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Regular Lie groups and a theorem of Lie-Palais

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Abstract. In 1984 Milnor had shown how to deduce the Lie-Palais theorem on integration of infinitesimal actions of finite-dimensional Lie algebras on compact manifolds from general theory of regular Lie groups modelled on locally convex spaces. We show how, in the case of effective action, one can eliminate from Milnor's argument the abstract Lie-Cartan theorem, making the deduction rather elementary.

1. Introduction

A well-known result dated back to Lie and finalized by Palais [7] states that every infinitesimal action of a finite-dimensional Lie algebra on a compact smooth manifold, X, is derived from a smooth action of a finite-dimensional Lie group on X. Milnor [4] gives a proof of this theorem based on theory of regular Lie groups. This proof, however, is only partial (in Milnor's own words), not being self-contained: by necessity, it invokes the abstract Lie-Cartan theorem. One definitely cannot hope to circumvent this landmark result while proving the Lie-Palais theorem (which simply turns into the Lie-Cartan theorem in the degenerate case of a trivial action). However, things are different in the most important particular case where the infinitesimal action is effective, that is, one deals with finite-dimensional Lie algebras of vector fields. We show how under this assumption the Lie-Cartan theorem can be eliminated from the Milnor's argument, which therefore becomes quite elementary in the sense that it does not invoke anything advanced beyond the scope of theory of regular Lie groups (like the structure theory of Lie algebras).

Call a Banach-Lie algebra coming from a Lie group *enlargeable.* The Lie-Cartan theorem essentially says that every finite-dimensional Lie algebra is enlargeable; in infinite dimensions non-enlargeable Banach-Lie algebras can be found [13]. However, a Banach-Lie algebra is enlargeable as soon as it admits a continuous monomorphism into an enlargeable Banach-Lie algebra [1, 11, 13].

For more general Lie groups and algebras the situation is much harder to deal with. A closed Lie subalgebra of the Lie algebra of a Fréchet-Lie group G

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need not have an associated Fréchet-Lie group even if G meets certain conditions of regularity [2, 4]. For some recent positive advances in this direction, see [3].

We show that the above result on Banach-Lie algebras can be pushed somewhat further to absorb a restricted version of the Lie-Palais theorem. Let $\mathfrak{h}$ be the Lie algebra of a regular Lie group modelled on a locally convex space, and let $\mathfrak{g}$ be a Banach-Lie algebra with finite-dimensional centre admitting a continuous monomorphism into $\mathfrak{h}$. Then $\mathfrak{g}$ is enlargeable. The result is deduced from regular Lie group theory by means of the concept of a free Banach-Lie algebra introduced by us earlier [8].

Since the group of diffeomorphisms of a compact manifold forms a regular Fréchet-Lie group, an application of our main result yields a theorem on integration of finite-dimensional Lie algebras of vector fields. The latter theorem thus merges fully in the realm of regular Lie group theory.

2. Preliminaries

We first recall a few basic facts about regular Lie groups. A C^∞ (Gâteaux) smooth Lie group G modelled on a bornological sequentially complete locally convex space, with a corresponding locally convex Lie algebra $\mathrm{Lie}\,(G)$, is called *regular* if every smooth path $v : I = [0,1] \to \mathrm{Lie}\,(G)$ has a left product integral $p\colon I \to G$, that is, a solution to the differential equation $Dp(t) = p(t) \cdot v(t)$, and if furthermore the correspondence $v \mapsto p(0)^{-1} \cdot p(1)$ defines a smooth map from the locally convex space $C^\infty(I, \mathrm{Lie}\,(G))$ to G. (This definition is taken from Milnor's survey [4], where regularity is understood in a somewhat less restrictive sense than by Kobayashi *et al* [2]. Notation is self-explanatory.)

The condition of regularity seems indispensable if one wants to derive substantial results, and all examples of Lie groups modelled on locally convex spaces known up to date are regular. Such are: the groups of diffeomorphisms of smooth compact manifolds, certain subgroups of these, the groups of currents (and their central extensions), Banach-Lie groups, and many more [2, 3, 4]. The following is a basic result about regular Lie groups.

Theorem 2.1. ([4]; cf. [2]) *Let G and H be smooth Lie groups modelled on bornological sequentially complete locally convex spaces. Let G be simply connected and H be regular. Then every continuous Lie algebra morphism $\mathrm{Lie}\,(G) \to \mathrm{Lie}\,(H)$ is tangent to a (necessarily unique) Lie group morphism $G \to H$.* ■

We also need a test for enlargability of Banach-Lie algebras. As is well known (cf., e.g., [12]), to every Banach-Lie algebra one can associate in a canonical way a group germ; to put it simply, there is in any Banach-Lie algebra a neighbourhood of the origin that can be converted via the Campbell-Hausdorff-Dynkin multiplication into a local analytic group. The following theorem can be found in [1, 11], and it also follows from the Cartan-Smith theorem as presented in [14] if one observes that every injective morphism of local groups from a local analytic group associated to a Banach-Lie algebra, restricted, if necessary, to an open local subgroup, is automatically a submersion.

Theorem 2.2. *Let $\mathfrak{g}$ be a Banach-Lie algebra such that there exist a group G, a neighbourhood, U, of the origin in $\mathfrak{g}$, and a map $\phi: U \to G$ which is an injective morphism of local groups. Then $\mathfrak{g}$ is enlargeable. Moreover, a subgroup of G generated by $\phi(U)$ can be given a unique structure of a Banach-Lie group associated to $\mathfrak{g}$ in such a way that ϕ becomes the restriction of an exponential map.* ∎

3. Free Banach-Lie algebras

Theorem 3.1. [8, 9] *Let E be a normed space. There exist a complete normed Lie algebra $\mathcal{FL}(E)$ and a linear isometry $i_E: E \hookrightarrow \mathcal{FL}(E)$ such that:*

1. *The image of i_E topologically generates $\mathcal{FL}(E)$.*
2. *For every complete normed Lie algebra $\mathfrak{g}$ and an arbitrary contracting linear operator $f: E \to \mathfrak{g}$ there exists a contracting Lie algebra homomorphism $\hat{f}: \mathcal{FL}(E) \to \mathfrak{g}$ with $\hat{f} \circ i_E = f$.*

The pair $(\mathcal{FL}(E), i_E)$ with the properties 1 and 2 is unique up to an isometrical isomorphism. ∎

We call $\mathcal{FL}(E)$ the *free Banach-Lie algebra* on E. It is shown in [9] that if $\dim E \geq 2$ then $\mathcal{FL}(E)$ is centreless. (An original proof of this fact presented in [8] is unsatisfactory.) The following is a direct corollary.

Theorem 3.2. *Every free Banach-Lie algebra on a normed space is enlargeable.* ∎

Let Γ be a set. We denote the Banach-Lie algebra $\mathcal{FL}(l_1(\Gamma))$ simply by $\mathcal{FL}(\Gamma)$ and call the free Banach-Lie algebra on a set Γ. It is easy to see that the above algebra possesses the following universal property: every map f sending Γ to the unit ball of a Banach-Lie algebra $\mathfrak{g}$ extends in a unique way to a contracting Lie algebra morphism from $\mathcal{FL}(\Gamma)$ to $\mathfrak{g}$. The well-known fact that a Banach space of density character $\leq \mathrm{Card}(\Gamma)$ is a factor space of $l_1(\Gamma)$ implies that an arbitrary Banach-Lie algebra is a factor Banach-Lie algebra of a free Banach-Lie algebra of the form $\mathcal{FL}(\Gamma)$ [8]. We denote by $\mathcal{G}(\Gamma)$ the simply connected Banach-Lie group attached to the free Banach-Lie algebra $\mathcal{FL}(\Gamma)$. It is couniversal among all connected Banach-Lie groups of density character $\leq \mathrm{Card}(\Gamma)$ (cf. [8]).

4. Extension of Lie-Palais theorem in the case of effective action

Lemma 4.1. *Let $\mathfrak{g}$ be a Banach-Lie algebra with finite-dimensional centre $\mathfrak{z}$. Let U be a neighbourhood of the origin in $\mathfrak{g}$ such that the Campbell-Hausdorff-Dynkin multiplication is defined on $U \times U$. Let f be a continuous map from U to a Hausdorff topological group G which is a morphism of local groups. Suppose* *that for every $x \in \mathfrak{z}$ there is a $\lambda \in \mathbb{R}$ with $\lambda x \in U$ and $f(\lambda x) \neq e_G$. Then $\mathfrak{g}$ is enlargeable.*

Proof. The restricted map $f|_{U \cap \mathfrak{z}}$ extends to a continuous abelian group homomorphism, $\tilde{f}: \mathfrak{z} \to G$. The set $\ker \tilde{f}$ forms a closed additive subgroup of a finite-dimensional vector space $\mathfrak{z}$, containing by hypothesis no one-dimensional linear subspaces; in other words, it is a discrete lattice, and there exists an open neighbourhood of zero, $V \subseteq U$, in $\mathfrak{z}$ such that $V \cap \ker \tilde{f} = \{0\}$. Choose an open $\tilde{U} \subset \mathfrak{g}$ with $0 \in \tilde{U} \cap \mathfrak{z} \subseteq V$. As a corollary, if $x \in \tilde{U} \setminus \{0\}$, then $f(x) \neq e_G$.

Let $\phi: \tilde{U} \to F$ be a universal morphism of a local group to a group. (That is, any other such morphism, $g: \tilde{U} \to A$, is a composition of ϕ and a group homomorphism $F \to A$; cf. [12, 14].) Let $x \in \tilde{U} \setminus \{0\}$; it suffices to prove that $\phi(x) \neq e_G$ and to apply Theorem 2.2. Now, if $x \notin \mathfrak{z}$, then x is separated from zero by the composition of the adjoint representation $x \mapsto \mathrm{ad}_x$ and the exponentiation $\mathrm{End}(\mathfrak{g}_+) \to \mathrm{GL}(\mathfrak{g}_+)$. (Both mappings preserve the Campbell-Hausdorff-Dynkin multiplication in suitable neighbourhoods of identity, and therefore determine morphisms of appropriate local groups.) If $x \in \mathfrak{z}$, then $x \in \tilde{U} \setminus \{0\}$ and $f(x) \neq e_G$. ∎

The following is the main result of our note.

Theorem 4.2. *Let $\mathfrak{g}$ be a Banach-Lie algebra having the finite-dimensional centre and admitting a continuous monomorphism into the Lie algebra of a regular Lie group. Then $\mathfrak{g}$ is enlargeable.*

Proof. Let H be a regular Lie group and $h: \mathfrak{g} \to \mathrm{Lie}(H)$ be a continuous Lie monomorphism. Denote by B the unit ball in $\mathfrak{g}$, and by i the contracting Lie algebra morphism $\mathcal{FL}(B) \to \mathfrak{g}$ extending the identity map $B \to B$. According to Theorem 2.1, the composition morphism $h \circ i: \mathcal{FL}(B) \to \mathrm{Lie}(H)$ is tangent to a morphism of Lie groups $\hat{i}: \mathcal{G}(B) \to H$. Denote $I = \ker(h \circ i)$, $J = \ker \hat{i}$, and let A be a subgroup of $\mathcal{G}(B)$ algebraically generated by the exponential image of I. Let G denote the Hausdorff topological group $\mathcal{G}(B)/J$, and let $f: \mathfrak{g} \to G$ be a map defined by $f(x + I) = (\exp x) \cdot J$. Since $A \subseteq J$ and the restriction of $\exp_{\mathcal{G}(B)}$ to a sufficiently small neighbourhood of the origin is a morphism of local groups, the map f is well-defined and is a morphism of local groups; diagrammatic search shows that it is continuous. The canonical group monomorphism $\psi: G \to H$ is continuous as well, and $\psi \circ \phi = \exp_H \circ h$. Since for every $x \in \mathrm{Lie}(H)$ one surely has $\exp_H(\lambda x) \neq e_H$ for an appropriate $\lambda \in \mathbb{R}$, and the centre of $\mathfrak{g}$ is finite-dimensional, the triple $(\mathfrak{g}, G, f)$ falls within the conditions of Lemma 4.1. ∎

Corollary 4.3. *Every continuous monomorphism from a Banach-Lie algebra with finite-dimensional centre to the Lie algebra of a regular Lie group is tangent to an appropriate Lie group morphism.*

Proof. Results from direct application of Theorems 4.2 and 2.1. ∎

As a corollary, we can now prove a restricted version of the Lie-Palais theorem by means of regular Lie group theory.

Theorem 4.4. (Lie-Palais) *Let M be a compact closed manifold and let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then every effective infinitesimal action of $\mathfrak{g}$ on M is derived from a smooth global action of a finite-dimensional Lie group on M.*

Proof. An effective infinitesimal action of $\mathfrak{g}$ upon M can be viewed as a Lie algebra monomorphism from $\mathfrak{g}$ to the algebra $\operatorname{vect} M$ of smooth vector fields on M endowed with the C^∞-topology. Clearly, the conditions of the Main Theorem 4.2 and Corollary 4.3 are satisfied, therefore there exists a Lie group G attached to $\mathfrak{g}$ (no Lie-Cartan theorem needed!), and the resulting Lie group morphism from G to the diffeomorphism group $\operatorname{diff} M$ (which is a regular Lie group with the Lie algebra $\operatorname{vect} M$, [2, 4]) determines a desired global smooth action. ∎

Remark 4.5.1. We stress that the proposed proof of the Lie-Palais theorem neither is simpler than nor is put forward as a substitute for the original proof contained in the classical Palais's Memoir [7]. Rather, it exhibits new links between (at least, three) different levels of Lie theory.

Remark 4.5.2. We do not know whether the condition upon the centre being finite-dimensional can be relaxed in the Main Theorem 4.2. However, our Example in [10] can be remade easily to show that the key Lemma 4.1 is no longer valid for Banach-Lie algebras with infinite-dimensional centre.

Remark 4.5.3. A reputed open problem, which might well fall within this circle of ideas, is that of existence of a (connected) infinite-dimensional Banach-Lie group acting effectively and transitively on a compact smooth manifold [5, 6].

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