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Finitely generated connected locally compact groups

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Introduction

Hofmann and Morris [6] proved that a locally compact connected group G has a finite subset generating a dense subgroup if and only if the weight $w(G)$ of G does not exceed $\mathfrak{c}$, the cardinality of the continuum. The minimum cardinality of such a topological generating set is an invariant of the group, is denoted by $\sigma(G)$, and is called the *topological rank* of G. For compact abelian groups of weight $\leq \mathfrak{c}$, this number is 1. It was also shown there that for any compact connected group G of weight $\leq \mathfrak{c}$, the invariant $\sigma(G) \leq 2$. Cleary and Morris [3] observed that $\sigma(\mathbb{R}^n) = n+1$, for $n \geq 1$ and proved the surprising result that for any compact connected group G with $w(G) \leq \mathfrak{c}$, the invariant $\sigma(G \times \mathbb{R}^n) = n+1$ for $n \geq 1$. Here we extend their result significantly. For example, if G is a compact connected group of weight $\leq \mathfrak{c}$ and L is a nonsingleton connected Lie group, then $\sigma(G \times L) = \sigma(L)$. For more general locally compact groups we bound the topological rank of a group in terms of an associated Lie group.

Partial results

We consider here Hausdorff topological groups only. For a subset $X \subseteq G$ we let $\langle X \rangle$ denote the group generated by X.

Lemma 1. *Let P and A denote topological groups, X a topological generating set of A and $\pi: X \to P$ a function such that $\pi(X)$ is a topological generating set of P. Set $G = P \times A$, $\mathrm{graph}(\pi) = \{(\pi(x), x) : x \in X\}$, and $H = \overline{\langle \mathrm{graph}(\pi) \rangle}$. Let pr_P and pr_A denote the projections. Then*

$$(*) \qquad \overline{\mathrm{pr}_P(H)} = P \quad \text{and} \quad \overline{\mathrm{pr}_A(H)} = A.$$

Proof. Since X, resp. $\pi(X)$, is a topological generating set of A, resp., P, then $\mathrm{pr}_P(\pi(X))$, resp. $\mathrm{pr}_A(X)$ topologically generates the group P, resp. A. Thus $\mathrm{pr}_P(H)$ and $\mathrm{pr}_A(H)$ are dense in P and A, respectively. ∎

Lemma 2. *Let P and A denote topological groups, and H a closed subgroup of $P \times A$ such that $(*)$ is satisfied. In the notation of* Lemma 1*:*

(i) *If P is compact, then $\mathrm{pr}_A|H\colon H \to A$ is an open surjective morphism of topological groups.*

(ii) $N \stackrel{\mathrm{def}}{=} \big((P \times \{1\}) \cap H\big) \times \big((\{1\} \times A) \cap H\big)$ *is normal in G.*

(iii) *Set $N_P = \{p \in P : (p,1) \in H\}$ and $A_N = \{a \in A : (1,a) \in H\}$. Then G/N can be identified with $P/N_P \times A/N_A$ and H/N with a closed subgroup thereof. Further, $H = G$ iff $N = G$.*

(iv) *If $N = \{1\}$ and P is compact, then there is a unique injective morphism $f\colon A \to P$ with dense image extending π such that $H = \{(f(a), a) : a \in A\}$.*

Proof. (i) Since P is compact, the projection pr_A is a proper map. Hence it maps the closed group H onto its image in such a fashion that the induced map $H/\big((P \times \{1\}) \cap H\big) \to A$ is an isomorphism of topological groups.

(ii) The normalizer of $(P\times\{1\})\cap H$ contains H (since $P\times\{1\}$ is normal in G) and $\{1\}\times A$ (since $\{1\}\times A$ centralizes $P\times\{1\}$). Since $\mathrm{pr}_P(H)$ is dense in P, then $H(\{1\}\times A)$ is dense in G. Hence $(P\times\{1\})\cap H$ is normal in G. Likewise $(\{1\}\times A)\cap H$. Thus N is normal.

(iii) is straightforward.

(iv) Since H meets the factor P trivially, H is the graph of a morphism $f\colon A \to P$. Since H is closed and P is compact, it is continuous. Since H meets A trivially, f is injective. By (i) the image of f is dense in P. Since $(\pi(x), x) \in H$ for all x, the morphism f extends π. Since X is a topological generating set, the extension is uniquely determined by this condition. ∎

Lemma 3. *Let P be a compact connected group and A a locally compact group. Assume that H is a closed subgroup of $G = P \times A$ satisfying $(*)$ and that a maximal compact connected abelian subgroup T of $P \times \{\mathbf{1}\}$ is contained in H. Then $H = G$.*

Proof. We consider N as in Lemma 2. By Lemma 2(ii) we have $T \subseteq N$. Set $T = T_1 \times \{\mathbf{1}\}$ and $N \cap (P \times \{\mathbf{1}\}) = N_1$. The maximal compact connected abelian subgroup T_1 of P is contained in N_1, which by Lemma 2(ii) is a normal subgroup of P. Since all maximal compact connected abelian subgroups are conjugate and their union covers P we conclude that $N_1 = P$. Hence $P \times \{\mathbf{1}\} \subseteq N \subseteq H$. Thus H contains the kernel of pr_A. Then from $(*)$ it follows that H is dense in G. As H is closed, $H = G$ follows. ∎

Lemma 4. *Let P be any compact connected group and T a maximal compact connected abelian subgroup. Let A denote a locally compact group with a topological generating set X. Assume that $X = Y \cup Z \subseteq A$ and $\pi\colon X \to P$ are such that $T \cup \pi(Z)$ is a topological generating set of P and that $\{\pi(y), y) : y \in Y\}$ topologically generates a subgroup containing $T \times \{\mathbf{1}\}$. Then $\mathrm{graph}(\pi) = \{(\pi(x), x) : x \in X\}$ is a topological generating set of $P \times A$.*

Proof. Let $H = \overline{\langle X\rangle}$. We must show $H = P \times A$. By Lemma 3 this is the case if the conditions of Lemma 3 hold. Since X is a topological generating set of A then $\mathrm{pr}_A(H)$ is dense in A, and since $\mathrm{pr}_P(H)$ contains T and $\pi(Z)$ it is dense in P. Thus $(*)$ holds. The remaining condition of Lemma 3 is satisfied by hypothesis. ∎

We note that every locally compact connected group G contains a characteristic subgroup G_k such that G_k contains the radical R and has no nontrivial compact simple homomorphic image, and is such that G_k/R contains no compact connected normal subgroup. We have $G = G_k$ iff G has no nontrivial compact connected simple group as a homomorphic image. We then say that G is *without compact factors.* Indeed if G is semisimple, then there is a closed semisimple Lie subgroup G_k containing all simple noncompact factors and no compact one. Then G is the product of G_k and the product of all compact simple factors. If G is not semisimple, then G_k is the full inverse image of $(G/R)_k$ in G.

We note that, in particular, if $f: G \to G^*$ is a quotient homomorphism, then $f(G_k) = G^*_k$.

Proposition 5. *Assume that P is any compact group whose identity component P_0 is contained in $\overline{P'}$, and A is a locally compact connected group. Let X be a topological generating set of A.*

(i) *If $\pi: X \to P$ is any function such that $\pi(X)$ is a topological generating set of P then* $\mathrm{graph}(\pi) = \{(\pi(x), x) : x \in X\}$ *is a topological generating set of a subgroup $H \subseteq P \times A$ containing $\{\mathbf{1}\} \times A_k$.*

(ii) *If A is without compact factors, then* $\mathrm{graph}(\pi)$ *topologically generates $P \times A$ for any π as in* (i).

Proof. We let H be the closure of the group generated by $\mathrm{graph}(\pi)$ and denote $P \times A$ by G. We claim $G = H$. Let N be as in Lemma 2. If $H/N = G/N$ then $H = G$. Now $(A/N_A)_k = (A_k N_A)/N_A$ and $(P/N_P)_0 = (P_0 N_P)/N_P$, the latter being contained in $\overline{(P/N_P)'} = (\overline{P'} N_P)/N_P$. Thus, without loss of generality we can assume $N = \{1\}$.

We claim that A is isomorphic to P and thus is compact. By Lemma 2, there is a unique injective morphism $f: A \to P$ extending π with dense image. Thus A is maximally almost periodic and is therefore $V \times C$ with a vector group V and a compact connected group C. Also, P is now a compact, connected group satisfying $P = \overline{P'}$. This last property implies that the identity component of the center of P is trivial. Hence $\mathrm{pr}_P(V)$ is a point and thus V is singleton. Hence $A = C$ is compact, and is mapped isomorphically onto P. Now $A \cong P$ is a compact connected semisimple group and thus $A_k = \{\mathbf{1}\}$. This proves (i).

In order to prove (ii) we note that if $A_k = A$, then after factoring N we have $A = \{\mathbf{1}\}$. After having factored N, we have $P = \{\mathbf{1}\}$. We conclude that $N = G$ and thus $H = G$ by Lemma 2(iii). ∎

Certain information obtained in the proof may be of independent interest: If $N = \{\mathbf{1}\}$ then $P \cong A$.

We shall say that a topological group G is (topologically) *perfect* if it satisfies $\overline{G'} = G$. (This holds for example for all locally compact connected groups without radical.)

As a corollary of Proposition 5 we have:

Proposition 6. *Let P be a compact group such that P_0 is perfect. If A is locally compact connected group without compact factors, and if $\sigma(P) \leq \sigma(A)$,* *then* $\sigma(P \times A) = \sigma(A)$. ∎

We shall now make a few observations about the abelian situation which we shall use presently.

If X is a subset of a locally compact abelian group A, let $F(X)$ denote the free (discrete) abelian group on X and $f: F(X) \to A$ the morphism extending the inclusion $X \to A$. The image is dense iff the adjoint morphism $\widehat{f}: \widehat{A} \to \widehat{F(X)} = \mathbb{T}^X$ is injective. Thus $\sigma(A)$ is the smallest among the cardinals of those sets X for which $\widehat{A}$ has a continuous injective image in $\mathbb{T}^X$. For instance, if $A = \mathbb{R}^n$, then $\widehat{A} \cong \mathbb{R}^n$, and $\mathbb{R}^n$ has a continuous injective image in $\mathbb{T}^{n+1}$ but not in $\mathbb{T}^n$. If A is compact and connected then $\widehat{A}$ is discrete and torsion free. It is (algebraically) isomorphic to a subgroup of $\mathbb{T}^X$ if (and only if) the rank of $\widehat{A}$ does not exceed $\mathfrak{c} \cdot \operatorname{card} X$, where $\mathfrak{c}$ is the cardinality of the continuum. Recall that the dimension of A in this case is the rank of $\widehat{A}$.

Lemma 7. *Assume that K is a compact connected abelian group of dimension $\leq \aleph_0$ and $A = \mathbb{R}^n \times K$ with $n \geq 1$. Let X be a topological generating set of A and assume that C is a compact abelian group with $\sigma(C) \leq \operatorname{card} X$. Then there is a function $\pi: X \to C$ such that $\{(\pi(x), x) : x \in X\}$ is a topological generating set of $C \times A$.*

Proof. Let $\widehat{f}: \widehat{A} \to \mathbb{T}^X$ be as in the remarks preceding the lemma. The image $V = \widehat{f}(\mathbb{R}^n \times \{1\})$ has a compact connected closure, and since $\widehat{f}$ is injective the index of V in $\overline{V}$ is of continuum cardinality. The rank of $\widehat{f}(\{0\} \times \widehat{K})$ is countable. Hence its divisible hull K^* is countable and torsion free. Since V is divisible, there is a subgroup W of $\mathbb{T}^X$ such that $\mathbb{T}^X = W \oplus V \oplus K^*$ algebraically. The the rank of W is at least $\mathfrak{c}$, but also it cannot be bigger since $X \subseteq A$ and $\operatorname{card} A = \mathfrak{c}$. Since W contains the torsion group of $\mathbb{T}^X$ and has the same rank as $\mathbb{T}^X$ it is algebraically isomorphic to $\mathbb{T}^X$ under an algebraic isomorphism $i: \mathbb{T}^X \to W$. Now let $\varphi: (\mathbb{T}^X)_d \oplus \widehat{A} \to \mathbb{T}^X$ be the injective morphism given by $\varphi(\alpha \oplus \beta) = i(\alpha) + \widehat{f}(\beta)$. Notice that the character group $A(X)$ of $(\mathbb{T}^X)_d$ is the free compact abelian group on the set X. (See [4].) The dual morphism $\widehat{\varphi}: F(X) \to A(X) \times A$ is given by $\widehat{\varphi}(g) = (\widehat{i}(g), f(g))$ and has dense image. It satisfies $\widehat{\varphi}(x) = (\widehat{i}(x), x)$ where $\{\widehat{i}(x) : x \in X\}$ is topologically generating in $A(X)$. Since $\sigma(C) \leq \operatorname{card}(X)$ there is a surjective homomorphism $\eta: A(X) \to C$. If we set $\pi = \eta \circ (\widehat{i}|X)$ then we have the desired function. ∎

We obtain at once the following variant of the preceding lemma:

Lemma 7′. *Suppose that K is a compact connected abelian group of dimension $\leq \aleph_0$ and $A = \mathbb{R}^n \times K$ with $n \geq 1$. Let $\{x_j : j \in J\}$ be a topological generating family in A and suppose that C is a compact abelian group with $\sigma(C) \leq \operatorname{card} J$. Then there is a function $\pi: J \to C$ such that $\{(\pi(j), x_j) : j \in J\}$ is a topological generating set of $C \times A$.* ∎

Lemma 8. *If P is a nonsingleton monothetic compact group, then $\sigma(P \times \mathbb{R}^n) = n + 1$.*

Proof. For $n = 0$ the assertion is trivial. If $n \geq 1$ then $\sigma(\mathbb{R}^n) = n+1$ by the remarks preceding Lemma 7. (There are other arguments for this conclusion: See [3].) Now Lemma 7 implies the conclusion (with $K = \{\mathbf{1}\}$). ■

Lemma 9. *Let A denote a locally compact group such that $A/\overline{A'}$ is second countable and connected. Assume that C is a compact abelian group satisfying $\sigma(C) \leq \sigma(A)$. Then for any generating set X of A there is a function $\pi: X \to P$ such that* graph(π) *is a generating set of $P \times A$.*

Proof. Let X denote a topological generating set of A and let $f: A \to A/\overline{A'}$ be the quotient map. Then $\{f(x) : x \in X\}$ is a topological generating family of $f(A)$. By Lemma $7'$, there is a function $\pi: X \to C$ such that $\{(\pi(x), f(x)) : x \in X\}$ topologically generates $C \times f(A)$. Let H be the closed subgroup generated by $\{(\pi(x), x) : x \in X\}$ in $C \times A$. We shall prove that $H = G \stackrel{\text{def}}{=} C \times A$ and thereby finish the proof.

Let $h = (\pi(x_1), x_1) \cdots (\pi(x_m), x_m)$ and $h' = (\pi(x'_1), x'_1) \cdots (\pi(x'_n), x'_n)$. Set $[a, b] = aba^{-1}b^{-1}$. Then $[h, h'] = (1, [x_1 \cdots x_m, x'_1 \cdots x'_n])$ since C is abelian. Hence $D = \{1\} \times \overline{A'} \subseteq H$. But if we identify $C \times f(A)$ with G/D, then $H/D = C \times f(A)$ because H/D contains all $(\pi(x), f(x))$, $x \in X$. Then $H = G$ follows. ■

Lemma 10. *Let P be any compact group with identity component $P_0 \subseteq \overline{P'}$ and assume $w(P_0) \leq \mathfrak{c}$. Also assume that A is a locally compact connected group with a second countable factor group A/A_k such that $\sigma(P) \leq \sigma(A)$ and $\sigma(P/P_0) \leq \sigma(A/A_k) - 2$. Then for any topological generating set X of A there is a function $\pi: X \to P$ such that* graph(π) *topologically generates $P \times A$.*

Proof. If $A = A_k$ then Proposition 5 proves the assertion. Now assume $A_k \neq A$. We let $f: A \to A/A_k$ denote the quotient map. If X is a topological generating set of A then $f(X)$ is a topological generating set of A/A_k. Since A/A_k is nonabelian, there are at least two elements y and z such that $f(y) \neq f(z)$.

Let T denote a maximal compact connected abelian subgroup of P. Then there is a $p \in P_0$ such that $T \cup \{p\}$ is a topological generating set of P_0 (see [6], Corollary 2.5). Since $w(P_0) \leq \mathfrak{c}$, then T is monothetic. If M is the closed subgroup generated by $f(y)$ in A/A_k, then there is a $t \in T$ such that $(t, f(y))$ topologically generates $T \times M$ by Lemma 7. Now the closed subgroup topologically generated by $\{(t, f(y)), (p, f(z))\} \cup (\{1\} \times f(X_0))$ with $X_0 = X \setminus \{y, z\}$ has dense projections into A and P_0 and thus generates $P_0 \times A/A_k$ by Lemma 3.

Define $\pi: X \to P$ by $\pi(y) = t$, $\pi(z) = p$, and such that $\pi(X_0)$ is a generating set of P modulo P_0. Such a choice is possible because $\sigma(P/P_0) \leq \sigma(A/A_k) - 2 \leq |X_0|$. Let H denote the closed subgroup generated in $G = P \times A$ by graph(π). Let N be defined as in Lemma 2. Then $\{\mathbf{1}\} \times A_k$ is contained in N by Proposition 5. Now H/N is topologically generated by $\{(\pi(x), f(x)) : x \in X\} = \{(t, f(y)), (k, f(z))\} \cup \{(\pi(x), f(x)) : x \in X_0\}$ and thus agrees with G/N by the preceding. Now $G = H$ follows. ■

Lemma 11. *Let $\theta: G \to H$ be an open surjective morphism of locally compact groups such that* $\ker\theta$ *is compact totally disconnected and contained in G_0. Then*

(i) *for each generating set X of H each subset X' of G with $\theta(X') = X$ is a generating subset of G, and*

(ii) $\sigma(G) = \sigma(H)$.

Proof. We begin by noting that (ii) follows from (i): If X is topological generating set of G with $\operatorname{card} X = \sigma(G)$, then $\theta(X)$ is a topological generating set of H, whence $\sigma(H) \le \operatorname{card} X = \sigma(G)$. The reverse inclusion, however, we deduce from (i) by taking a generating subset X of H with $\operatorname{card} X = \sigma(H)$ and considering a subset $X' \subseteq G$ such that $\theta|X': X' \to X$ is bijective. Then $\sigma(G) \le \operatorname{card} X' = \operatorname{card} X = \sigma(H)$ by (i).

Now we prove (i). Let X be a topological generating set of H. Let $X' \subseteq G$ be any subset such that $\theta(X') = X$. Let $K = \overline{\langle X' \rangle}$. Then $\theta(K) = H$ since $N \stackrel{\text{def}}{=} \ker\theta$ is compact. It follows that $\theta(K_0) = H_0$. Since N is compact, NK_0 is closed, and since $\theta: G \to H$ is a proper map, then $\theta|NK_0: NK_0 \to \theta(K_0)$ is a proper map. Thus $NK_0/N \cong \theta(K_0) = H_0 \cong G_0/N$. We conclude $G_0 = NK_0$. The homogeneous spaces G_0/K_0 and $N/(N \cap N_0)$ are homeomorphic, and the latter is totally disconnected, as N is totally disconnected. The former is connected as G_0 is connected. Thus they are singleton and $K_0 = G_0$. Now let $\Theta: G/G_0 \to H/H_0$ be defined by $\Theta(gG_0) = \theta(g)H_0$. Then Θ is an isomorphism because $G/G_0 \cong (G/N)/(G_0/N) \cong H/H_0$. Now $\Theta(K/G_0) = \theta(K)/H_0 = H/H_0 = \Theta(G/G_0)$. Since Θ is an isomorphism we conclude $K/G_0 = G/G_0$ and thus $K = G$. Hence X' is a generating set of G.

The principal result

At last we are able to state and prove the main result.

Theorem 12. *Let P be compact group and A a locally compact group satisfying the following hypotheses:*

(a) *A is connected,*

(b) $w(A) \le \aleph_0$ *(that is, A is second countable),*

(c) $w(P) \le \mathfrak{c}$,

(d) $\sigma(P) \le \sigma(A)$, *and*

(e) $\sigma(A) \ge 2$ *and* $\sigma(P/P_0) \le \sigma(A) - 2$.

Then

(i) *for every generating set X of A there is a function $\pi: X \to P$ such that $\{(\pi(x), x) : x \in X\}$ is a generating set of $P \times A$, and*

(ii) $\sigma(P \times A) = \sigma(A)$.

Proof. (i) implies (ii): If we consider X with $\operatorname{card} X = \sigma(A)$ then $\sigma(P \times A) \le \operatorname{card}(\{(\pi(x), x) : x \in X\}) = \operatorname{card}(X) = \sigma(A)$. Conversely, if Y is a generating set of $P \times A$ with cardinality $\sigma(P \times A)$, then $\operatorname{pr}_A(Y)$ is a generating set of A of cardinality $\le \operatorname{card}(Y) = \sigma(P \times A)$. Hence $\sigma(A) \le \sigma(P \times A)$.

Proof of (i):

Case 1: P is abelian. Then the assertion is true by Lemma 9 in view of (a,b,d).

Case 2: The identity component $Z_0(P)$ of the center of P is trivial. Then by Theorem 1.3 in [5] we know that $P_0 = (\overline{P'})_0 \subseteq \overline{P'}$. Then the assertion follows from Lemma 10 in view of (a,b,c,d,e).

Case 3: P is arbitrary. By the Theorem of Lee [10] there is a compact zero dimensional group D of P such that $P = P_0D$. The set $(P_0)'D$ is a compact subgroup B since $(P_0)'$ is a compact characteristic subgroup of P_0 and thus is normal in P. We have $P = Z_0(P)B$ (see [5], Theorem 1.3) and $D \stackrel{\text{def}}{=} Z_0(P) \cap B$ is a compact totally disconnected central subgroup of P contained in P_0. Now A, as a locally compact connected group, is σ-compact. Hence $P \times A$ is σ-compact. Hence Lemma 11 applies to the quotient map $\theta\colon (P \times A) \to \frac{P\times A}{D\times\{\mathbf{1}\}}$. Assume that we have a generating set X of A and a function $\pi_0\colon X \to P/D$ such that graph(π_0) is a generating set for $(P/D)\times A \cong \frac{P\times A}{D\times\{\mathbf{1}\}}$. Now define $\pi\colon X \to P$ such that $\pi(x)D = \pi_0(x)$ for all $x \in X$. Then graph$(\pi) = \{(\pi(x), x) : x \in X\}$ maps onto graph(π_0). By Lemma 11 then graph(π) is a generating set of $P \times A$.

We therefore can replace P by P/D and thus assume that P is the direct product of the subgroups $Z_0(P)$ and B. Thus $G = Z_0(P) \times B \times A$. Let X be a generating set of A. By Case(1) we find a function $\pi_1\colon X \to Z_0(P)$ such that $\{(\pi_1(x), \mathbf{1}, x) : x \in X\}$ is a generating set of $Z_0(P) \times \{\mathbf{1}\} \times A$. Now we notice that

$$B/B_0 \to \frac{Z_0(B) \times B \times A}{Z_0(B) \times B_0 \times A} \cong P/P_0.$$

Hence $\sigma(B/B_0) = \sigma(P/P_0)$ and all hypotheses are satisfied which allow us to apply Lemma 10 with $Z_0(P) \times \{\mathbf{1}\} \times A$ in place of A and B in place of P. This yields a function $\pi_2\colon X \to B$ such that $\{(\pi_1(x), \pi_2(x), x) : x \in X\}$ is a generating set for $Z_0(P) \times B \times A$. If we recall $P = \mathbb{Z}_0(P) \times B$ and define $\pi\colon X \to P$ by $\pi(x) = (\pi_1(x), \pi_2(x))$ we see that this is the assertion of the theorem. ∎

Note that if P is connected then the condition $\sigma(P/P_0) \leq \sigma(A) - 2$ is trivially satisfied and by [6] conditions (c) and (e) imply (d). In particular, for the case that A is a connected Lie group, we obtain:

Corollary 13. *Let P denote a compact connected group of weight $\leq \mathfrak{c}$ and L a nonsingleton connected Lie group. Then*

(i) *for every generating set X of A there is a function $\pi\colon X \to P$ such that* graph(π) *is a generating set of $P \times A$, and*

(ii) $\sigma(P \times L) = \sigma(L)$.

Proof. (i) If L is compact abelian, then L and $P \times L$ are monothetic and the assertion is true. Otherwise $\sigma(L) \geq 2$ and Theorem 12 applies. (ii) follows from (i) ∎

We note that assumptions of the sort of conditions (d) and (e) are necessary in order for the conclusion to hold. The topological rank of the group $\mathbb{Z}(2)^n \times \mathbb{R}$ is $\min\{2, n\}$ which is large if n is large while $\sigma(R) = 2$.

It is interesting to realize that condition (b) cannot be replaced by the condition $w(A) \leq \mathfrak{c}$ without invalidating the theorem. In order to understand this let us consider the following example:

Let P be any nonsingleton compact connected group with $w(P) \leq \mathfrak{c}$. Let $K = \widehat{\mathbb{R}_d}$ denote the universal solenoid. Set $A = \mathbb{R} \times K$ and observe $w(P) \leq \mathfrak{c} = w(K) = w(A)$ and $\sigma(P) = 1$, $\sigma(A) = 2$. Thus hypotheses (a,c,d,e) of Theorem 12 are satisfied. The prescription of a generating set X of two elements of A is equivalent to giving a morphism of locally compact groups $e: \mathbb{Z}^2 \to A$ with dense image and that is tantamount to giving an injective morphism $\widehat{e}: \widehat{A} \to \mathbb{T}^2$. But $\widehat{A} = \mathbb{R} \times \mathbb{R}_d$. Hence we can choose $\widehat{e}$ to be an isomorphism of $\widehat{A}$ onto a full torsion free complement of the torsion group $(\mathbb{Q}/\mathbb{Z})^2$ of $\mathbb{T}^2$. The prescription of any $\pi: X \to P$ is the specification of two elements $p, q \in P$. Let $\alpha: \mathbb{Z}^2 \to P$ denote the unique morphism sending $(1,0)$ to p and $(0,1)$ to q. Define $\delta: \mathbb{Z}^2 \to P \times A$ by $\delta(t) = (\alpha(t), e(t))$. Now $\{(\pi(x), x) : x \in X\}$ is a generating set of $P \times A$ iff δ has dense image. But that means exactly that

$$\widehat{\delta}: \widehat{P} \times \widehat{A} = \widehat{P} \times \mathbb{R} \times \mathbb{R}_d \to \mathbb{T}^2, \quad \widehat{\delta}(r,s,t) = \widehat{\alpha}(r) + \widehat{e}(s,t)$$

is injective. But since $\widehat{e}(\mathbb{R} \times \mathbb{R}_d)$ is a complement of the torsion subgroup of $\mathbb{T}^2$ and $\alpha(\widehat{A})$ would have to be a nonzero torsion free subgroup meeting $\widehat{e}(\mathbb{R} \times \mathbb{R}_d)$ trivially, this is patently impossible. Notice that we can even take $P = \mathbb{T}$. ∎

We saw that $\sigma(\mathbb{R}^n) = n + 1$ for $n \geq 1$. So Corollary 13 yields:

Corollary 14. (CLEARY and MORRIS [3]) *Let P be a compact connected group of weight $\leq \mathfrak{c}$. Then $\sigma(P \times \mathbb{R}^n) = n + 1$, if $n \geq 1$.* ∎

Iwasawa Pairs

There is a "classical" theorem by IWASAWA [8] which says that a locally compact connected group contains a compact normal subgroup N and a local Lie group U commuting elementwise with N such that $(n,u) \mapsto nu: N \times U \to NU$ is a homeomorphism onto an identity neighborhood. Thus there is a Lie group L with an identity neighborhood isomorphic to U, and we obtain an injective morphism $\varphi: L \to G$ such that $(n,g) \mapsto n\varphi(g) : N \times L \to G$ is a surjective homomorphism which, by the Open Mapping Theorem, is also open. Let us formalize these remarks as follows:

Lemma 15. *Let G denote a locally compact connected group. Then there is a compact normal subgroup N and a connected Lie group L and an injective morphism $\varphi: L \to G$ such that*

(i) $[N, \varphi(L)] = \{\mathbf{1}\}$,

(ii) $G = N\varphi(L)$, *and*

(iii) *there is an identity neighborhood U in L such that $(n,u) \mapsto n\varphi(u) : N \times U \to N\varphi(U)$ is a homeomorphism onto an identity neighborhood of $\mathbf{1}$ such that $[N, \varphi(U)] = \{\mathbf{1}\}$.*

Proof. (i) and (ii) are consequences of (iii), and (iii) is the IWASAWA's Local Product Theorem (see [8]). ∎

Definition 16. If G is a locally compact connected group, then an *Iwasawa pair* is a pair $(N, \varphi: L \to G)$ with a morphism $\varphi: L \to G$ satisfying the conditions of Lemma 15. ∎

Lemma 17. *Let G denote a locally compact connected group with weight $w(G) \leq \mathfrak{c}$. If $(N, \varphi: L \to G)$ is an Iwasawa pair, then $\sigma(N_0) \leq 2$.*

Proof. From $w(N_0) \leq w(G) \leq \mathfrak{c}$ we conclude $\sigma(N_0) \leq 2$. (See [7], 4.13.) ∎

We observe that G/N is a Lie group locally isomorphic to L. Let $p: G \to G/N$ denote the quotient map. The kernel of the covering $\Phi = p \circ \varphi: L \to G/N$ is $K = \varphi^{-1}(N \cap \varphi(L))$.

For an abelian group A we set $\operatorname{rank} A = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} A)$ and call this number the *torsion free rank* of A.

Lemma 18. *Let $\theta: G \to H$ be a surjective morphism of connected locally compact groups and write $N = \ker \theta$. Assume that S is a closed subgroup of G.*

(i) *If $G = NS$ and N is countable, then $S = G$.*
(ii) *If N is finite and $\theta(S)$ is dense, then $S = G$*
(iii) *If N is finite, then $\sigma(G) = \sigma(H)$.*
(iv) *If θ is a covering homomorphism of connected Lie groups then $\sigma(G) \leq \sigma(H) + \operatorname{rank}(\ker \theta)$.*

Proof. (i) If $\bigcup_{n \in N} nS = NS = G$, then the locally compact space G is a countable union of homeomorphic closed subsets nS. Then by the Baire Category Theorem, one of them has inner points. Thus the subgroup S has nonempty interior and so is open. Hence it is also closed. As G is connected, $S = G$ follows.

(ii) If N is finite, then NS is a closed subgroup. Since $\theta(S)$ is denseit follows that NS is dense. Thus $NS = G$.

(iii) Let $Y \subseteq H$ denote a generating set in the sense that $H = \overline{\langle Y \rangle}$ and let $X \subseteq \theta^{-1}(Y)$ be any subset such that $\theta|X: X \to Y$ is bijective. Define $S = \overline{\langle X \rangle}$. Then $\theta(S)$ is dense in H. As N is finite, (ii) applies and proves the assertion.

(iv) Now N is finitely generated abelian. Its torsion group M is finite. By (iii) we have $\sigma(G) = \sigma(G/M)$ and the covering morphism $\Theta: G/M \to H$, $\Theta(gM) = \theta(g)$ has a free kernel of rank $\operatorname{rank} N$ which is also $\sigma(N)$. Since always $\sigma(G/M) \leq \sigma(H) + \sigma(\ker \Theta)$ (cf. [3]), the assertion follows. ∎

Conclusion (iii) also follows from Lemma 11.

For the quotient map $f: \mathbb{R} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ we have $\sigma(\mathbb{R}) = 2$, $\sigma(\mathbb{T}) = 1$ and $\operatorname{rank}(\mathbb{Z}) = 1$. This shows that the inequality in (iv) is sharp.

For an Iwasawa pair $(N, \varphi: L \to G)$ we write $\beta(N, \varphi) = \operatorname{rank}(\ker \Phi)$ and call it its *Betti number*.

Proposition 19. *If G is a noncompact locally compact connected group of weight not exceeding the cardinality of the continuum, then for any Iwasawa pair $(N, \varphi: L \to G)$,*

$$\sigma(G) \leq \sigma(L) \leq \sigma(G/N) + \beta(N,\varphi) \leq \sigma(G) + \beta(N,\varphi).$$

Proof. Since G is not compact then L is not compact, in particular, L is nonsingleton. Then Corollary 13 shows $\sigma(N_0 \times L) = \sigma(L)$. We consider the surjective morphism $\pi: N \times L \to G$, $\pi(n,g) = n\varphi(g)$. In Lemma 17 of [7] we show that the arc component G_a of the identity in G is $N_a\varphi(L)$ where N_a is the arc component of $\mathbf{1}$ in N. But G_a is dense in G and $G_a \subseteq N_0\varphi(L) = \pi(N_0 \times L)$. Thus π maps $N_0 \times L$ onto a dense subgroup of G. Hence any topological generating set X of $N_0 \times L$ is mapped onto a topological generating set $\pi(X)$ of G. Therefore, $\sigma(G) \leq \operatorname{card}\pi(X) \leq \operatorname{card} X = \sigma(N_0 \times L) = \sigma(L)$.

Notice that we always have $\sigma(G/N) \leq \sigma(G)$.

In order to prove the remaining inequality we consider the quotient map $q: G \to G/N$ and the morphism $f \overset{\text{def}}{=} q \circ \varphi: L \to G \to G/N$. Now $\ker f = \{g \in L : \varphi(g) \in N\}$. If $U \subseteq L$ is as in Lemma 15(iii), then $\varphi(U) \cap N = \{1\}$ and thus $U \cap \ker f = \{\mathbf{1}\}$. Thus $\ker f$ is discrete. Because of $G = N\varphi(L)$ the morphism f is surjective, and since L is connected, hence σ-compact, it is open. Thus f is a covering homomorphism. Now Lemma 18(iv) applies and shows the assertion. ∎

In order to illustrate the situation by an example we define $N = \widehat{\mathbb{Q}/\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$ where $\mathbb{Z}_p$ is the additive group of p-adic integers. Then $q: \mathbb{Q}/\mathbb{Z} \to \mathbb{T} = \mathbb{R}/\mathbb{Z}$ gives an injective morphism $\widehat{q}: \mathbb{Z} \to N$ with dense image. Write $i: \mathbb{Z} \to N \times \mathbb{R}$, $i(n) = (\widehat{q}(n), -n)$. Then i is an injective morphism with discrete image (since the projection of the image onto $\mathbb{R}$ is discrete). Define $G = (N \times \mathbb{R})/(\operatorname{im} i)$ and let $p: N \times \mathbb{R} \to G$ denote the quotient homomorphism. Since the compact space $N \times [0,1]$ maps onto G under p we know that G is compact. The dual of the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathbb{N} \times \mathbb{R} \xrightarrow{p} G \to 0$$

is the exact sequence

$$0 \to \widehat{G} \xrightarrow{\widehat{p}} (\mathbb{Q}/\mathbb{Z}) \times \mathbb{R} \xrightarrow{\widehat{i}} \mathbb{R}/\mathbb{Z} \to 0$$

with $\widehat{i}(q + \mathbb{Z}, r) = q - r + \mathbb{Z}$. Thus $\operatorname{im}\widehat{p} = \ker\widehat{i} = \{(q + \mathbb{Z}, r) : q - r \in \mathbb{Z}\} = \{(q + \mathbb{Z}, q) : q \in \mathbb{Q}\} \cong \mathbb{Q}$. It follows that G is the rational solenoid $\cong \widehat{\mathbb{Q}}$. The image $N^* = p(N \times \{0\})$ in G is an isomorphic copy of N, and $(N^*, \varphi: \mathbb{R} \to G)$, $\varphi(r) = p(0,r)$ is an Iwasawa pair for G. Since $G = \widehat{\mathbb{Q}}$, then G is monothetic; that is, $\sigma(G) = 1$. We know $\sigma(\mathbb{R}) = 2$. We have $G/N \cong \mathbb{R}/\mathbb{Z}$ and $\ker(\mathbb{R} \to G \to G/N) = \{r \in \mathbb{R} : \varphi(r) = p(0,r) \in N^*\} = \{r \in \mathbb{R} : (\exists n \in \mathbb{Z})\, (0,r) \in (\widehat{q}(n), -n) + (N \times \{0\}\} = \mathbb{Z}$. Thus $\beta(N^*, \varphi) = 1$. Thus the second inequality of Proposition 19 is sharp.

Remarks and Open Questions

There is a result by KURANISHI [9] saying that every connected semisimple Lie group has topological rank 2. This suggests the following question:

Question A. *If G is a perfect connected nondegenerate Lie group is $\sigma(G) = 2$?*

An analysis of the topological rank of a connected Lie group requires an answer to the following question:

Question B. *What is the topological rank of a solvable connected Lie group?*

In this context the following result of CLEARY [1] is relevant:

Proposition 20. *If G is a connected nilpotent Lie group then $\sigma(G) = \sigma(G/\overline{G'})$.* ■

Note that in Proposition 20 $G/\overline{G'} \cong \mathbb{R}^n \times \mathbb{T}^m$ and thus $\sigma(G) = n + 1$.

Finally we mention that in [6,7] we discussed the cardinal invariant $s(G)$ for a locally compact group defined as the minimum cardinal of a suitable subset X of G where X is a topological generating subset of G such that X is discrete and closed in $G \setminus \{\mathbf{1}\}$. Trivially, $\sigma(G) \leq s(G)$. Further it was shown in [6] that for a locally compact connected group G of weight $\leq \mathfrak{c}$, the cardinal $s(G)$ is finite and so equals $\sigma(G)$. This leaves open the question whether these two cardinals are always equal. They are not as the example below demonstrates.

Let X be a set of cardinality $2^{\mathfrak{c}}$. Now let $G = A(X)$ be the free compact abelian group on the discrete set X [4]. Then $\widehat{G} = (\mathbb{T}^X)_d$ and thus $w(G) = \operatorname{rank} \widehat{G} = 2^{\mathfrak{c} \cdot 2^{\mathfrak{c}}} = 2^{2^{\mathfrak{c}}}$. In [6] we showed for a compact connected group H with $w(H) > c$ that $s(H)^{\aleph_0} = w(H)^{\aleph_0}$. Thus $s(G)^{\aleph_0} = 2^{2^{\mathfrak{c}}}$. On the other hand, X is a topological generating subset of G. Hence $\sigma(G) \leq \operatorname{card} X = 2^{\mathfrak{c}}$. Since $(2^{\mathfrak{c}})^{\aleph_0} = 2^{\mathfrak{c}}$ we conclude that $\sigma(G) < s(G)$.

If $\sigma(G)$ is infinite, then it coincides with the density $d(G)$. The relation $d(G) = \log w(G)$ was proved by COMFORT and ITZKOWITZ in [1].

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