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# Stabilizers of Lattices in Lie Groups

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**Abstract.** Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$, containing a lattice $\Gamma$. We shall write $\mathrm{Aut}(G)$ for the group of all smooth automorphisms of $G$. If $A$ is a closed subgroup of $\mathrm{Aut}(G)$ we denote by $\mathrm{Stab}_A(\Gamma)$ the stabilizer of $\Gamma$ in $A$; for example, if $G$ is $\mathbb{R}^n$, $\Gamma$ is $\mathbb{Z}^n$, and $A$ is $\mathrm{SL}(n,\mathbb{R})$, then $\mathrm{Stab}_A(\Gamma)=\mathrm{SL}(n,\mathbb{Z})$. The latter is, of course, a lattice in $\mathrm{SL}(n,\mathbb{R})$; in this paper we shall investigate, more generally, when $\mathrm{Stab}_A(\Gamma)$ is a lattice (or a uniform lattice) in $A$.

## Introduction

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$, containing a lattice $\Gamma$; so $\Gamma$ is a discrete subgroup and $G/\Gamma$ has finite $G$-invariant measure. We shall write $\mathrm{Aut}(G)$ for the group of all smooth automorphisms of $G$, and $M(G) = \{\alpha \in \mathrm{Aut}(G)\colon \mathrm{mod}(\alpha) = 1\}$ for the group of measure-preserving automorphisms of $G$ (here $\mathrm{mod}(\alpha)$ is the common ratio $\mathrm{measure}(\alpha(F))/\mathrm{measure}(F)$, for any measurable set $F \subset G$ of positive, finite measure). If $A$ is a closed subgroup of $\mathrm{Aut}(G)$ we denote by $\mathrm{Stab}_A(\Gamma)$ the stabilizer of $\Gamma$ in $A$, in other words $\{\alpha \in A\colon \alpha(\Gamma) = \Gamma\}$. The main question of this paper is:

When is $\mathrm{Stab}_A(\Gamma)$ a lattice (or a uniform lattice) in $A$?

We point out that the thrust of this question is whether the stabilizer $\mathrm{Stab}_A(\Gamma)$ is cocompact or of cofinite volume, since in any event, in all cases of interest, the stabilizer is discrete (see Prop. 1.1).

Our first result, for mixed groups, applies to all the classical groups. We prove (Cor. 1.9) that if the radical of $G$ is a vector group $V$, and the Levi factor $S$ has no compact part and acts irreducibly on the complexification $V^{\mathbb{C}}$, then $\mathrm{Stab}_{M(G)}(\Gamma)$ is a lattice in $M(G)$ for any lattice $\Gamma$ in $G$. The same is true if $M(G)$ is replaced by $I(G)$, since for this class of groups $I(G)$ has finite index in $M(G)$ (1.7). In the course of the proof we obtain a generalization (see Th. 1.6) of the fact that every derivation of a semisimple Lie algebra is inner.

In Section 3 we turn to nilpotent groups $G$. Here our main results concern the Iwasawa nilpotent parts of the simple, non-compact, real, rank-1 simple Lie groups: the real vector groups $\mathbb{R}^n$, their complex analogues, the Heisenberg groups $N_n$, their quaternionic analogues $H_n$, and the exceptional case $n = 1$ of the

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groups $C_n$ built on the Cayley numbers. In all these cases, stabilizers of log-lattices (lattices whose log is a lattice in the Lie algebra) in $G$ are lattices in $M(G)$. Nevertheless, there is an interesting dichotomy between $\mathbb{R}^n$ and $N_n$ on the one hand, and $H_n$ and $C_1$ on the other.[1] In the case of real vector groups and Heisenberg groups, the stabilizer is a non-uniform lattice in $M(G)$ (2.4). When $G$ is $H_n$ or $C_1$, on the other hand, stabilizers of log-lattices turn out actually to be *uniform* lattices in $M(G)$ (see Prop. 2.5 for $H_n$ with $n \leq 2$ and for $C_1$, and the forthcoming thesis of Paolo Barbano for the general case).

We conclude with various classes of examples and counterexamples. First we consider a class of nilpotent groups $G$, and a connected subgroup $A$ satisfying $I(G) \subset A \subset M(G)$ (strict containment) in which $\mathrm{Stab}_A(\Gamma)$ is a uniform lattice in $A$ whenever $\Gamma$ is a compatible log-lattice (see Prop. 2.10). Then we give examples (2.12, 2.13) to show that our results on Heisenberg groups cannot be extended to arbitrary unitriangular groups or to arbitrary two-step nilpotent groups.

## 1. Lattices in Mixed Groups

In connection with the main question of the paper, stated in the Introduction, we shall actually only consider subgroups $A$ of $M(G)$, since $M(G)$ is the largest subgroup of $\mathrm{Aut}(G)$ in which it makes sense to consider this question. More precisely, if $A$ is a closed subgroup of $\mathrm{Aut}(G)$ and $\mathrm{Stab}_A(\Gamma)$ a lattice in $A$, then $A$ is contained in $M(G)$. Indeed, if $\alpha \in \mathrm{Aut}(G)$ and $\alpha(\Gamma) = \Gamma$, then $\alpha$ preserves the (finite) measure of a fundamental domain for $\Gamma$. Thus $\alpha$ is measure-preserving, so $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma) \subset M(G)$. Therefore the lattice $\mathrm{Stab}_A(\Gamma)$ is contained in the closed normal subgroup $A \cap M(G)$ of $A$, so mod maps $A/A \cap M(G)$ onto a compact subgroup of $\mathbb{R}_+^\times$. It follows that $A \subset M(G)$.

We begin by fixing some notation. If $G$ is a connected Lie group, then $\mathrm{Aut}(G)$ and $M(G)$ are also Lie groups. If $\alpha^\circ$ denotes the differential in $\mathrm{Aut}(\mathfrak{g})$ of $\alpha \in \mathrm{Aut}(G)$, then the map $\alpha \mapsto \alpha^\circ$ is an injection of $\mathrm{Aut}(G)$ to $\mathrm{Aut}(\mathfrak{g})$. We denote by $\mathcal{M}(\mathfrak{g})$ the group of Lebesgue-measure preserving automorphisms of $\mathrm{Aut}(\mathfrak{g})$: $\mathcal{M}(\mathfrak{g}) = \{\beta \in \mathrm{Aut}(\mathfrak{g}) \colon |\det(\beta)| = 1\}$. The differential map takes $M(G)$ into $\mathcal{M}(\mathfrak{g})$. If $G$ is simply connected, the differential map is surjective, and in particular, $M(G)^\circ = \mathcal{M}(\mathfrak{g})$.

Relative to any basis of $\mathfrak{g}$, $\mathcal{M}(\mathfrak{g})$ is defined by the polynomials induced by the Lie algebra structure, together with the polynomial $\det^2$, so $\mathcal{M}(\mathfrak{g})$ is a real linear algebraic group (the group of real points of a linear algebraic group defined over $\mathbb{R}$). Of course, if the Lie algebra $\mathfrak{g}$ has rational structure constants, then these polynomials have rational coefficients, so $\mathcal{M}(\mathfrak{g})$ is defined over $\mathbb{Q}$. We denote by $\mathcal{D}(\mathfrak{g})$ the Lie algebra of derivations of $\mathfrak{g}$; $\mathcal{D}(\mathfrak{g})$ is the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$. We further denote by $\mathcal{D}_0(\mathfrak{g})$ the subalgebra of derivations of trace 0; $\mathcal{D}_0(\mathfrak{g})$ is the Lie algebra of $\mathcal{M}(\mathfrak{g})$.

Although we shall not use this fact below, we note in Proposition 1.1 below that in most cases of interest, the stabilizer of a lattice is always a discrete subgroup of $\mathrm{Aut}(G)$. Tits [18] introduced the notion of *automorphism of*

[1] This dichotomy is seen in other areas as well; for example, the corresponding simple groups satisfy Kazhdan's property T iff the nilpotent parts are $H_n$ and $C_1$ (see, for example, the table on p. 47 of [2]).

*bounded displacement*: an automorphism $\alpha \in \mathrm{Aut}(G)$ has bounded displacement if $\{\alpha(g)g^{-1} : g \in G\}$ has compact closure. Many groups have no non-trivial automorphisms of bounded displacement, for example, groups $G$ whose radical is simply connected and of exponential type (so that the radical is a *type* $E$ group), and whose Levi factor has no compact part [15, (1.1) and (1.3)]. For a simply connected nilpotent group, the proposition below is due to Malcev ([13, Th. 3]).

**Proposition 1.1.** *Let $G$ be a connected Lie group with no non-trivial automorphisms of bounded displacement. If $\Gamma$ is a lattice in $G$, then $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ is a discrete subgroup of $\mathrm{Aut}(G)$.*

**Proof.** Let $U$ be a neighborhood of 1 in $G$ such that $U \cap \Gamma = (1)$, and let $F = \{\gamma_1, \dots, \gamma_n\}$ be a finite[2] generating set for $\Gamma$. Then $W(F,U) = \{\alpha \in \mathrm{Aut}(G) \colon \gamma^{-1}\alpha\gamma \in U$ for all $\gamma \in F\}$ is a neighborhood of 1 in $\mathrm{Aut}(G)$. If $\alpha \in W(F,U) \cap \mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$, then $\gamma_i^{-1}\alpha\gamma_i \in U \cap \Gamma$ for all $i = 1, \dots, n$, so $\alpha = 1$ on $\Gamma$. Since $\Gamma$ is a lattice it now follows from [8, Lemma 13] that $\alpha$ is of bounded displacement. Hence $\alpha = 1$. ∎

Our first result applies to the large class of groups $G$ which satisfy the "density" condition $B(G) = Z(G)$, where $B(G)$ is the set of elements of $G$ with *bounded* conjugacy classes; such groups were extensively studied in, for example, [8, 9, 15]. For these groups, we answer the main question of the paper for the group $A = I(G)$ of inner automorphisms.

**Theorem 1.2.** *Let $G$ be a connected Lie group satisfying $B(G) = Z(G)$. If $\Gamma$ is a lattice in $G$, then $\mathrm{Stab}_{I(G)}(\Gamma)$ is a lattice in $I(G)$.*

**Proof.** Since $I(G)$ is closed in $\mathrm{Aut}(G)$ [7, Th. 2], the differential $\alpha \mapsto \alpha^\circ$, which maps $I(G)$ onto $\mathrm{Ad}(G)$, is an isomorphism, and the image of the stabilizer of $\Gamma$ is clearly $\mathrm{Ad}(N_G(\Gamma))$. Thus it suffices to prove that $\mathrm{Ad}(N_G(\Gamma))$ is a lattice in $\mathrm{Ad}(G)$.

We first note that if $G$ is a connected Lie group, then the condition $B(G) = Z(G)$ implies the analogous condition for the simply connected covering group $\tilde{G}$: $B(\tilde{G}) = Z(\tilde{G})$ [9, p. 257]. Furthermore, by Corollary 5 of [9], $\mathrm{Ad}(\Gamma)$ is discrete in $\mathrm{GL}(\mathfrak{g})$, and in the closed subgroup $\mathrm{Ad}(G)$ [7, Th. 2]. Pushing the measure forward, we see that $\mathrm{Ad}(\Gamma)$ is a lattice in $\mathrm{Ad}(G)$. Let $N$ denote the closed subgroup $N_G(\Gamma)$. Since $\mathrm{Ad}(N) \supset \mathrm{Ad}(\Gamma)$, to show that $\mathrm{Ad}(N)$ is a lattice it suffices to show that it is discrete.[3]

[2] For the reader's convenience we complete the sketch of the proof in [17, 6.18] that lattices in a connected Lie group are finitely generated. For a semisimple group, a sketch is given in [17, 13.21]. For an arbitrary Lie group $G$, let $G = RCS$ where $R = \mathrm{rad}(G)$, $C$ is the compact part of a Levi factor, and $S$ is semisimple with no compact part. Then by an extension of a result of H.C. Wang and one of Mostow ([7, Prop. 8]), $\Gamma \cap RC$ is a *uniform* lattice in the normal subgroup $RC$. Hence by [5, chap. VII, §3, Lemme 3] $\Gamma \cap RC$ is finitely generated, while $\Gamma \bmod RC$ is a lattice in a factor group of $S$, and therefore also finitely generated (as above). Thus $\Gamma$ is itself finitely generated.

[3] If we assumed that $G$ has no non-trivial automorphisms of bounded displacement, then we could already conclude from Proposition 1.1 that $\mathrm{Ad}(N)$ is discrete, since it is the stabilizer in $\mathrm{Ad}(G)$ of $\mathrm{Ad}(\Gamma)$.

Now since $N$ normalizes $\Gamma$, $\Gamma$ is a *uniform* lattice in $N$. Furthermore, the Euclidean component group $N_0$ normalizes, hence centralizes, $\Gamma$. By [9, Th. 3], this implies that $N_0 \subset Z(G)$. Thus $Z(G)\Gamma$ is an open subgroup of finite index in $N$. It now follows that $\mathrm{Ad}(\Gamma)$ has finite index in $\mathrm{Ad}(N)$, so in particular $\mathrm{Ad}(N)$ is discrete. Therefore $\mathrm{Ad}(N)$ is a lattice in $\mathrm{Ad}(G)$. ∎

**Corollary 1.3.** *Let $G$ be a connected Lie group $G$ whose radical is simply connected and of type E, and whose Levi factor has no compact part. If $\Gamma$ is a lattice in $G$, then $\mathrm{Stab}_{I(G)}(\Gamma)$ is a lattice in $I(G)$.*

**Proof.** As noted above, $G$ has no non-trivial automorphisms of bounded displacement, hence automatically satisfies $B(G) = Z(G)$. ∎

If $G$ is itself semisimple, then $I(G)$ has finite index in $\mathrm{Aut}(G)$, so we get:

**Corollary 1.4.** *If $G$ is a connected semisimple Lie group without compact part, then $\mathrm{Stab}_{\mathrm{Aut}(G)}(\Gamma)$ is a lattice in $\mathrm{Aut}(G)$.*

In the remainder of this section we shall consider a class of generalized motion groups. These groups will be constructed as semidirect products, as follows. Let $\mathfrak{h}$ be a Lie algebra, $\mathfrak{s}$ a subalgebra of $\mathcal{D}(\mathfrak{h})$ containing $\mathrm{ad}(\mathfrak{h})$, and $\mathfrak{g}$ the semidirect sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$, with bracketing given by

$$[(h, X), (k, Y)] = ([h, k] + Xk - Yh, [X, Y]). \tag{1}$$

If $H$ is a connected and simply connected Lie group with Lie algebra (isomorphic to) $\mathfrak{h}$, we can identify $\mathrm{Aut}(H)$ with $\mathrm{Aut}(\mathfrak{h})$ via the differential. If we denote by $S$ the analytic subgroup of $\mathrm{Aut}(H)$ corresponding to $\mathfrak{s}$, then $S \supset I(H)$, and $\mathfrak{g}$ is the Lie algebra of the connected Lie group $G = H \times_{\eta} S$ (semidirect product). $G$ is unimodular iff $H$ and $S$ are both unimodular, and $\mathfrak{s} \subset \mathcal{D}_0(\mathfrak{h})$. In particular, $G$ is unimodular if $\mathfrak{s}$ is semisimple.

For later use we need to determine the derivations of $\mathfrak{g}$ which leave $\mathfrak{h}$ invariant. Consider a linear map $D \in \mathfrak{gl}(\mathfrak{g})$ which leaves $\mathfrak{h}$ stable. For such a map we can write

$$D = \begin{pmatrix} D_{\mathfrak{h}} & \varphi \\ 0 & D_{\mathfrak{s}} \end{pmatrix}$$

where $D_{\mathfrak{h}} = D|_{\mathfrak{h}}$ and $D_{\mathfrak{s}} = D|\mathfrak{h}$ are endomorphisms of $\mathfrak{h}$ and $\mathfrak{s}$, respectively, and $\varphi$ is a linear map of $\mathfrak{s}$ to $\mathfrak{h}$.

The following proposition can be proven by means of formal calculations.

**Proposition 1.5.** *A linear map $D$ which leaves $\mathfrak{h}$ invariant is a derivation of $\mathfrak{g}$ if and only if both of the following conditions hold:*

1. *$D_{\mathfrak{h}}$ and $D_{\mathfrak{s}}$ are derivations, and $\varphi$ is a 1-cocycle;*
2. *$[D_{\mathfrak{h}}, X] = D_{\mathfrak{s}}(X) + \mathrm{ad}_{\mathfrak{h}}(\varphi(X))$ for all $X \in \mathfrak{s}$.*

The next result can be viewed as a generalization of the classical fact that every derivation of a semisimple Lie algebra is inner.

**Theorem 1.6.** *Suppose $\mathfrak{s}$ is a semisimple algebra which acts irreducibly on $\mathfrak{h}^{\mathbb{C}}$ (in the case of real scalars) or on $\mathfrak{h}$ (in the case of complex scalars). Then every derivation $D$ of trace 0 of $\mathfrak{g}$ which leaves $\mathfrak{h}$ invariant is inner.*

**Proof.** By semisimplicity of $\mathfrak{s}$, $D_{\mathfrak{s}} = \mathrm{ad}_{\mathfrak{s}}(X_0)$ for some $X_0 \in \mathfrak{s}$, while $\varphi(X) = -X(h_0)$ for some $h_0 \in \mathfrak{h}$ (Whitehead's lemma). It follows that

$$\begin{aligned}[D_{\mathfrak{h}}, X] &= D_{\mathfrak{s}}(X) + \mathrm{ad}_{\mathfrak{h}}(\varphi(X)) \\ &= [X_0, X] + [\mathrm{ad}(h_0), X]\end{aligned} \tag{2}$$

for all $X \in \mathfrak{s}$. Thus $L = D_{\mathfrak{h}} - X_0 - \mathrm{ad}(h_0)$ centralizes $\mathfrak{s}$, so by Schur's lemma is a multiple of the identity. On the other hand, since $\mathrm{tr}(D_{\mathfrak{h}}) = \mathrm{tr}(D) - \mathrm{tr}(D_{\mathfrak{s}})$ and $\mathfrak{s}$ is semisimple and contains $\mathrm{ad}(\mathfrak{h})$, the trace of each term in the sum $L$ is 0. Thus $D_{\mathfrak{h}} = X_0 + \mathrm{ad}(h_0)$, from which one sees easily that $D = \mathrm{ad}(h_0, X_0)$. ∎

**Corollary 1.7.** *Under the same hypotheses, if $\mathfrak{h}$ is characteristic in $\mathfrak{g}$, then $I(G)$ has finite index in $M(G)$ (so $M(G)_0 = I(G)$).*

**Proof.** The algebra $D_0(\mathfrak{g})$ is the Lie algebra of $\mathcal{M}(\mathfrak{g})$, so Theorem 1.6 implies that $\mathcal{M}(\mathfrak{g})_0 = \mathrm{Ad}(G)$. Since $\mathcal{M}(\mathfrak{g})$ is a real algebraic group, it has only finitely many components, so $\mathrm{Ad}(G)$ has finite index in $\mathcal{M}(\mathfrak{g})$. Now the differential map $\alpha \mapsto \alpha^\circ$ is injective, so $I(G)$ has finite index in $M(G)$. ∎

**Remark 1.8.** These results hold, for example, whenever $\mathfrak{h} = V$ is a vector space, since then $\mathfrak{h}$ is characteristic in $\mathfrak{g}$. It may be noted, however, that if the scalars are real, and $\mathfrak{s}$ merely acts irreducibly on $V$, then the conclusions need not be true. For example, let $V$ be the real vector space $\mathbb{H} = \mathbb{R}^4$, regarded as the algebra of quaternions, and let $S$ be the semisimple group $\mathrm{SU}(2)$, identified with the unit sphere in $\mathbb{H}$. Since $S$ acts transitively on itself by left multiplication, both it and its Lie algebra $\mathfrak{s}$ act irreducibly on $\mathbb{H}$. But any right multiplication by an element of $S$ commutes with $\mathfrak{s}$, so we can find nontrivial linear maps $\rho$ with trace 0 which commute with $\mathfrak{s}$. Then $\left(\begin{smallmatrix}\rho & 0\\ 0 & 0\end{smallmatrix}\right)$ is an outer derivation of trace 0 on $\mathfrak{g}$.

**Corollary 1.9.** *Let $G = V \times_\eta S$, where $V$ is a real vector space, and $S$ is a semisimple subgroup of $\mathrm{GL}(V)$ without compact factors, which acts irreducibly on $V^{\mathbb{C}}$. Then $\mathrm{Stab}_{M(G)}(\Gamma)$ is a lattice in $M(G)$ for all lattices $\Gamma$ in $G$.*

**Proof.** ¿From Corollary 1.3 we know that $\mathrm{Stab}_{I(G)}(\Gamma)$ is a lattice in $I(G)$. But by Corollary 1.7, $I(G)$ has finite index in $M(G)$. ∎

## 2. Lattices in Nilpotent Groups

In this section we consider stabilizers of log-lattices in nilpotent groups. If $\mathbf{G}$ is a linear algebraic group defined over the field $k$, we denote by $X_k(\mathbf{G})$ the group of $k$-rational characters of $\mathbf{G}$, by $\mathbf{G}_u$ the unipotent radical of $\mathbf{G}$, and by $\mathbf{G}^0$ the Zariski connected component. If $\mathfrak{g}$ is a linear Lie algebra $\subset \mathfrak{gl}(V)$, we denote by $\mathfrak{g}_n$ its nilradical, that is, the operators in $\mathrm{rad}(\mathfrak{g})$ which are nilpotent on $V$. Similarly, if $G$ is a linear Lie group, we denote by $G_u$ its unipotent radical, in other words, the normal analytic subgroup whose Lie algebra is $\mathfrak{g}_n$, and by $G_0$ its Euclidean connected component. If $\mathbf{G}$ is defined over $\mathbb{R}$, and $G = \mathbf{G}_{\mathbb{R}}$, then the component group $G/G_0$ is finite ([17, p. 10]).

Our first result in this section is a consequence of the celebrated theorems of Borel–Harish-Chandra [4] and Mostow–Tamagawa[16].

**Theorem 2.1.** *Let $\mathbf{G}$ be a linear algebraic group, defined over $\mathbb{Q}$, and let $\mathfrak{g}$ be the Lie algebra of the group of real points $\mathbf{G}_{\mathbb{R}}$. If $\mathrm{rad}(\mathfrak{g}) = \mathfrak{g}_n$, then $\mathbf{G}_{\mathbb{R}}/\mathbf{G}_{\mathbb{Z}}$ has finite invariant measure. If in addition $\mathfrak{g}/\mathfrak{g}_n$ is of compact type, then $\mathbf{G}_{\mathbb{R}}/\mathbf{G}_{\mathbb{Z}}$ is compact.*

**Proof.** For the proof, we first need some observations about Lie groups and real algebraic groups. Let $G$ be a linear Lie group, and $\mathfrak{g}$ its Lie algebra. The hypothesis that $\mathrm{rad}(\mathfrak{g}) = \mathfrak{g}_n$ (or equivalently that $\mathfrak{g}/\mathfrak{g}_n$ is semisimple) is equivalent to the assertion that $G_0/G_u$ is semisimple. Furthermore, the additional hypothesis that $\mathfrak{g}/\mathfrak{g}_n$ be of compact type is then equivalent, by Weyl's theorem, to the assertion that $G_0/G_u$ is compact.

Specializing now to the case $G = \mathbf{G}_{\mathbb{R}}$, we note that in this case $G_u = (\mathbf{G}_{\mathbb{R}})_u$ actually equals $(\mathbf{G}_u)_{\mathbb{R}}$. In fact, $(\mathbf{G}_u)_{\mathbb{R}}$ is connected and unipotent [17, p. 10], hence contained in the radical of $\mathbf{G}_{\mathbb{R}}$, and therefore in $(\mathbf{G}_{\mathbb{R}})_u$. Conversely, $\mathrm{rad}(\mathbf{G}_{\mathbb{R}}) \subset \mathrm{rad}(\mathbf{G})$, so $(\mathbf{G}_{\mathbb{R}})_u = \mathrm{rad}(\mathbf{G}_{\mathbb{R}})_u \subset \mathrm{rad}(\mathbf{G})_u = \mathbf{G}_u$; thus $(\mathbf{G}_{\mathbb{R}})_u \subset (\mathbf{G}_u)_{\mathbb{R}}$. Now by [17, p. 11], if $\mathbf{G} = \mathbf{G}_u\mathbf{H}$ (semidirect), with $\mathbf{H}$ reductive, then $(\mathbf{G}_{\mathbb{R}})_0$ is the semidirect product $(\mathbf{G}_u)_{\mathbb{R}}(\mathbf{H}_{\mathbb{R}})_0 = (\mathbf{G}_{\mathbb{R}})_u(\mathbf{H}_{\mathbb{R}})_0$. Thus the hypothesis $\mathrm{rad}(\mathfrak{g}) = \mathfrak{g}_n$ is in this case equivalent to semisimplicity of $(\mathbf{H}_{\mathbb{R}})_0$ (or $\mathbf{H}_{\mathbb{R}}$), and the additional hypothesis that $\mathfrak{g}/\mathfrak{g}_n$ be of compact type is equivalent to compactness of $(\mathbf{H}_{\mathbb{R}})_0$ (or $\mathbf{H}_{\mathbb{R}}$).

One further general observation about $\mathbf{G}$ is that the Euclidean connected components of $(\mathbf{G}^0)_{\mathbb{R}}$ and of $\mathbf{G}_{\mathbb{R}}$ are the same: $((\mathbf{G}^0)_{\mathbb{R}})_0 = (\mathbf{G}_{\mathbb{R}})_0$ (so that $\mathbf{G}_{\mathbb{R}}$ and $(\mathbf{G}^0)_{\mathbb{R}}$ have the same Lie algebra). In fact, the group $(\mathbf{G}_{\mathbb{R}})_0$ is Zariski connected, hence contained in $\mathbf{G}^0$; since it consists of real points, it must be contained in $(\mathbf{G}^0)_{\mathbb{R}}$; but since it is Euclidean connected, it must therefore be contained in $((\mathbf{G}^0)_{\mathbb{R}})_0$. The reverse inclusion is obvious.

Now suppose $\mathrm{rad}(\mathfrak{g}) = \mathfrak{g}_n$. To prove that $\mathbf{G}_{\mathbb{R}}/\mathbf{G}_{\mathbb{Z}}$ has finite invariant measure, we need to show that $\mathbf{G}^0$ has no non-trivial $\mathbb{Q}$-rational characters [4, Th. 9.4]. If $\chi \in X_{\mathbb{C}}(\mathbf{G}^0)$, then $\chi$ must be trivial on $(\mathbf{G}_{\mathbb{R}})_0 = (\mathbf{G}_{\mathbb{R}})_u(\mathbf{H}_{\mathbb{R}})_0$, since a rational character kills unipotent groups and connected semisimple groups. Since $[(\mathbf{G}^0)_{\mathbb{R}} : ((\mathbf{G}^0)_{\mathbb{R}})_0]$ is finite, $\chi$ sends $(\mathbf{G}^0)_{\mathbb{R}}$ to a finite subgroup of $\mathbb{C}^{\times}$. Now by a result of Rosenlicht, $(\mathbf{G}^0)_{\mathbb{R}}$ is Zariski dense in $\mathbf{G}^0$ ([17, p. 10]), and finite subgroups are Zariski closed, so $\chi$ maps the connected group $\mathbf{G}^0$ into a finite subgroup. Therefore $\chi$ is trivial, proving the first assertion of the Proposition.

Next suppose that $\mathfrak{g}/\mathfrak{g}_n$ is semisimple and of compact type. To prove that $\mathbf{G}_{\mathbb{R}}/\mathbf{G}_{\mathbb{Z}}$ is compact, we must show that all unipotent elements in $\mathbf{G}_{\mathbb{Z}}$ are actually in $\mathbf{G}_u$ ([4, Th. 11.8], [16, Th., p. 453]). From the previous remarks, we know that $(\mathbf{G}_{\mathbb{R}})_0/(\mathbf{G}_{\mathbb{R}})_u = (\mathbf{H}_{\mathbb{R}})_0$ is compact, so also $\mathbf{G}_{\mathbb{R}}/(\mathbf{G}_{\mathbb{R}})_u$. Therefore if $u$ is any unipotent in $\mathbf{G}_{\mathbb{Z}}$, or even in $\mathbf{G}_{\mathbb{R}}$, the image of $u$ by the quotient morphism is a unipotent element of a compact group. Hence the image of $u$ is both unipotent and semisimple, and therefore $u \in (\mathbf{G}_{\mathbb{R}})_u$. ■

The converses in 2.1 are not true. For a counterexample to the first assertion, we may take the $\mathbb{Q}$-group $\mathbf{G} = \mathrm{SO}(2, \mathbb{C})$; then $\mathbf{G}_{\mathbb{R}} = \mathrm{SO}(2, \mathbb{R})$ is a compact abelian group, so $\mathbf{G}_{\mathbb{Z}}$ is a (uniform) lattice, but $\mathfrak{g}_n = (0)$. For a counterexample to the second assertion we may take a semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$, such that $\mathbf{G}_{\mathbb{R}}$ is not compact (so $\mathfrak{g}$ is not of compact type), but $\mathbf{G}_{\mathbb{R}}/\mathbf{G}_{\mathbb{Z}}$ is compact [17, Th. 14.2].

We shall now consider the case of a simply connected nilpotent Lie group G that contains a lattice $\Gamma$. If $\mathfrak{g}_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-span of $\log(\Gamma)$, then by Malcev's theorem ([17, p. 34]), $\mathfrak{g}_{\mathbb{Q}}$ generates $\mathfrak{g}$ over $\mathbb{R}$, and the structure constants of $\mathfrak{g}$, relative to any $\mathbb{Q}$-basis of $\mathfrak{g}_{\mathbb{Q}}$, are rational. In particular, $\mathfrak{g}_{\mathbb{Q}}$ is a $\mathbb{Q}$-*structure* for $\mathfrak{g}$, that is, a $\mathbb{Q}$-Lie subalgebra which generates $\mathfrak{g}$ over $\mathbb{R}$. Thus every lattice in $G$ generates a $\mathbb{Q}$-structure of $\mathfrak{g}$. Our results in this section are valid for special lattices called *log-lattices*, that is, lattices $\Gamma$ for which $\log(\Gamma)$ is a lattice in $(\mathfrak{g}, +)$. ¿From [14, Th. 2] it follows if $G$ contains a lattice, then it must actually contain a log-lattice.

As we have observed in the beginning of the paper, if we choose any basis in $\mathfrak{g}_{\mathbb{Q}}$, then the group of differentials $M(G)^{\circ}$ of the measure-preserving automorphisms of $G$ is the real linear algebraic group $\mathcal{M}(\mathfrak{g})$ (defined over $\mathbb{Q}$, in our case). $\mathcal{M}(\mathfrak{g})$ is the set of real points of a linear algebraic group $\mathbf{M} = \mathbf{M}(\mathfrak{g})$. The $\mathbb{Q}$-algebraic group structure on $\mathbf{M}$ is independent of the choice of basis in $\mathfrak{g}_{\mathbb{Q}}$, since any two such bases are conjugate by an element of $\mathrm{GL}(n, \mathbb{Q})$. Furthermore, as we noted earlier, the Lie algebra of $\mathcal{M}(\mathfrak{g})$ is the algebra $\mathcal{D}_0(\mathfrak{g})$ (or simply $\mathcal{D}_0$) of derivations of trace 0. More generally, if $A$ is a closed subgroup of $M(G)$, and $\mathcal{A} = \{\alpha^{\circ} \colon \alpha \in A\}$ denotes the set of differentials, then the Lie algebra of $\mathcal{A}$ is an algebra of derivations $\mathfrak{a} \subset \mathcal{D}_0$.

**Theorem 2.2.** *Suppose $G$ is a simply connected nilpotent group whose Lie algebra $\mathfrak{g}$ has a $\mathbb{Q}$-structure $\mathfrak{g}_{\mathbb{Q}}$, and $\Gamma$ is a log-lattice in $G$ such that $\log(\Gamma) \subset \mathfrak{g}_{\mathbb{Q}}$. Let $A$ be a closed subgroup of $M(G)$, and assume that $A^{\circ}$ is the set of real points of a linear algebraic group $\mathbf{A} \subset \mathbf{M}$ defined over $\mathbb{Q}$ (for example, $A = M(G)$ satisfies this hypothesis). Then*

1. *If $\mathrm{rad}(\mathfrak{a}) = \mathfrak{a}_n$, then $\mathrm{Stab}_A(\Gamma)$ is a lattice in $A$.*

2. *If in addition $\mathfrak{a}/\mathfrak{a}_n$ is of compact type, then $\mathrm{Stab}_A(\Gamma)$ is uniform in $A$.*

**Proof.** We shall show first that the map $\alpha \mapsto \alpha^{\circ}$ induces an equivariant diffeomorphism of $A/\mathrm{Stab}_A(\Gamma) \to \mathbf{A}_{\mathbb{R}}/\mathbf{A}_{\mathbb{Z}}$. Indeed, if $\mathcal{S}$ denotes the stabilizer of $\log(\Gamma)$ in $\mathcal{A}$, then the isomorphism $\alpha \mapsto \alpha^{\circ}$ of $M(G)$ onto $\mathcal{M}(\mathfrak{g})$ takes $A$ onto $\mathcal{A}$ and $\mathrm{Stab}_A(\Gamma)$ to $\mathcal{S}$, because of the relation $\alpha \circ \exp = \exp \circ \alpha^{\circ}$. Now if $\beta \in \mathfrak{gl}(\mathfrak{g})$, then $\beta$ stabilizes $\log(\Gamma)$ iff it maps the basis elements of $\log(\Gamma)$ to another basis of $\log(\Gamma)$. If $\beta \in \mathcal{A}$, then $\det \beta$ is a unit, so $\beta$ stabilizes $\log(\Gamma)$ iff it has integer matrix entries (with respect to a fixed basis of $\log(\Gamma)$). Thus $\mathcal{S} = \mathcal{A}_{\mathbb{Z}} = \mathbf{A}_{\mathbb{Z}}$, and by hypothesis, $\mathcal{A} = \mathbf{A}_{\mathbb{R}}$. Now the result follows from 2.1. ■

**Corollary 2.3.** *If $D(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$ for every $D \in \operatorname{rad}(\mathfrak{a})$, then $\operatorname{Stab}_A(\Gamma)$ is a lattice in $A$. If the inclusion actually holds for every $D \in \mathfrak{a}$, then $\operatorname{Stab}_A(\Gamma)$ is a uniform lattice.*

**Proof.** If $D$ is a derivation such that $D(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$, then $D^i(\mathfrak{g}) \subset \mathfrak{g}^i$ for each term $\mathfrak{g}^i$ in the descending central series. Since $\mathfrak{g}$ is a nilpotent Lie algebra, it follows that $D$ is a nilpotent transformation. Therefore the first hypothesis implies that $\operatorname{rad}(\mathfrak{a}) = \mathfrak{a}_n$. The second hypothesis implies that every derivation in $\mathfrak{a}$ is nilpotent, so $\mathfrak{a}$ is a nilpotent Lie algebra. The corresponding analytic group $\mathcal{A}_0$ is therefore nilpotent, so all its lattices are uniform [13]. Since $\mathcal{A}_0$ has finite index in $\mathcal{A}$, all lattices in $\mathcal{A}$ and therefore $A$ are also uniform. ∎

Our next results concern the Iwasawa nilpotent parts of the simple, non-compact, real, rank-1 simple Lie groups. These nilpotent parts have been classified in [6, Th. 1.1 and Th. 4.2], and consist of the real vector groups $\mathbb{R}^n$, the Heisenberg groups $N_n$, their quaternionic analogues $H_n$, and the exceptional case $n = 1$ of the groups $C_n$ built on the Cayley numbers. In the class of real vector groups, $G = \mathbb{R}^n$, stabilizers in $M(G)$ of lattices in the group are of course non-uniform lattices in $M(G)$. In the Heisenberg groups, we shall see below (2.4) that stabilizers of log-lattices are also non-uniform lattices in $M(G)$ . Turning to the last two classes of groups, we prove in Proposition 2.5 that for $G = H_n$ with $n = 1$ or $n = 2$, and for the exceptional case $C_1$, stabilizers of log-lattices are actually uniform lattices in $M(G)$.

**Proposition 2.4.** *Let $G$ be a Heisenberg group. If $\Gamma$ is any log-lattice in $G$, then the stabilizer $S = \operatorname{Stab}_{M(G)}(\Gamma)$ is a non-uniform lattice in $M(G)$.*

**Proof.** Assuming $G$ is $2n+1$-dimensional, we shall write elements of $G$ as matrices

$$\begin{pmatrix} 1 & x_1 & \dots & x_n & z \\ & 1 & & 0 & y_1 \\ & & \ddots & & \vdots \\ & 0 & & 1 & y_n \\ & & & & 1 \end{pmatrix}$$

with $x_i, y_i, z \in \mathbb{R}$. Then $\mathfrak{g}$ has a basis consisting of $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$ where $X_i$ is defined by the condition $x_i = 1$ and all other off-diagonal matrix entries are 0 (similarly for $Y_i$ and $Z$). The only non-trivial bracketing relations are given by $[X_i, Y_i] = Z$. A direct calculation shows that relative to this basis, the general derivation $D$ of trace 0 has matrix given by

$$D = \begin{pmatrix} A & B & 0 \\ C & -A^t & 0 \\ v & w & 0 \end{pmatrix}$$

where $C$ and $B$ are symmetric, and $v$ and $w$ are vectors in $\mathbb{R}^n$. Thus the algebra $\mathcal{D}_0$ is isomorphic to the semidirect sum $\mathbb{R}^{2n} \oplus \mathfrak{sp}(n, \mathbb{R})$. $\mathfrak{sp}(n, \mathbb{R})$ is semisimple of non-compact type, and the radical of $\mathcal{D}_0$ consists of the nilpotent operators

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v & w & 0 \end{pmatrix}.$$

By Corollary 2.2 the stabilizer of $\Gamma$ in $M(G)$ is a lattice.

To see that this stabilizer is non-uniform, we consider first the lattice $\Gamma_k$ defined by the conditions $x_i, y_i \in \mathbb{Z}$, and $z \in \frac{1}{k}\mathbb{Z}$, with $k$ even. Then $\log\Gamma_k$ is a lattice, generated over $\mathbb{Z}$ by the basis $\mathcal{B}_k = \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \frac{1}{k}Z\}$. From the description of $\mathcal{D}_0$ given above, it is clear that the identity component of $\mathcal{M}(\mathfrak{g})$ is the set of matrices of the form

$$\beta = \left(\begin{array}{c|c} \sigma & 0 \\ \hline v \quad w & 1 \end{array}\right)$$

with $\sigma \in \mathrm{SP}(n, \mathbb{R})$ and $v, w \in \mathbb{R}^n$. Furthermore, as in the proof of Corollary 2.2, $\beta$ is in the stabilizer of $\log\Gamma_k$ iff it is in $(\mathcal{M}(\mathfrak{g})_0)_{\mathbb{Z}}$. Hence the orbit of $\log\Gamma_k$ under $\mathcal{M}(\mathfrak{g})_0$ is diffeomorphic to $(\mathrm{SP}(n,\mathbb{R})/\mathrm{SP}(n,\mathbb{Z})) \times \mathbf{T}^{2n}$. This orbit is of course non-compact, since $\mathrm{SP}(n,\mathbb{Z})$ has many non-trivial unipotents. Since $\mathcal{M}(\mathfrak{g})$ has only finitely many components, it follows that $\mathcal{M}(\mathfrak{g})/\operatorname{Stab}_{\mathcal{M}(\mathfrak{g})}(\log\Gamma_k)$ is not compact either.

Now in general, by [1], any lattice $\Gamma$ in $G$ is isomorphic under an automorphism $\alpha$ of $G$ to a lattice of the form $\Gamma_{k,\vec{r}}$ for some $k \in \mathbb{Z}^+$ and some $\vec{r} = (r_1, \ldots, r_n) \in (\mathbb{Z}^+)^n$ satisfying $r_1 = 1, r_i | r_{i+1}$ for all $i$: $\Gamma_{k,\vec{r}}$ is defined by the conditions $x_i \in r_i\mathbb{Z}$, $y_i \in \mathbb{Z}$, and $z \in \frac{1}{k}\mathbb{Z}$. Since $M(G)$ is normal and conjugation by $\alpha$ carries $\operatorname{Stab}_{M(G)}(\Gamma)$ to $\operatorname{Stab}_{M(G)}(\Gamma_{k,\vec{r}})$, the quotient spaces are diffeomorphic. Therefore it is enough to prove that the stabilizer of $\Gamma_{k,\vec{r}}$ is not uniform when $\Gamma_{k,\vec{r}}$ is a log-lattice, that is, when $k$ is even. In this case $\log\Gamma_{k,\vec{r}}$ is the lattice generated over $\mathbb{Z}$ by the basis $\mathcal{B}_{k,\vec{r}} = \{r_1X_1, \ldots, r_nX_n, Y_1, \ldots, Y_n, \frac{1}{k}Z\}$. If $\phi$ is the isomorphism of $\mathfrak{g}$ taking $r_iX_i$ to $X_i$ and leaving all other elements of $\mathcal{B}_{k,\vec{r}}$ fixed (so $\phi$ takes $\mathcal{B}_{k,\vec{r}}$ to $\mathcal{B}_k$), then conjugation by $\phi$ is a $\mathbb{Q}$-rational isomorphism $\Phi$ of the $\mathbb{Q}$-group $\mathcal{M}(\mathfrak{g})$ relative to the basis $\mathcal{B}_k$, to the $\mathbb{Q}$-group $\mathcal{M}(\mathfrak{g})$ relative to the basis $\mathcal{B}_{k,\vec{r}}$. Furthermore, since by [17, Cor. 10.14.ii] $\Phi$ takes arithmetic subgroups to arithmetic subgroups, $\Phi(\operatorname{Stab}_{\mathcal{M}(\mathfrak{g})}(\log\Gamma_k))$ is commensurable with $\operatorname{Stab}_{\mathcal{M}(\mathfrak{g})}(\log\Gamma_{k,\vec{r}})$, since the stabilizers are the sets of $\mathbb{Z}$-points of $\mathcal{M}(\mathfrak{g})$ relative to the $\mathcal{B}_k$ and $\mathcal{B}_{k,\vec{r}}$, respectively. Since we have already shown that $\operatorname{Stab}(\log\Gamma_k)$ is a non-uniform lattice, the same is true of $\operatorname{Stab}(\log\Gamma_{k,\vec{r}})$. This completes the proof. ∎

Next we shall examine the groups $G = H_n$, which are the Iwasawa nilpotent parts of the symplectic rank-1 groups, and $G = C_1$, the Iwasawa nilpotent part of the exceptional simple rank-1 group. We begin with $H_n$ and its Lie algebra $\mathfrak{h}_n$.

For the construction, we let $\mathbb{H}$ denote the algebra of quaternions over $\mathbb{R}$, and $\mathbf{P}$ denote the subspace of pure quaternions, generated by $i$, $j$, and $k$. Then $\mathbf{P} = \operatorname{Im}(\mathbb{H})$, with $\operatorname{Im}(q) = (q - \bar{q})/2$. For any $n \geq 1$, we define

$$\mathfrak{g} = \mathfrak{h}_n = \mathbf{P} \oplus \mathbb{H}^n,$$

a vector space of dimension $4n+3$, with operations given as follows:

$$[(p,v),(q,w)] = (\mathrm{Im}\langle v,w\rangle, 0),$$

where

$$\langle v,w\rangle = \bar{v}_1 w_1 + \ldots \bar{v}_n w_n$$

for $v=(v_1,\ldots,v_n)$ and $w=(w_1,\ldots,w_n)$ in $\mathbb{H}^n$. We note that if $\mathbb{H}$ is replaced by $\mathbb{C}$, then the analogous construction gives the Heisenberg algebra of dimension $2n+1$ (if $\mathbb{H}$ is replaced by $\mathbb{R}$, then we get $\mathbb{R}^{n+1}$!). It is easy to see that $\mathbf{P} = [\mathfrak{g},\mathfrak{g}]$, and is the center of $\mathfrak{g}$, so that $\mathfrak{g}$ is two-step nilpotent.

Similarly, we can define two-step nilpotent groups $C_n$ by analogy with the groups $H_n$, by replacing the associative algebra $\mathbb{H}$ by the non-associative algebra $\mathbb{K}$ of Cayley numbers. $C_1$ is the Iwasawa nilpotent part of the exceptional rank-1 simple group.

**Proposition 2.5.** *Let $G = H_n$, with $n=1$ or $n=2$, or $G=C_1$, and let $\Gamma$ be a log-lattice in $G$. Then the stabilizer $\mathrm{Stab}_{M(G)}(\Gamma)$ is a uniform lattice in $M(G)$.*[4]

**Proof.** For each of the three groups mentioned, one can exhibit an explicit decomposition $\mathcal{D}_0(\mathfrak{g}) = (\mathcal{D}_0)_n \oplus \mathfrak{s}$ (semidirect sum), where $(\mathcal{D}_0)_n$ is the nilradical, and $\mathfrak{s}$ is a compact semisimple algebra[5]. The result then follows from Corollary 2.2. ∎

We turn now to another class of semidirect products, in which the group of automorphisms $A$ is nilpotent, but in contrast with the examples discussed earlier, here $A$ turns out to be essentially different from both $I(G)$ and $M(G)$. We shall prove that for these groups, the $A$-orbit spaces of log-lattices are compact (see Proposition 2.10).

The groups $G$ under consideration are of the form $G = V \times_\eta S$, where $V$ is an n-dimensional vector space with $n>1$, and $S$ is a 1-parameter group in $\mathrm{SL}(V)$, say $S=(e^{tC})_{t\in\mathbb{R}}$ with $\mathrm{tr}\,C=0$. $G$ is unimodular since $\mathrm{tr}\,C=0$. As usual, we write $\mathfrak{s}$ $(=\mathbb{R}C)$ for the Lie algebra of $S$, and we identify $V$ with its Lie algebra. Thus $\mathfrak{g} = V\oplus\mathfrak{s}$ (semidirect sum). We define an algebra $\mathfrak{a}$ of derivations of trace 0 on $\mathfrak{g}$, as follows:

$$\mathfrak{a} = \{D\in\mathcal{D}_0(\mathfrak{g})\colon D(V)\subset V, \mathrm{tr}_V(D_V)=0\}. \tag{3}$$

Thus $\mathfrak{a}$ consists of the derivations of trace 0 which preserve $V$, *and have trace 0 on $V$*.[6]

[4] This result has been proved in general by P. Barbano, and will appear in his forthcoming thesis.

[5] In fact, one can show that $\mathfrak{h}_1/(\mathfrak{h}_1)_n = \mathfrak{sp}(1)\oplus\mathfrak{sp}(1)$, $\mathfrak{h}_2/(\mathfrak{h}_2)_n = \mathfrak{sp}(2)\oplus\mathfrak{sp}(1)$ and $\mathfrak{c}_1/(\mathfrak{c}_1)_n = \mathfrak{so}(7)$.

[6] If the degree of nilpotence of $C$ is greater than 2, then the condition $D(V)\subset V$ in the definition of $\mathfrak{a}$ is redundant, since $V$ is actually a characteristic ideal of $\mathfrak{g}$. In fact, one can show that in the present situation, if we describe a linear map $D\in\mathrm{End}(\mathfrak{g})$ as a block matrix

$$\begin{pmatrix} A & v \\ w^t & \delta \end{pmatrix},$$

**Lemma 2.6.** *If $D \in \mathfrak{gl}(\mathfrak{g})$, then $D \in \mathfrak{a}$ iff $D$ equals*

$$D(L,\varphi) = \begin{pmatrix} L & \varphi \\ 0 & 0 \end{pmatrix} \tag{4}$$

*where $L \in \mathfrak{sl}(V)$, and $[L,C] = 0$.*

**Proof.** If $D$ is a linear operator on $\mathfrak{g}$ leaving $V$ invariant, and we decompose $D$ as in Proposition 1.5 (replacing $\mathfrak{h}$ by $V$), then $D_{\mathfrak{s}}$ is given by $D_{\mathfrak{s}}(X) = \lambda X$ where $\lambda$ is a constant (since $\mathfrak{s}$ is 1-dimensional). If in addition $\operatorname{tr} D = 0$, then $\lambda = -\operatorname{tr}_V(D_V)$. Now it is easy to see from Proposition 1.5 that if $D$ leaves $V$ invariant and has trace 0, then $D$ is a derivation iff

$$[D_V, C] = -\operatorname{tr}(D_V)C.$$

Therefore such a $D$ is in $\mathfrak{a}$ iff $[D_V, C] = 0$. ■

Henceforth we shall assume that $C$ is a nilpotent operator. In this case, the algebra $\mathfrak{a}$ is essentially different from both $\operatorname{ad}(\mathfrak{g})$ and $\mathcal{D}_0(\mathfrak{g})$.

**Proposition 2.7.** *If $C$ is a non-zero nilpotent operator, then both of the inclusions $\operatorname{ad}(\mathfrak{g}) \subset \mathfrak{a} \subset \mathcal{D}_0(\mathfrak{g})$ are strict.*

**Proof.** It is easy to see that $\operatorname{ad}(v,X) = D(X,\hat{v})$, where $\hat{v}$ is the map $Y \mapsto -Y(v)$ on $\mathfrak{s}$. Since $X$ is a multiple of $C$, $X$ has trace 0 and commutes with $C$. Therefore $\operatorname{ad}(\mathfrak{g}) \subset \mathfrak{a}$. Now $\dim \operatorname{ad}(\mathfrak{g}) = \dim \mathfrak{g} - \dim \mathfrak{z}(\mathfrak{g})$, and an easy calculation shows that $\mathfrak{z}(\mathfrak{g}) = \ker C$. Since $C$ is nilpotent, we have

$$\dim \operatorname{ad}(\mathfrak{g}) = n + 1 - \dim \ker C \leq n.$$

On the other hand,

$$\dim(\mathfrak{a}) = \dim\{D(L,\varphi) \colon L \in \mathfrak{sl}(V), [L,C] = 0, \varphi \in \operatorname{Hom}_{\mathbb{R}}(\mathfrak{s}, V)\}.$$

Since $D(L,\varphi) = 0$ iff $L = \varphi = 0$, and $\mathfrak{s}$ is 1-dimensional, $\dim \mathfrak{a} = \dim \mathcal{Z}_{\mathfrak{sl}(V)}(C) + n$ ($\mathcal{Z}$ denotes the centralizer). Since $C$ centralizes itself, $\dim \mathfrak{a} \geq n+1$. Therefore $\operatorname{ad}(\mathfrak{g})$ is a proper subset of $\mathfrak{a}$.

For the other inclusion, we observe that by the Jacobson–Morozov theorem [11, p. 100], there is an $H \in \mathfrak{gl}(V)$ with $[H,C] = 2C$. Set $H_1 = H + \lambda I$ with $\lambda$ chosen so that $\operatorname{tr}(H_1) = -2$. Then $[H_1, C] = -\operatorname{tr}(H_1)C$, so by Lemma 2.6 and its proof, the map

$$D = \begin{pmatrix} H_1 & 0 \\ 0 & -\operatorname{tr}(H_1) \end{pmatrix}$$

is a derivation of $\mathfrak{g}$ with trace 0 (leaving $V$ invariant, in fact), but is not in $\mathfrak{a}$. ■

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with $A \in \operatorname{End}(V)$, $v, w \in V$, and $\delta \in \mathbb{R}$, then $D$ is a derivation iff $w \perp \operatorname{im} C$, $(w \cdot y)Cx = (w \cdot x)Cy$ for all $x, y \in V$, and $[A, C] = \delta C$. If C has degree of nilpotence $> 2$, then we can find $y \in \operatorname{im} C$ but not in $\ker C$. The second condition then shows that if $D$ is a derivation, then $w = 0$, so $D(V) \subset V$.

¿From now on we assume that $C$ is of the form

$$C = \begin{pmatrix} 0 & c_{12} & \dots & * \\ 0 & 0 & \ddots & * \\ \vdots & \vdots & & c_{n-1,n} \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad c_{k-1,k} \neq 0 \text{ for } k = 1, \dots, n \tag{5}$$

with respect to a fixed basis $\{e_1, \dots, e_n\}$ of $V$. Since $C$ is a nilpotent operator, the group $G = V \times_\eta (e^{tC})$ is nilpotent. We denote by $\mathfrak{n}_n \subset \mathfrak{sl}(V)$ the Lie algebra of upper niltriangular operators on $V$ with respect to this basis.

**Lemma 2.8.** *Let $C$ be of the form* (5). *If $X \in \mathfrak{sl}(V)$ and $[X, C] = 0$ then $X \in \mathfrak{n}_n$.*

**Proof.** Because $[X, C] = 0$ it follows that $[X, C^k] = 0$ for all positive $k$ and hence $X$ leaves each $\ker C^k$ invariant. But $\{\ker C^k\colon k = 1, \dots, n\}$ form a flag. This means that $X = (x_{ij})$ is upper triangular, so for each $k$,

$$Xe_k = x_{kk}e_k \mod < e_1, \dots, e_{k-1} > .$$

Thus from the hypothesis on $C$ we get

$$CXe_k = c_{k-1,k}x_{kk}e_{k-1} \operatorname{mod} < e_1, ..., e_{k-2} >,$$

while

$$XCe_k = Xc_{k-1,k}e_{k-1} = c_{k-1,k}x_{k-1,k-1}e_{k-1} \operatorname{mod} < e_1, \dots, e_{k-2} >.$$

Because $[X, C] = 0$ and $c_{k-1,k} \neq 0$, we see that $x_{kk} = x_{k-1,k-1}$ for all $k$. Since $\operatorname{tr} X = 0$, each $x_{kk} = 0$. ■

**Lemma 2.9.** *Let $C$ be of the form* (5). *If the entries $c_{ij}$ are rational for all $i, j$, then $G$ has a log-lattice which intersects $V$ in a lattice of $V$.*

**Proof.** The map

$$(v, e^{tC}) \mapsto \begin{pmatrix} e^{tC} & v \\ 0 & 1 \end{pmatrix}$$

is an isomorphism of $G$ onto a subgroup $H$ of the unitriangular group in $\mathrm{GL}(n + 1, \mathbb{R})$. $H$ is a real algebraic group defined over $\mathbb{Q}$, because $C$ is niltriangular with rational entries. Furthermore, the Zariski closure $\mathbf{H}$ of $H$ is unipotent, so $\mathbf{H}_\mathbb{R}/\mathbf{H}_\mathbb{Z}$ is compact. Now $H = \mathbf{H}_\mathbb{R}$ ([17, p. 9]). Furthermore, $H_\mathbb{Z} = \mathbf{H}_\mathbb{Z}$ intersected with the image of $V$ is the integer points of this image, and is therefore a lattice. Pulling back $H_\mathbb{Z}$ to $G$ gives a lattice satisfying the condition. ■

We are now ready for the main result for this class of nilpotent semidirect product groups.

**Proposition 2.10.** *Let $V$ be a real $n$-dimensional vector space ($n > 1$), $S = (e^{tC})$ with $C$ of the form* (5) *with $c_{ij}$ rational for all $i, j$. Let $G = V \times_\eta S$, and $A$ be the subgroup of* $\mathrm{Aut}(G)$ *defined by*

$$A = \{\alpha \in M(G)\colon \alpha(V) \subset V, \alpha|_V \in M(V)\}.$$

*Then* $\mathrm{Stab}_A(\Gamma)$ *is a uniform lattice in $A$, for every log-lattice $\Gamma$ in $G$ which intersects $V$ in a lattice of $V$.*

**Proof.** We observe first by (3) that the Lie algebra of $\mathcal{A} = A^\circ$ is $\mathfrak{a}$, since as one sees easily, $\mathcal{A}$ is defined by the conditions

$$\beta \in \mathcal{M}(\mathfrak{g}), \quad \beta(V) \subset V, \quad \det_V^2 \beta = 1.$$

Now by Lemma 2.8, $\mathcal{Z}_{\mathfrak{sl}(V)}(C) \subset \mathfrak{n}_n$. It follows from Lemma 2.6 that $\mathfrak{a} \subset \mathfrak{n}_{n+1}$, so that $\mathcal{A}$ is a subgroup of the *unipotent* group of order $n+1$. The conditions defining $\mathcal{A}$ show that it is a real linear algebraic $\mathbb{Q}$-group relative to the $\mathbb{Q}$-structure defined by any log-lattice which intersects $V$ in a lattice. The result now follows from Corollary 2.2, since $\mathfrak{a}$ is nilpotent. ∎

Our final results furnish examples of nilpotent groups where the stabilizer $\mathrm{Stab}_{M(G)}(\Gamma)$ is not a lattice at all. These examples show that there are significant limitations on the possibility of extending Corollary 2.2. The first example (2.12) is the unitriangular group ($n \geq 4$), and the second (2.13) is even two-step nilpotent. We begin with a lemma.

**Lemma 2.11.** *Let $G \subset \mathrm{GL}(V)$ be a linear algebraic group, defined over $\mathbb{Q}$. If $\mathfrak{g}$ is the Lie algebra of $G_\mathbb{R}$, and $\mathfrak{g}$ has a $\mathbb{Q}$-rational weight when acting on $V_\mathbb{R}$, then $G^0$ has a non-trivial $\mathbb{Q}$-rational character.*

**Proof.** We assume given a $\mathbb{Q}$-form $V_\mathbb{Q} \subset V_\mathbb{R}$ of $V$, and (by hypothesis) also a non-zero semi-invariant vector $v \in V_\mathbb{Q}$ satisfying $Xv = \lambda(X)v$ for all $X \in \mathfrak{g}$, with $\lambda$ non-trivial. If $v = v_1$ is extended to a basis of $V_\mathbb{Q}$, then the matrices $(X_{ij})$ of elements of $\mathfrak{g}$ satisfy $X_{i1} = 0$ for $i > 1$. By the proof of Theorem 2.1, $\mathfrak{g}$ is the Lie algebra of $(G^0)_\mathbb{R}$. Exponentiation therefore shows that the matrices $(g_{ij})$ of elements of $(G^0)_\mathbb{R}$ also satisfy $g_{i1} = 0$ for $i > 1$. But $(G^0)_\mathbb{R}$ is Zariski dense in $G^0$, so the same relation holds for the latter group. Thus the map $g \mapsto g_{11}$ is a non-trivial $\mathbb{Q}$-rational character of $G^0$. ∎

**Example 2.12.** Let $G$ be the group of $n \times n$ real unitriangular matrices, for $n \geq 4$. We shall show that there are log-lattices in $G$ whose stabilizer in $M(G)$ is not a lattice, i.e. is a discrete subgroup, but has infinite covolume (of course, such lattices cannot exist when $n = 3$). We believe that the underlying reason that these stabilizers are not lattices is that $M(G)$ is not unimodular, but we have not proven that. Instead, we give a direct proof.

The Lie algebra $\mathfrak{g}$ of $G$ consists of all $n \times n$ upper triangular matrices with 0's on the diagonal, and so has a basis

$$\mathcal{B} = \{E_{12}, E_{23}, \ldots, E_{n-1,n}, E_{13}, \ldots, E_{n-2,n}, \ldots, E_{1n}\}.$$

Here $E_{ij}$ is the matrix with a single 1 in the $i,j$ position, and 0's elsewhere. The Lie brackets in $\mathfrak{g}$ are as follows:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$$

for $i < j$ and $k < l$. In particular, the basis $\mathcal{B}$ generates a $\mathbb{Q}$-structure for $\mathfrak{g}$. Now the center of $\mathfrak{g}$ is 1-dimensional, generated by $E_{1n}$, so for any $D \in \mathcal{D}$, and $Z \in \mathfrak{z}(\mathfrak{g})$, $D(Z) = \lambda(D)Z$, with $\lambda(D) \in \mathbb{R}$. In view of Lemma 2.11 applied to the Lie algebra $\mathcal{D}_0$, and the Borel–Harish-Chandra theorem [4], to prove that not all stabilizers of log-lattices are themselves lattices, it is sufficient to show that for $n \geq 4$, $\lambda(D) \neq 0$ for some $D \in \mathcal{D}_0$.

To see this, choose $n-1$ arbitrary real numbers $d_1, \ldots, d_{n-1}$, and define a derivation $D$ as follows:

$$D(E_{ij}) = \left(\sum_{p=i}^{j-1} d_p\right) E_{ij}$$

for $1 \leq i < j \leq n$. It is easy to see that $D$ is a derivation, for any choice of the $d_i$'s. Furthermore,

$$\lambda(D) = \sum_{p=1}^{n-1} d_p,$$

while

$$\operatorname{tr}(D) = (n-1)d_1 + 2(n-2)\sum_{p=2}^{n-2} d_p + (n-1)d_{n-1}.$$

For $n \geq 4$ these two forms are clearly linearly independent, so we can choose the $d_i$'s so that $\operatorname{tr}(D) = 0$ and $\lambda(D) \neq 0$.

The Iwasawa nilpotent parts of rank-1 simple groups are two-step nilpotent (or abelian). In our last example, we show that it is not simply the low degree of nilpotence that causes stabilizers of log-lattices to be lattices themselves.

**Example 2.13.** Let $G$ be the simply connected two-step nilpotent group whose Lie algebra $\mathfrak{g}$ is the direct sum of the Heisenberg Lie algebra and $\mathbb{R}$. Let $Z$ span $[\mathfrak{g}, \mathfrak{g}]$, $\{Z, W\}$ span $\mathcal{Z}(\mathfrak{g})$, and let $\{Z, W, X, Y\}$ be a basis for $\mathfrak{g}$ with $[X, Y] = Z$. It is easy to see that with respect to this basis any $D \in \mathcal{D}_0$ is of the form

$$D = \begin{pmatrix} \alpha & \beta & \\ 0 & -2\alpha & * \\ & 0 & \tilde{D} \end{pmatrix}$$

where $\alpha = \operatorname{tr}\tilde{D}$ and $\tilde{D}$ is an arbitrary element of $\mathfrak{gl}(2, \mathbb{R})$. The same argument as in Example 2.12 shows that $G$ contains a log-lattice whose stabilizer in $M(G)$ is not a lattice.

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