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Weyl groups of disconnected Lie groups

Karl-Hermann Neeb

Introduction

Let G be a real Lie group and $\mathfrak{g}$ its Lie algebra. If one studies the exponential function $\exp : \mathfrak{g} \to G$ and fibrations of the set of regular elements of G over the set of all Cartan subalgebras of $\mathfrak{g}$, then one needs a general concept of a *Cartan subgroup* and a *Weyl group.* So let $\mathfrak{h} \subseteq \mathfrak{g}$ be a *Cartan subalgebra*, i.e., a nilpotent subalgebra which is self-normalizing. Then we consider the adjoint action of the complexification $\mathfrak{h}_{\mathbb{C}}$ on the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ given by $\mathrm{ad} : \mathfrak{h}_{\mathbb{C}} \to \mathrm{der}\, \mathfrak{g}_{\mathbb{C}}, \ \mathrm{ad}(X)(Y) = [X, Y]$.

For $\alpha \in \mathfrak{h}_{\mathbb{C}}^*$ we set

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} := \bigcap_{X \in \mathfrak{h}_{\mathbb{C}}} \bigcup_{n \in \mathbb{N}} \ker \big(\mathrm{ad}\, X - \alpha(X)\, \mathrm{id} \big)^n,$$

and call it the α*-root space.* We write

$$\Delta := \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) := \{\alpha \in \mathfrak{h}_{\mathbb{C}}^* \setminus \{0\} \colon \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}$$

for the set of *roots.*

Now let $N(\mathfrak{h}) := \{g \in G \colon \mathrm{Ad}(g)\mathfrak{h} = \mathfrak{h}\}$ be the normalizer of $\mathfrak{h}$ in G. Note that for $g \in N(\mathfrak{h})$ and $\gamma := \mathrm{Ad}(g)\,|_{\mathfrak{h}_{\mathbb{C}}}$ we have that

$$\mathrm{Ad}(g)\mathfrak{g}_{\mathbb{C}}^{\alpha} = \mathfrak{g}_{\mathbb{C}}^{\gamma.\alpha},$$

where $\gamma.\alpha = \alpha \circ \gamma^{-1}$. So $N(\mathfrak{h})$ acts on the finite set Δ such that the effectivity kernel of this action is given by

$$C(\mathfrak{h}) := \{g \in N(\mathfrak{h}) \colon (\forall \alpha \in \Delta)\alpha \circ \mathrm{Ad}(g)\,|_{\mathfrak{h}} = \alpha\}.$$

This group is called the *Cartan subgroup* associated to $\mathfrak{h}$. We note in passing that this definition of a Cartan subgroup is consistent with the standard definitions for semisimple Lie groups or algebraic groups (cf. [3]).

We define the *Weyl group* as

$$\mathcal{W} := \mathcal{W}(G, \mathfrak{h}) := N(\mathfrak{h})/C(\mathfrak{h}).$$

This group is naturally isomorphic to the image of $N(\mathfrak{h})$ in the group of permutations of Δ. It follows in particular that it is finite.

So it is very natural to ask which finite groups may occur as Weyl groups. In this note we prove the following theorem:

Theorem 1. *Let Γ be a finite group. Then there exists a real Lie group G and a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that*

$$\mathcal{W}(G, \mathfrak{h}) \cong \Gamma.$$

$\blacksquare$

This contrasts the following result for connected Lie groups (cf. [3, V.12]) which shows that the Weyl groups of connected Lie groups are Weyl groups of semisimple groups.

Theorem 2. *Let G be a connected Lie group and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan algebra. Then there exists a Levi decomposition $\mathfrak{g} \cong \mathfrak{r} \rtimes \mathfrak{s}$ such that $\mathfrak{h} \cap \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{s}$ and*

$$\mathcal{W}(G, \mathfrak{h}) \cong \mathcal{W}(S, \mathfrak{h} \cap \mathfrak{s}),$$

where $S = \langle \exp \mathfrak{s} \rangle$ is the analytic subgroup corresponding to $\mathfrak{s}$. $\blacksquare$

Realization of finite groups as Weyl groups

In this section we prove Theorem 1. First we strip off the Lie algebra context to see what we have to prove about finite groups.

Let $\mathfrak{h}$ be an abelian real Lie algebra, $\Delta \subseteq \mathfrak{h}^* \setminus \{0\}$ a finite generating subset, and

$$\Gamma := \mathrm{Aut}(\Delta) := \{\gamma \in \mathrm{Gl}(\mathfrak{h}^*) : \gamma.\Delta = \Delta\}.$$

We define a representation of $\mathfrak{h}$ on $V := \mathbb{R}^\Delta$ by

$$X.v_\alpha = \alpha(X) v_\alpha \qquad \forall X \in \mathfrak{h}, \alpha \in \Delta,$$

where $v_\alpha(\beta) = \delta_{\alpha,\beta}$. In addition, we define an action of Γ on V by

$$\gamma.v_\alpha = v_{\gamma.\alpha} \qquad \forall \gamma \in \Gamma, \alpha \in \Delta.$$

Let $X \in \mathfrak{h}$, $\alpha \in \Delta$, and $\gamma \in \Gamma$. Then

$$\gamma.(X.v_\alpha) = \alpha(X) v_{\gamma.\alpha}$$

and

$$(*) \qquad X.(\gamma.v_\alpha) = (\gamma.\alpha)(X) v_{\gamma.\alpha} = \alpha(\gamma.X) v_{\gamma.\alpha},$$

where Γ acts on $\mathfrak{h} \cong (\mathfrak{h}^*)^*$ via

$$\gamma.X := (\gamma^*)^{-1}.X.$$

Let H be a simply connected group with $\mathbf{L}(H) \cong \mathfrak{h}$. Then the representation of $\mathfrak{h}$ on V integrates to a representation of H on V with

$$\exp(X).v_\alpha = e^{\alpha(X)} v_\alpha \qquad \forall X \in \mathfrak{h}, \alpha \in \Delta.$$

Now $(*)$ shows that the actions of H and the action of Γ on $H \cong \mathfrak{h}$ define an action of the semidirect product $H \rtimes \Gamma$. So we have a representation of the group $H \rtimes \Gamma$ on V and we can form the Lie group

$$G := V \rtimes (H \rtimes \Gamma) \cong (V \rtimes H) \rtimes \Gamma.$$

Lemma 2. *The Lie group G has the following properties:*

(i) $\mathfrak{h}$ *is a Cartan algebra of* $\mathfrak{g} := \mathbf{L}(G)$.

(ii) $Z(G) = \{\mathbf{1}\}$ *and* $\mathrm{Ad}(G) \cong G$.

(iii) $N(\mathfrak{h}) = \mathrm{Ad}(H \rtimes \Gamma)$.

(iv) $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \Delta$.

(v) $C(\mathfrak{h}) = \mathrm{Ad}(H)$.

(vi) $\mathcal{W}(G, \mathfrak{h}) \cong \Gamma$.

Proof. (i) Since $\mathfrak{h}$ is abelian, it remains to show that $\mathfrak{h}$ is self-normalizing in $\mathfrak{g} \cong V \rtimes \mathfrak{h}$. This follows immediately from the assumption that $0 \notin \Delta$ which entails that

$$V_{\text{fix}} := \{v \in V : \mathfrak{h}.v = \{0\}\} = \{0\}.$$

(ii) Let $g = (v, h, \gamma)$ and $\alpha \in \Delta$. Then

$$g(v_\alpha, \mathbf{1}, \mathbf{1})g^{-1} = (h\gamma.v_\alpha, \mathbf{1}, \mathbf{1}) = (e^{\gamma.\alpha(\log h)}.v_{\gamma.\alpha}, \mathbf{1}, \mathbf{1}).$$

If $\mathrm{Ad}(g) = \mathbf{1}$, then this calculation shows that $\gamma = \mathbf{1}$ and $\alpha(\log h) = 0$ for all $\alpha \in \Delta$. Since Δ spans $\mathfrak{h}^*$, it follows that $\log h = 0$, i.e., $h = \mathbf{1}$. So $g = (v, \mathbf{1}, \mathbf{1})$ and since H acts on V without non-zero fixed points, it follows that $v = 0$. This proves that $\ker \mathrm{Ad} = \{\mathbf{1}\}$ and in particular that $Z(G) = \{\mathbf{1}\}$.
(iii) It is clear that $H \rtimes \Gamma$ normalizes $\mathfrak{h}$. The other implication follows from (i).
(iv) This is immediate from (i) and the construction of $\mathfrak{g}$.
(v) Using (iv) and $(*)$, we see that the action of $H \rtimes \Gamma$ on $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ corresponds to the action of Γ on Δ. Hence the pointwise stabilizer coincides with H.
(vi) This is a consequence of (iii) and (v). ■

Given a real vector space $\mathfrak{h}$ and a generating subset $\Delta \subseteq \mathfrak{h}^* \setminus \{0\}$, Lemma 2 provides a Lie group G such that $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{h}$ with $\mathcal{W}(G, \mathfrak{h}) \cong \Gamma$.

So we have the following problem: Given a finite group Γ. Find a vector space $\mathfrak{h}$ and $\Delta \subseteq \mathfrak{h}^* \setminus \{0\}$ such that

$$\Gamma \cong \mathrm{Aut}(\Delta) = \{\gamma \in \mathrm{Gl}(V) : \gamma(\Delta) = \Delta\}.$$

Thus the proof of Theorem 1 is completed by the following proposition.

Proposition 3. *Let G be a finite group and $A := \mathbb{R}[G]$ its group algebra. Then there exists a finite subset $\Delta \subseteq A^*$ such that*

$$\mathrm{Aut}(\Delta) \cong G,$$

where G acts on A by left translations.

Proof. We identify $\mathbb{R}[G]$ with $\mathbb{R}^G$. Then we have a basis given by the functions $\varepsilon_g(g') = \delta_{gg'}$. We also consider A as a Hilbert space with respect to normalized Haar measure, i.e., $\{\varepsilon_g : g \in G\}$ is an orthonormal basis. Using this Hilbert space structure, we identify A with its dual A^*.

We choose an injective function $m: G \to \mathbb{N}$ with $m(g) > 2$ for all $g \in G$ and set $n(g, g') := m(g^{-1}g')$. We define the finite set

$$\Delta := \{\varepsilon_g : g \in G\} \cup \{\varepsilon_g + n(g, g')\varepsilon_{g'} : g, g' \in G\}.$$

Let $C := \sum_{g \in G} \mathbb{R}^+ \varepsilon_g$ denote the cone of positive functions in $A = \mathbb{R}^G$. This is a simplicial cone in the real vector space A. Let $\gamma \in \mathrm{Gl}(A)$ with $\gamma(\Delta) = \Delta$. Then $\gamma(C) = C$ is a consequence of the fact that C is the smallest cone containing Δ. Thus γ preserves the set $\{\mathbb{R}^+\varepsilon_g : g \in G\}$ of one-dimensional faces of C. It follows that for each $g \in G$ the element $\gamma(\varepsilon_g)$ is contained in a ray $\mathbb{R}^+\varepsilon_{g'}$ for some $g' \in G$. On the other hand $\mathbb{R}^+\varepsilon_{g'} \cap \Delta = \varepsilon_{g'}$ so that $\gamma(\varepsilon_g) = \varepsilon_{g'}$. Thus there exists a bijective mapping

$$\eta : G \to G \quad \text{with} \quad \gamma(\varepsilon_g) = \varepsilon_{\eta.g}.$$

We claim that γ is left multiplication by $\eta.\mathbf{1}$. To see this, let $g, g' \in G$ and $f_{g,g'} := \varepsilon_g + n(g, g')\varepsilon_{g'}$. Then $\gamma(f_{g,g'})$ is an element in the two-dimensional face spanned by $\varepsilon_{\eta.g}$ and $\varepsilon_{\eta.g'}$ which is contained in Δ, so

$$\gamma.f_{g,g'} \in \{f_{\eta.g,\eta.g'}, f_{\eta.g',\eta.g}\}.$$

Since

$$\gamma.f_{g,g'} = \varepsilon_{\eta.g} + n(g, g')\varepsilon_{\eta.g'},$$

the equality

$$\gamma.f_{g,g'} = f_{\eta.g',\eta.g} = \varepsilon_{\eta.g'} + n(\eta.g', \eta.g)\varepsilon_{\eta.g}$$

is excluded by the fact that the coefficient $n(\eta.g', \eta.g)$ of $\varepsilon_{\eta.g}$ is greater than 1.

Whence

$$\varepsilon_{\eta.g} + n(g, g')\varepsilon_{\eta.g'} = \gamma.f_{g,g'} = f_{\eta.g,\eta.g'} = \varepsilon_{\eta.g} + n(\eta.g, \eta.g')\varepsilon_{\eta.g'}.$$

This leads to

$$m(g^{-1}g') = n(g, g') = n(\eta.g, \eta.g') = m\big((\eta.g)^{-1}\eta.g'\big)$$

and, by the injectivity of m,

$$g^{-1}g' = (\eta.g)^{-1}\eta.g',$$

i.e., $g(\eta.g)^{-1} = g'(\eta.g')^{-1}$. With $g' = \mathbf{1}$ this identity yields $g(\eta.g)^{-1} = (\eta.\mathbf{1})^{-1}$, or, equivalently,

$$\eta.g = (\eta.\mathbf{1})g \qquad \forall g \in G.$$

This proves that $\mathrm{Aut}(\Delta)$ consists of left multiplications in $\mathbb{R}[G]$.

If, conversely, $g \in G$, then the left multiplication $\lambda_g : x \mapsto gx$ preserves Λ because it preserves the set $\{\varepsilon_g : g \in G\}$ and

$$n(\lambda_g.x, \lambda_g.y) = m(x^{-1}g^{-1}gy) = m(x^{-1}y) = n(x, y)$$

shows that it also preserves the set $\{f_{g,g'} : g, g' \in G\}$. ∎

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Fachbereich Mathematik
Technische Hochschule Darmstadt
Schloßgartenstraße 7
D-6100 Darmstadt, Germany
neeb@mathematik.th-darmstadt.de

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