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The Brauer algebra and the Birman-Wenzl-Murakami algebra

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1. Introduction

Knowledge of finite dimensional representations and their morphisms is a useful tool in the study of classical semisimple Lie groups. Fundamental works in this field were done by R. Brauer [B] and H. Weyl [W] in the thirties. Investigations of q-deformed simple Lie groups lead to similar questions. The appearing algebras are closely related with representations of braid groups. They were extensively studied by V. Jones, J. Birman [BW] and H. Wenzl [We] in the eighties. In Sections 2 and 3 we recall the definition and the some properties of the Brauer algebra. In Sections 4 and 5 we discuss the q-deformed version of this algebra, the so-called Birman-Wenzl-Murakami algebra. In our discussion we restrict ourselves to the orthogonal groups. The results for the symplectic groups are completely similar. In Section 6 we show how the latter algebra occurs in classification of bicovariant bimodules and differential calculi on quantum groups.

2. Brauer-Weyl Duality for Classical Simple Lie Groups

Let $\rho\colon G \to GL(V)$ and $\tau\colon G \to GL(W)$ finite dimensional representations of the semisimple Lie group G on vector spaces V and W, respectively. Recall that the *intertwining space* of ρ and τ is the vector space

$$\mathrm{Mor}(\rho,\tau) = \{T \in L(V,W) : \tau(g)T = T\rho(g) \quad \text{for } g \in G\}.$$

In case $\tau = \rho$ the space $\mathrm{Mor}(\rho,\tau)$ is obviously an algebra. It is called the *centralizer algebra* of ρ and is denoted by $\mathrm{Mor}(\rho)$. The Brauer-Weyl duality establishes the relation between irreducible subrepresentations of ρ and properties of the algebra $\mathrm{Mor}(\rho)$. To be more precise, $\mathrm{Mor}(\rho)$ is semisimple, the invariant subspaces of V are in 1-1-correspondence with the right ideals of $\mathrm{Mor}(\rho)$, and the irreducible subrepresentations of ρ are in 1-1-correspondence with the minimal right ideals of $\mathrm{Mor}(\rho)$.
Let G be a subgroup of the general linear group over the vector space $V = C^N$, $u\colon G \to GL(V)$ the fundamental representation (embedding) of G and let $\rho = u \otimes \cdots \otimes u$ (f times). In this situation we write $B_f(G)$ for $\mathrm{Mor}(\rho)$. Obviously, $B_f(G)$ is a subalgebra of $L(V \otimes \cdots \otimes V)$. For $\pi \in S_f$, let P_π denote the linear

operator on the tensor space $V^{\otimes f}$ which permutes the order of vectors according to π. It is easily seen that the operators P_π belong to $B_f(G)$. That is, we have

$$\mathrm{alg}\{P_\pi : \pi \in S_f\} = \mathrm{lin}\{P_\pi : \pi \in S_f\} \subseteq B_f(G) \tag{1}$$

for all subgroups G of $GL(V)$. For the groups $G = GL(N, C)$ and $G = SL(N, C)$ we have even equality in (1). This is a classical result of H. Weyl (1937).

Now let us consider the orthogonal groups $G = O(N, C)$. We begin with the special case $f = 2$. Since the fundamental representation u coincides with its contragredient representation , the trivial representation is contained in $u \otimes u$ with multiplicity one. Hence, the intertwining spaces $\mathrm{Mor}(u \otimes u, 1)$ and $\mathrm{Mor}(1, u \otimes u)$ are one-dimensional. Non-trivial elements of these spaces are the linear mappings $B_\bullet = (b_{rs})$ and $C^\bullet = (c^{rs})$ resp., where $b_{rs} = c^{rs} = \delta^r_s$ for $r, s = 1, \ldots, N$. Composing both mappings we get an element of $B_2(G) = \mathrm{Mor}(u \otimes u, u \otimes u)$, the so-called trace operator $E = C^\bullet \cdot B_\bullet$. Let P denote the flip operator on the tensor product, i.e. $P(x \otimes y) = y \otimes x$. The operators E, P, and the identity I generate a three dimensional algebra: $E^2 = N \cdot E$, $E \cdot P = P \cdot E = E$, $P^2 = I$.

Throughout we denote by $T_{i,i+1} \in L(V \otimes \cdots \otimes V)$ the operator $I \otimes \cdots \otimes \underbrace{T}_{i,i+1} \otimes \cdots \otimes I$, for any T in $L(V \otimes V)$.

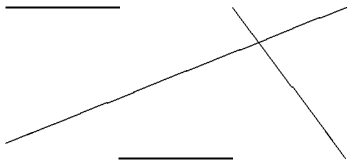
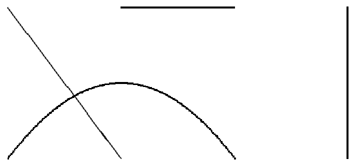
Now let $f \geq 2$ be arbitrary. It is easy to see that for all $T \in B_2(G)$ the operator $T_{i,i+1}$ belongs to $B_f(G)$ for $i = 1, \ldots, f-1$. Hence,

$$\mathrm{alg}\{E_{12}, P_\pi : \pi \in S_f\} \subseteq B_f(G)\,. \tag{2}$$

Another classical result of H. Weyl states that for the group $G = O(N, C)$ we have again equality in (2). The same is true for the odd special orthogonal groups $G = SO(2n+1, C)$, but not for $SO(2n, C)$.

3. THE BRAUER ALGEBRA D_f

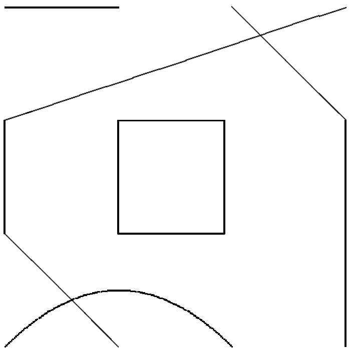
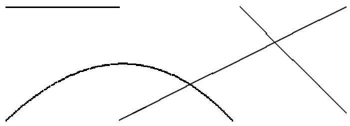
We will first define the Brauer algebra D_f over the field of rational functions $C(x)$. For $f = 0$ let $D_0 = C(x)$. For $f > 0$, a linear basis of the $C(x)$ algebra D_f is given by graphs with f edges and $2f$ vertices, arranged in two lines of f vertices each. In these graphs each edge belongs to exactly two vertices and each vertex belongs to exactly one edge. Two examples for graphs in D_4 are

$a =$  $b =$  .

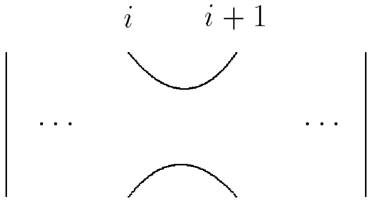
It is easy to see that we have $2f-1$ possibilities to join the first vertex with another one, then $2f-3$ possibilities for the next one and so on. So the dimension of D_f is $(2f-1) \cdot (2f-3) \cdot \ldots \cdot 3 \cdot 1$. To define the multiplication in D_f, it is enough to define the product $a\,b$ for two graphs a and b. This is done similarly as with braids by the following rules.

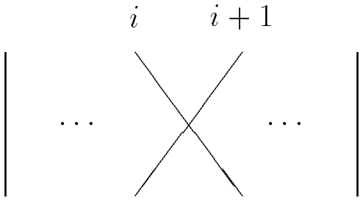
1. Draw b below a.
2. Connect the i-th upper vertex of b with the i-th lower vertex of a.
3. Let d be the number of cycles in the graph obtained in 2. and let c be this graph without the cycles. Then we define $a \cdot b = x^d \cdot c$.

Example.

$a \cdot b =$  $= x \cdot$  .

We will call an edge *horizontal* if it joins two vertices in the same row. Note that there are as many horizontal edges in the upper row as there are in the lower one. Whenever a graph has no horizontal edges, it can be regarded as a permutation π connecting the i-th upper vertex to the $\pi(i)$-th lower vertex. It is easy to check that the multiplication of graphs is compatible with the composition of permutations under this identification. Therefore it is obvious that D_f contains $C(x)S_f$ as a subalgebra. Let e_i and g_i for $i = 1, \ldots, f-1$ denote the graphs

$e_i :=$ 

, $g_i :=$ 

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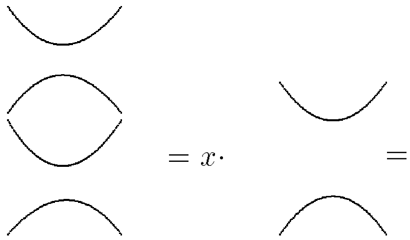
so these $2f-2$ elements generate the algebra D_f. The following relations are immediately clear from the above pictures.

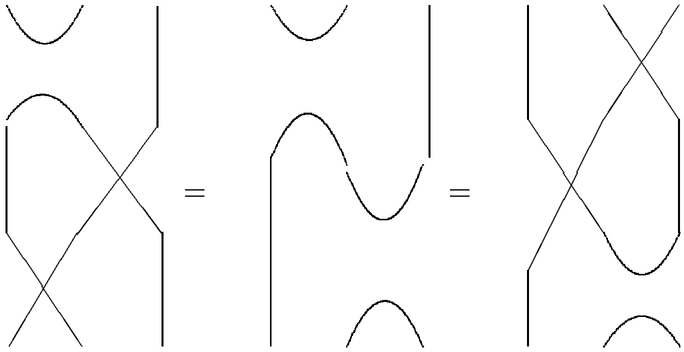
$$\begin{array}{lrcll}
\underline{\text{Type 1}} & e_i^2 & = & x \cdot e_i \,, & \\
 & e_i g_i & = & g_i e_i = e_i \,, & \\
 & g_i^2 & = & 1 & \forall\, i = 1, \ldots, f-1 \,. \\
\underline{\text{Type 2}} & e_i e_{i+1} e_i & = & e_i \,, & \\
 & e_{i+1} e_i e_{i+1} & = & e_{i+1} \,, & \\
 & e_i g_i g_{i+1} & = & e_i e_{i+1} = g_{i+1} g_i e_{i+1} \,, & \\
 & e_{i+1} g_i g_{i+1} & = & e_{i+1} e_i = g_i g_{i+1} e_i \,, & \\
 & g_{i+1} g_i g_{i+1} & = & g_i g_{i+1} g_i & \forall i = 1, \ldots, f-2 \,. \\
\underline{\text{Type 3}} & e_i e_j & = & e_j e_i \,, &
\end{array}$$

$$
\begin{aligned}
e_i g_j &= g_j e_i \,, \\
g_i g_j &= g_j g_i \qquad \forall i,j \quad mit \quad |i-j| \geq 2\,.
\end{aligned}
$$

The type 1.1 relations look like



The type 2.3 relations may be given diagrammatically as follows:



The diagrams for the type 3 relations are obvious. This list of relations is sufficient to give a second definition of the Brauer algebra.

Definition. The *Brauer algebra* D_f is the quotient of the free $C(x)$ algebra with $2f-2$ generators $\{e_i,\, g_i\}$ by the two sided ideal generated by the relations of types 1, 2 and 3.

Due to $e_2 = g_1 g_2 e_1 g_2 g_1$, $e_3 = g_2 g_3 e_2 g_3 g_2$ etc. D_f is already generated by the single element e_1 and the set $\{g_i\}$. The two sided ideal in D_f generated by e_1 will be denoted by I_f. Then $D_f/I_f \cong C(x)S_f$. Let $D_f(x)$ be the **C** algebra with unit which is defined similarly as D_f where x is a fixed complex parameter. The importance of the algebra D_f stems from the following result of H. Weyl [W].

Proposition 1. *Let E and P be the trace resp. the flip operators in $L(C^N \otimes C^N)$ defined as above. Then the mapping $e_i \mapsto E_{i,i+1}$, $g_i \mapsto P_{i,i+1}$ can be extended to a representation*

$$\rho_f : D_f(N) \to B_f(O(N,C))$$

of the Brauer algebra into the centralizer of $O(N,C)$.

Proposition 2. *The representation ρ_f is surjective if and only if $N \geq 3$.*

Proposition 3. *The representation ρ_f is faithful if and only if $N \geq f$. The algebra $D_f(N)$ is semisimple if and only if $N \geq f-1$.*

The first proposition goes back to H. Weyl [W], while the second and the third one are due to W. P. Brown [Br].

4. The Quantum Groups $SO_q(N)$

Now we want to transfer the Brauer-Weyl duality to the q-deformed simple Lie groups. In the remainder of this section we suppose q is not a root of unit. Let A be one of the Hopf algebras for the quantum groups B_n or D_n as defined in [FRT] by means of the matrices $\hat{R} \in L(C^N \otimes C^N)$ and $C \in L(C^N)$. Recall that A is the quotient $C\langle u_i^j : i,j = 1,\ldots,N\rangle / J$ of the free algebra generated by the N^2 entries of the matrix $u = (u_j^i)$ by the two sided ideal J generated by the relations

$$\begin{aligned} \hat{R}(u \otimes u) &= (u \otimes u)\hat{R} & (3) \\ \text{and} \qquad u^t\, C\, u &= u\, C\, u^t = C\,. & (4) \end{aligned}$$

As usual we consider $u = (u_j^i)$ as the fundamental representation of A. Then, in the language of representation theory (3) means that $\hat{R} \in \mathrm{Mor}(u \otimes u)$. This is the starting point for the analogy of Brauer-Weyl duality for the quantum group A. The fact that A is a deformation of the classical Lie group is reflected in the theory of representations . Since, by the above assumption q is not a root of unity, the representation theory of the quantum group A is, roughly speaking, similar to the representation theory of the corresponding classical group, cf.[Ro] and [L]. To be a little more precise, the tensor product representations $u \otimes \cdots \otimes u$ split into irreducible subrepresentations exactly as their classical analogs, i.e. we have the same dimensions and the same multiplicities as in the classical case. Thus, in particular, $\mathrm{Mor}(u \otimes u)$ is three-dimensional. Hence the element $\hat{R}^3 \in \mathrm{Mor}(u \otimes u)$ is a linear combination of I, $\hat{R}$ and $\hat{R}^2$. This is indeed true, since we have the cubic relation, (see [FRT]):

$$(\hat{R} - qI)(\hat{R} + q^{-1}I)(\hat{R} - q^{1-N}I) = 0\,.$$

The matrix $\hat{R}$ fulfills another important relation, the so-called *Yang-Baxter equation*

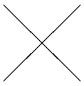
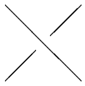
$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}\,.$$

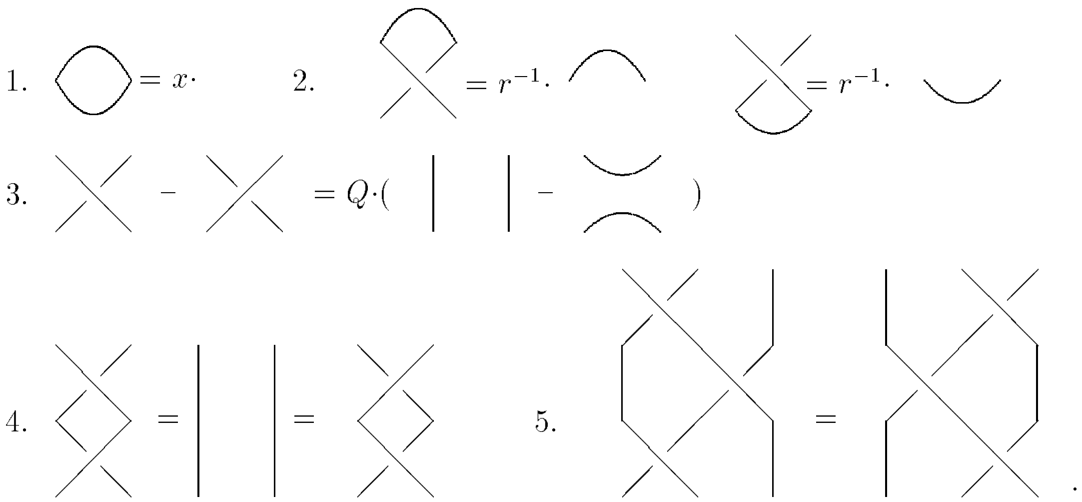
This relation ensures that the dimension of $\mathrm{Mor}(u \otimes u \otimes u)$ is indeed 15 as in the classical case and not higher.

5. The Birman-Wenzl-Murakami Algebra C_f

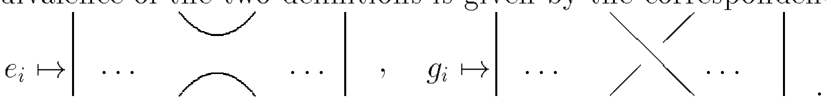
Definition. The *Birman-Wenzl-Murakami algebra* C_f, BWM algebra for short, is the quotient $C(q,r)\langle g_i, e_i : i = 1,\cdots,f-1\rangle / J$ of the free $C(q,r)$ algebra with unit and generators $\{g_i, e_i\}$ by the two sided ideal J generated by the relations of types $1'$, 2 and 3, where types 2 and 3 are as given above and type $1'$ is defined below.

$$\begin{aligned}
\underline{\text{Type } 1'} \qquad e_i^2 &= x\cdot e_i\,, \qquad x = 1+\frac{r-r^{-1}}{q-q^{-1}}\,,\\
e_ig_i &= g_ie_i = r^{-1}e_i\,,\\
g_i^2 &= 1+Qg_i-Qr^{-1}e_i\,, \qquad Q=q-q^{-1} \qquad \forall i=1,\ldots,f-1\,.
\end{aligned}$$

An equivalent definition of this algebra for $f>0$ as a braid algebra can be given as follows. We take as a linear basis of C_f the $(2f-1)\cdot\ldots\cdot3\cdot1$ braids which we get from the graphs of the Brauer algebra D_f when we always replace  by . As usual we get the product of two braids by joining. The reduction to basis elements can be obtained by taking into account the following relations



The relations 4. and 5. are nothing but the Reidemeister moves 2 and 3 from knot theory, see [K], the first Reidemeister move is slightly modified in relation 2. The equivalence of the two definitions is given by the correspondence



As in classical case we can rewrite the defining relation (4) with the tensors

$B_\bullet = (b_{rs}) =$  , $C^\bullet = (c^{rs}) =$  as follows:

$$B_\bullet u\otimes u = 1\cdot B_\bullet \quad , C^\bullet\cdot 1 = u\otimes uC^\bullet\,.$$

Obviously, $K := C^\bullet B_\bullet$ is an element of $\mathrm{Mor}(u\otimes u)$. This matrix K satisfies the identity $\hat R-\hat R^{-1} = Q(Id-K)$, cf. relation 3. The following result due to N. Yu. Reshetikhin [R] is the quantum version of Proposition 1.

Proposition 4. *Set $r=q^{N-1}$. The mapping $g_i\mapsto \hat R_{i,i+1}$, $e_i\mapsto K_{i,i+1}$ can be extended to a representation*

$$C_f(q,r)\to B_f(SO_q(N))$$

of the BWM-Algebra $C_f(q,r)$ into the centralizer of $SO_q(N)$.

6. Application to the Classification of Differential Calculi

In this section we briefly indicate how the BWM algebra appears in order to classify bicovariant bimodules and differential calculi on A. For this we first recall some notations and some facts, see e.g. [S] or [Sch] for more details.
We suppose that (Γ, d) is a bicovariant differential calculus over one of the Hopf algebras A for the quantum groups of type B or D. Let Φ_L and Φ_R denote the corresponding left resp. right actions of A on Γ and let (ω_{ij}) , $i,j \in I$, with $I = \{1,\ldots,N\}$ a basis of the vector space of left invariant forms in Γ. As shown in [W1], there exist linear functionals $f^{\,ab}_{\,rs}$, $a,b,r,s \in I$, on A and elements $v^{\,ab}_{\,rs}$, $a,b,r,s \in I$, of A such that the right module structure and the right comodule structure on Γ are given by

$$\begin{aligned} \omega_{rs} a &= (f^{xy}_{rs} * a)\cdot\omega_{xy} && (5) \\ \text{and} \quad \Phi_R(\omega_{rs}) &= \omega_{xy} \otimes v^{xy}_{rs} && (6) \end{aligned}$$

(We always sum over repeated indices.) The Theorems 2.4 and 2.5 in [W1] give a complete description of bicovariant bimodules Γ in terms of these functionals $f^{\,ab}_{\,rs}$ and elements $v^{\,ab}_{\,rs}$. Define a linear mapping $T = (T^{\,abc}_{\,rst}) \in L(C^N \otimes C^N \otimes C^N)$ by

$$T^{abc}_{rst} := f^{bc}_{rs}(u^a_t)\,.$$

Then it can be shown that $T \in \mathrm{Mor}(u \otimes u \otimes u)$, cf. [S]. Let $D(T)$, $tr^1_1(T)$ and $M(T)$ be defined by

$$\begin{aligned} D(T)^{abc}_{rst} &:= b_{rx} T^{xab}_{sty} c^{yc} && (7) \\ tr^1_1(T)^{bc}_{st} &:= b_{zx} T^{xbc}_{yst} c^{zy} && (8) \\ M(T)^{abc}_{rst} &:= b_{xy} (\hat{R}^{-1})^{ax}_{rz} T^{ybc}_{vst} b^{zv}\,. && (9) \end{aligned}$$

Using this notation the fact that the functionals $f^{\,ab}_{\,rs}$ annihilate the ideal J defined by (3) and (4) is reflected in the following equations

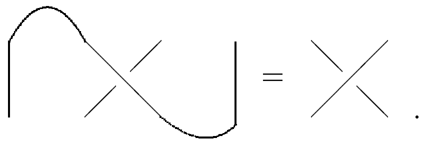
$$\begin{aligned} \hat{R}_{12} T_{234} T_{123} &= T_{234} T_{123} \hat{R}_{34} && (10) \\ T\cdot D(T) &= D(T)\cdot T = I\,. && (11) \end{aligned}$$

In a similar way, the fact that the differential d annihilates the ideal J can be expressed in terms of T by two other equations:

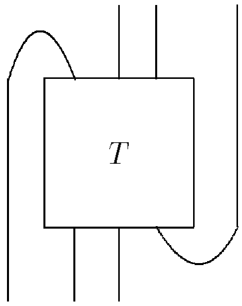
$$\begin{aligned} tr^1_1(T) &= -I && (12) \\ M(T) - T\hat{R}_{23} &= K_{12}\hat{R}_{23} - \hat{R}^{-1}_{12}\,. && (13) \end{aligned}$$

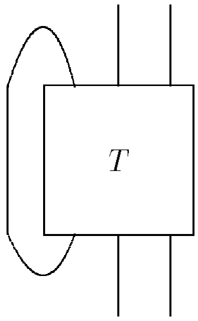
Conversely, equations (10)–(13) give a complete description of an N^2-dimensional bicovariant differential calculus on the quantum group A.

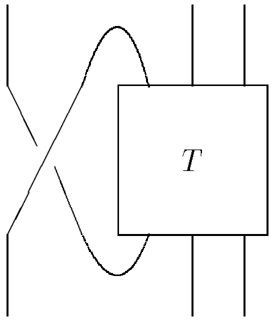
The aim of this section is to write the three transformations $D(T)$, $tr^1_1(T)$ and $M(T)$ in terms of the BWM algebra. For this we recall a crucial property of the matrix $\hat{R}$, see [FRT (1.10)], namely the relation

$$b_{rx}\hat{R}^{xa}_{sy}c^{yb} = \hat{R}^{-1}{}^{ab}_{rs}$$ or graphically  .

We conclude that the right hand sides of (7)–(9) belong to Mor($u\otimes u\otimes u$) and Mor($u\otimes u$) resp. In the braid algebras $C_3(q,r)$ and $C_2(q,r)$ the three transformations read as follows:

$D(T) =$ 

, $tr_1^1(T) =$ 

, $M(T) =$ 

That is, the four equations (10)–(13) needed to classify bicovariant bimodules and bicovariant differential calculi can be lifted into the BWM algebra. This has been an essential step in the classification problem investigated in [Sch].

References

[BW] Birman, J., and H. Wenzl, *Braids, link polynomials and a new algebra*, Trans. Amer. Math. Soc. **313** (1989), 249–273.

[B] Brauer, R., *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. **38** (1937), 857–872.

[Br] Brown, W. P., *The semisimplicity of* ω_f^n, Ann. of Math. **63** (1956), 324–335.

[FRT] Faddeev, L. D., N. Yu. Reshetikhin, and N. Yu.Takhtajan, *Quantization of Lie groups and Lie algebras*, Algebra and Analysis **1** (1987), 178–206 (Russian).

[K] Kauffman, L. H., *On Knots*, Princeton Univ. Press, 1987.

[L] Lusztig, G., *Quantum Deformations of certain Simple Modules over Enveloping Algebras*, Adv. Math. **70** (1988), 237–249.

[R] Reshetikhin, N. Yu., *Quantized Universal Enveloping Algebras, the Young-Baxter Equation and Invariants of Links I*, Preprint LOMI E-4-87, Leningrad, 1988.

[Ro] Rosso, M., *Finite Dimensional Representations of the Quantum Analog of the Enveloping Algebra of a Complex Simple Lie algebra*, Comm. Math. Phys. **117** (1988), 581–593.

[S] Schmüdgen, K.,*Covariant Differential Calculi on Quantum Spaces*, Preprint- Nr. 24/1991.

[Sch] Schüler, A., *Covariant Bimodules and Differential Calculi on Quantum Groups of Type B, C, D*, Seminar Sophus Lie **2** (1992), 115–121.

[W] Weyl, H., *The classical groups*, Princeton Univ. Press, Princeton, 1939.

[We] Wenzl, H., *On the structur of Brauers centralizer algebras*, Ann. of Math. **128** (1988), 173–193.

[W1] Woronowicz, S. L., *Differential calculus on compact matrix pseudogroups (quantum groups)*, Comm. Math. Phys. **122** (1989), 125–170.

[W2] —, *Compact matrix pseudogroups*, Comm. Math. Phys. **111**, 1987, 613–665.

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