f \in F_4$ and for all $g \in F_2$ there exists $m \in M_0$ such that $g(fkf^{-1})g = (fkf^{-1})m$. But this is obvious because f and g commute, and there exists $m \in M_0$ such that $gkg = km$.

∎

Theorem 5.9. $\mathrm{Fix}(F_4, X_1) = W$.

Proof. Since Σ_0 is of type $(BC)_l$, all roots of the form $\alpha = \sum_{j=0}^{l-1} \nu_j(\alpha)\delta_j$ with $\nu_j(\alpha) \in 2\mathbb{Z}$ (i.e. even roots) are such that $\nu_0(\alpha) = \pm 2$, as it is clear from the list at the end of Section 2. More precisely, all the non–vanishing coefficients of an even root must be equal to ± 2, but for each fixed $\delta_j \neq \delta_0$ there is an even root α for which $\nu_j(\alpha) = 0$. This is the second reason for choosing δ_0 as we did.
Let now $\langle k \rangle \in \mathrm{Fix}(F_4, X_1)$, $\overline{H} \in \mathfrak{a}_0$ a regular element, and $Y = \mathrm{Ad}\, k\overline{H}$. Then Y is fixed by all elements of F_4. According to the remark following Theorem 3.3, we may write:

$$Y = Y_0 + \sum_{\alpha \in E} Y_\alpha,$$

where E is the set of even roots in Σ_0. Therefore, if $f = \exp iA_0$

$$Y_\alpha = e^{i\alpha(A_0)} Y_\alpha.$$

Take now $A_0 = \frac{\pi}{2} H_0$ (so that $(\exp iA_0)^2 = f_0$) and observe that:

$$\alpha(A_0) = \sum_{j=0}^{l-1} \nu_j(\alpha)\delta_j(\frac{\pi}{2}H_0) = \frac{\pi\nu_0(\alpha)}{2} = \pm\pi.$$

Thus $Y_\alpha = -Y_\alpha = 0$ and $Y \in \mathfrak{a}_0$. Hence $k \in M_0'$, that is $\langle k \rangle \in W$, which shows that $\mathrm{Fix}(F_4, X_1) \subset W$. The reverse inclusion is obvious. ∎

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