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# A globality theorem for Lie-wedges that are bounded by a hyperplane-ideal

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### Abstract

We consider a Lie group $G$ containing a normal subgroup $N \trianglelefteq G$ such that $G/N \cong \mathbb{R}$, i.e the Lie algebra $\mathfrak{n}$ is a hyperplane ideal in $\mathfrak{g}$. A natural question that arises in this context is the following: Suppose we are given a Lie-wedge $W$ which is contained in a halfspace bounded by $\mathfrak{n}$. Under which conditions is $W$ global in $G$? We will prove globality for all pointed wedges $W$ such that there exists another wedge $W' \subseteq \mathfrak{n}$ which is global in $N$ and satisfies $(W \cap \mathfrak{n}) \supseteq int_{\mathfrak{n}}(W') \cup \{0\}$. Especially our result applies to the groups and Lorentzian wedges considered by Levichev and Levicheva in this volume. As another application, we solve the globality problem of the Heisenberg-algebra, i.e. we give a complete characterization of all Lie-wedges that are global in the Heisenberg-group.

## 1. The globality theorem

***Theorem 1.1.*** *Let $G$ be a Lie group, $N \trianglelefteq G$ a normal subgroup such that $G/N \cong \mathbb{R}$. Let $W \subseteq \mathfrak{n}$ be a Lie wedge which is global in $N$ and $X \in \mathfrak{g} \setminus \mathfrak{n}$ such that the edge $H(W) = W \cap (-W)$ is $ad\,X$-invariant. We define the following set:*

$$\mathcal{W} = \left\{ W' \subseteq \mathfrak{n} \mid W' \text{ global in } N \text{ and } (\exists \varepsilon > 0)(\forall t \in [0, \varepsilon]) e^{t\,ad\,X} W \subseteq W' \right\}.$$

*If $\bigcap \mathcal{W} = W$ then the Lie wedge $W + \mathbb{R}^+ X$ is global in $G$.*

**Proof.** The $ad\,X$-invariance of $H(W)$ implies that $e^{t\,ad\,Y} X \in X + H(W)$ for all $Y \in H(W)$. Since $W$ is a Lie wedge and $H(W) = H(W + \mathbb{R}^+ X)$ it follows that $W + \mathbb{R}^+ X$ is invariant under the adjoint action of its edge, hence a Lie wedge. Let us denote with $\alpha(t)$ the inner automorphism of $G$ induced by $\exp(tX)$, i.e. $\alpha(t)(g) = \exp(tX) g \exp(-tX)$ and $\alpha(t)(\exp Y) = \exp(Ad(\exp(tX)Y) = \exp(e^{t\,ad\,X} Y)$. We use the notation $\prod_{i=1}^n g_i = g_1 g_2 \cdots g_n$ and set for $t \geq 0$

$$S_t = \left\{ \left( \prod_{i=1}^n \alpha(t_i)(g_i) \right) \exp(tX) \;\middle|\; t_i \in [0, t],\ g_i \in \exp(W) \right\} \tag{1}$$

and $S = \bigcup_{t \geq 0} S_t$. We will prove the following:

(i) $S$ is a semigroup because for $s,t \geq 0$ we have $S_t S_s \subseteq S_{t+s}$. Indeed, if $g = (\prod_{i=1}^{n} \alpha(t_i)(g_i)) \exp(tX) \in S_t$ and $h = (\prod_{j=1}^{m} \alpha(s_j)(h_j)) \exp(sX) \in S_s$ then

$$\begin{aligned} gh &= \left(\prod_{i=1}^{n} \alpha(t_i)(g_i)\right) \exp(tX) \left(\prod_{j=1}^{m} \alpha(s_j)(h_j)\right) \exp(sX) \\ &= \left(\prod_{i=1}^{n} \alpha(t_i)(g_i)\right) \alpha(t) \left(\prod_{j=1}^{m} \alpha(s_j)(h_j)\right) \exp((s+t)X) \\ &= \left(\prod_{i=1}^{n} \alpha(t_i)(g_i)\right) \left(\prod_{j=1}^{m} \alpha(t+s_j)(h_j)\right) \exp((s+t)X) \in S_{t+s}. \end{aligned}$$

(ii) $L(S) \supseteq W + \mathbb{R}^+X$ for obviously $\exp(W) \cup \exp(\mathbb{R}^+X) \subseteq S$.

(iii) $L(S) \subseteq W + \mathbb{R}^+X$. This is the non-trivial part of the proof. Let $Y \in L(S)$ then there exist sequences $m_j \in \mathbb{R}^+$ and $g_j \in S$ such that $\lim_j g_j = 1$ and $Y = \lim_j m_j \log g_j$. Now $g_j = a_j \exp(t_j X)$ with $t_j \geq 0$, $\lim_j a_j = 1$ and $\lim_j t_j = 0$ because $G/N \cong \mathbb{R}$. Let an arbitrary $W' \in \mathcal{W}$ be given, then there is an $\varepsilon > 0$ such that $e^{s\,ad\,X} W \subseteq W'$ for all $s \in [0,\varepsilon]$. Since $\lim_j t_j = 0$ we have $t_j < \varepsilon$ for all sufficiently large $j$. For these $j$ we have

$$a_j = \prod_i \alpha(s_i)\exp(w_i) = \prod_i \exp(e^{s_i\,ad\,X} w_i) \in \langle \exp(W') \rangle .$$

Applying the formula for the Campbell-Hausdorff-multiplication $*$ we obtain

$$\begin{aligned} m_j \log g_j &= m_j \log(a_j \exp(t_j X)) = m_j(\log a_j * t_j X) \\ &= \underbrace{m_j \log(a_j)}_{\in \mathfrak{n}} + m_j t_j X + m_j r_j \end{aligned} \tag{2}$$

with $r_j \in \mathfrak{n}$ and $\|r_j\| \leq |t_j|\, \|\log a_j\|$ for a suitable norm on $\mathfrak{g}$. Now the existence of $Y = \lim m_j \log g_j$ implies that $m_j t_j \geq 0$ converges, hence is bounded. Thus $\|m_j r_j\| \leq |m_j t_j|\, \|\log a_j\| \to 0$ since $a_j \to 1$, and we conclude that $m_j \log a_j$ also converges. Now the globality of $W'$ in $N$ yields $\lim_j m_j \log a_j \in W'$ which implies $Y \in W' + \mathbb{R}^+X$. Since $W' \in \mathcal{W}$ was arbitrary, it follows that

$$L(S) \subseteq \bigcap_{W' \in \mathcal{W}} (W' + \mathbb{R}^+X) = W + \mathbb{R}^+X, \tag{3}$$

which proves (iii).

From (i)–(iii) we deduce the globality of $W + \mathbb{R}^+X$ in $G$, because we have proved the existence of a semigroup $S$ with the prescribed tangent wedge. ■

***Remark 1.2.*** *The sets $S_t$ defined in the previous proof are bigger than they ought to be. Indeed, it would be sufficient to take*

$$\widetilde{S}_t = \{\alpha(t_1)(g_1)\cdots\alpha(t_n)(g_n)\exp(tX) \mid 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq t,\ g_i \in \exp(W)\}$$

*in order to obtain a semigroup via $\widetilde{S} = \bigcup_{t\geq 0} \widetilde{S}_t$. But this does not change the proof of (iii). The main difference is that with our choice, the sets $T_t = S_t \exp(-tX) \subseteq N$ are subsemigroups of $N$ whereas the corresponding sets $\widetilde{T}_t = \widetilde{S}_t \subseteq N$ need not be subsemigroups of $N$. If $N$ is abelian then obviously $S_t = \widetilde{S}_t$ because we may rearrange the factors $\alpha(t_i)\exp(w_i)$ such that $t_i \leq t_{i+1}$, but in the non-abelian case $S_t \setminus \widetilde{S}_t$ may be non-empty.*
*The assumption that $\bigcap \mathcal{W} = W$ may not be weakened which will be shown by a counter example.*

## 2. A counterexample

The following counterexample shows, that we cannot weaken the assumption that $\bigcap \mathcal{W} = W$. We take $N$ to be the Heisenberg group and $G = N \rtimes \mathbb{R}$ the four-dimensional oscillator group where $\mathbb{R}$ acts on $N$ by rotations around the center $Z(N)$. So we may identify $G$ with the set $\mathbb{C} \times \mathbb{R} \times \mathbb{R}$ where the multiplication is given by

$$(v, s, r)(w, t, r') = \left(v + e^{ir}w,\ s + t + \frac{1}{2}\Im(\overline{v}e^{ir}w), r + r'\right). \tag{4}$$

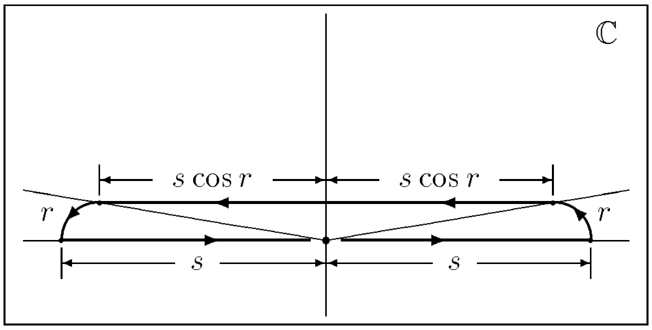
We may identify $\mathfrak{g}$ with the set $\mathbb{C} \times \mathbb{R} \times \mathbb{R}$ and let $P = (1, 0, 0)$, $Q = (i, 0, 0)$, $Z = (0, 1, 0)$, $R = (0, 0, 1)$, then $\{P, Q, Z\}$ is a standard base for the Heisenberg algebra $\mathfrak{n}$, i.e. $[P, Q] = Z$, $Z$ is central. Now we consider the following wedge:

$$W = \mathbb{R}^+(Z - P) + \mathbb{R}^+(Z + P). \tag{5}$$

Then $W$ is global in $N$ because $W + (-W)$ is an abelian subalgebra of $\mathfrak{n}$, but every wedge in $\mathcal{W}$ wedge has central elements in its interior, hence generates the whole group $N$ (see [1, V.4.10]), so $\mathcal{W} = \{\mathfrak{n}\}$ and $\bigcap \mathcal{W} = \mathfrak{n} \neq W$. The non-globality of $W + \mathbb{R}^+R$ in $G$ follows from the following proposition:

***Proposition 2.1.*** *Let us denote with $S_r$ the slice $\langle \exp(W + \mathbb{R}^+R)\rangle \cap \mathbb{C} \times \mathbb{R} \times \{r\}$, then for every $r > 0$ the set $Z(N)(0, 0, 2r) = \{0\} \times \mathbb{R} \times \{2r\}$ is contained in $S_{2r}$, hence $Z(N) \subseteq \overline{\langle \exp W + \mathbb{R}^+R\rangle}$.*

**Proof.** Let $r > 0$ be given. Since $S_r(0, 0, r' - r) \subseteq S_{r'}$ for $r \leq r'$ we only have to consider small $r > 0$, so we may assume w.l.o.g. that $r < \pi/2$, i.e. $\sin(2r) > 0$. With respect to the $\mathbb{C}$-factor we will steer along the following path in order to reach all points in $\{0\} \times \mathbb{R} \times \{2r\}$:



For $s \geq 0$ and $z \geq 0$ we compute:

$$\begin{aligned}
S_{2r} &\ni (0,z,0)(s,s,0)(0,0,r)(-2s\cos r, 2s\cos r, 0)(0,0,r)(s,s,0)\\
&= (s,s+z,0)(0,0,r)(-2s\cos r, 2s\cos r, 0)(se^{ir}, s, r)\\
&= (s,s+z,0)(0,0,r)\left(-2s\cos r + se^{ir}, 2s\cos r + s + \Im(-s\cos r se^{ir}), r\right)\\
&= (s,s+z,0)(0,0,r)\left(\underbrace{-s\cos r + is\sin r}_{-se^{-ir}}, s + 2s\cos r - \frac{s^2}{2}\sin(2r), r\right)\\
&= (s,s+z,0)\left(-s, s+2s\cos r - \frac{s^2}{2}\sin(2r), 2r\right)\\
&= \left(0, z+2s+2s\cos r - \frac{s^2}{2}\sin(2r) + \frac{1}{2}\Im(-s^2), 2r\right)\\
&= \left(0, z+2s(1+\cos r) - s^2\frac{\sin(2r)}{2}, 2r\right).
\end{aligned}$$

Now $\lim_{s\to\infty} z + 2s(1+\cos r) - s^2\frac{\sin(2r)}{2} = -\infty$ thus $\{0\}\times\mathbb{R}\times\{2r\} \subseteq S_{2r}$ for any $0 < r < \pi/2$ which proves our claim. ∎

As an observation, we still prove that our example is in some sense as bad as possible.

***Proposition 2.2.*** *For any $r > 0$ we have $S_r = N \times \{r\}$.*

**Proof.** First we will prove $\mathrm{pr}_{\mathbb{C}}(S_{2r}) = \mathbb{C}$: We have for $0 \leq t \leq r$ and $s \geq 0$:

$$(0,0,t)(s,s,0)(0,0,r-t) \in S_r \text{ and } (0,0,t)(-s,s,0)(0,0,r) \in S_r.$$

Hence $\mathrm{pr}_{\mathbb{C}}(S_r) \supseteq C_r^1 \cup C_r^2$ with

$$C_r^1 = \{v \mid \arg(v) \in [0,r]\} \text{ and } C_r^2 = -C_r^1 = \{v \mid \arg(v) \in [\pi, \pi+r]\}.$$

Thus $\mathrm{pr}_{\mathbb{C}}(S_{2r}) \supseteq (C_r^1 + e^{ir}C_r^2) \cup (C_r^2 + e^{ir}C_r^1)$. Now

$$C_r^2 + e^{ir}C_r^1 = \{\arg(v) \in [\pi,\pi+r]\} + \{\arg(v) \in [r,2r]\} = \{\arg(v) \in [r,\pi+r]\},$$

i.e. a halfspace $E_+$, and because of $C_r^2 = -C_r^1$ we have $C_r^1 + e^{ir}C_r^2 = -E_+$ which proves $\mathrm{pr}_{\mathbb{C}}(S_{2r}) \supseteq \mathbb{C}$. From the previous proposition we know that $\{0\}\times\mathbb{R}\times\{r\} \subseteq S_r$ so that we may deduce $S_{3r} = H \times \{r\}$ by right-multiplication with appropriate elements of $\{0\}\times\mathbb{R}\times\{r\}$. ∎

This counterexample is minimal with respect to the dimension of $G$, because if the hyperplane ideal $\mathfrak{n}$ has dimension $\leq 2$ then it is either abelian or the two-dimensional non-abelian algebra. In both cases every wedge $W \subseteq \mathfrak{n}$ is global in $N$, (see [1, V.4.13] for the non-abelian case). So if we take a pointed wedge $W$, then $\bigcap \mathcal{W} = W$ and if the edge $H(W) \neq \{0\}$ then $W$ must be a halfspace. Since $H(W)$ is $ad\,X$-invariant and $\mathfrak{n} \setminus H(W)$ is disconnected, $W$ must be $ad\,X$-invariant itself which implies $W \in \mathcal{W}$. Thus there is no counterexample of lower dimension. It is also possible to construct a nilpotent counterexample, just take $\mathfrak{n} \rtimes_D \mathbb{R}$ with $\mathfrak{n}$ the Heisenberg-algebra, $W$ as in the former example and $D \in Der(\mathfrak{n})$ is

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the base $\{P, Q, Z\}$.

## 3. Applications of the globality Theorem

The following corollary ([1, VI.5.2]) enables us to prove the globality in many cases:

***Corollary 3.1.*** *Suppose $W_2 \subseteq \mathfrak{g}$ is global in $G$ and $W_1 \subseteq W_2$ satisfies*

*(i) $W_1 \cap H(W_2) \subseteq H(W_1)$.*

*(ii) The analytical subgroup $\langle \exp H(W_1) \rangle$ is closed in $G$.*

*Then $W_1$ is also global in $G$.*

***Remark 3.2.*** *If $W_2$ is pointed, then these conditions are trivially satisfied.*

We will now give a sufficient condition for the equality $\bigcap \mathcal{W} = W$.

***Definition 3.3.*** *Let $V$ be a vector space and $W_1, W \subseteq V$ two wedges. We say that $W_1$ is* surrounding *$W$ (in $V$), if $W \setminus H(W) \subseteq int_V(W_1)$ or equivalently $W \subseteq int_V(W_1) \cup H(W)$.*

***Proposition 3.4.*** *Assume $W \subseteq \mathfrak{n}$ is pointed and there exists a pointed, global wedge $W_1 \subseteq \mathfrak{n}$ surrounding $W$ in $\mathfrak{n}$. Then $W = \bigcap \mathcal{W}$.*

**Proof.** If $v \in \mathfrak{n} \setminus W$ then there exists a wedge $W_2$ surrounding $W$ with $v \notin W_2$. Therefore $W' := W_1 \cap W_2$ is global and since $W'$ surrounds $W$, we can find an $\varepsilon > 0$ such that $e^{[0,\varepsilon]\, ad X} \subseteq W'$ no matter what the action of $ad X$ on $\mathfrak{n}$ is. Thus $W' \in \mathcal{W}$ and $v \notin \bigcap \mathcal{W}$ which proves the assertion. ■

So we may state

***Corollary 3.5.*** *Let $G$ be a Lie group containing a normal subgroup $N$ such that $G/N \cong \mathbb{R}$. Let $W \subseteq \mathfrak{n}$ be a pointed wedge such that there exists a global wedge $W_1$ surrounding $W$ in $\mathfrak{n}$. Then for every $X \in \mathfrak{g} \setminus \mathfrak{n}$ the Lie wedge $W + \mathbb{R}^+ X$ is global in $G$.*

**Proof.** Since $W$ is pointed, contained in $W_1$, it is global in $N$ and its edge $H(W) = \{0\}$ is trivially $ad X$-invariant. Proposition 3.4 yields $W = \bigcap \mathcal{W}$, so we may apply Theorem 1.1. ■

The next question which arises is: Suppose we are given a Lie wedge bounded by the hyperplane ideal $\mathfrak{n}$, i.e. $W \subseteq \mathfrak{n} + \mathbb{R}^+ X$ for an $X \in \mathfrak{g} \setminus \mathfrak{n}$. When are these wedges global? In view of corollary 3.1 one should try to find a suitable wedge $W' \subseteq \mathfrak{n}$ satisfying the assumptions of Theorem 1.1 such that $W' + \mathbb{R}^+ X \supseteq W$. If we assume that $W$ is pointed then the intersection $W \cap \mathfrak{n}$ will play an important role.

## 4. A Lemma on pointed wedges

If $(X,d)$ is a metric space, we may endow the set $\mathcal{C}(X)$ of compact subsets with the so-called Vietoris topology which is defined through the metric

$$d(A,B) = \max\left(\max\{d(a,B) \mid a \in A\}, \max\{d(b,A) \mid b \in B\}\right).$$

where $d(a,B) = \min\{d(a,b) \mid b \in B\}$ is the distance of $a$ from $B$.

***Proposition 4.1.*** *Assume $X$ is a metric space and $(K_n)_{n\in\mathbb{N}} \subseteq X$ a decreasing sequence of compact sets. Let $K = \bigcap_{n\in\mathbb{N}} K_n$, then $K_n \to K$ in the Vietoris topology.*

***Proposition 4.2.*** *Suppose $f\colon X \to X$ is a continous function and $K_n \to K$ in the Vietoris topology. Then $f(K_n) \to f(K)$ in the Vietoris topology.*

***Lemma 4.3.*** *Let $V$ be a vector space, $C \subseteq V$ a pointed, (closed, convex) cone, $H$ a supporting hyperplane of $C$ and the cone $\widetilde{C} \subseteq H$ is surrounding $C \cap H$ in $H$, i.e. $int_H(\widetilde{C}) \supseteq (C \cap H) \setminus \{0\}$. Then there exists an $x \in V \setminus H$ such that $C \subseteq \widetilde{C} + \mathbb{R}^+x$.*

**Proof.** Let $\alpha \in H^\perp \cap C^*$ and pick an arbitrary $x_0$ with $\alpha(x_0) = 1$. Pick $\omega \in int(C^*)$ and let $K = C \cap \omega^{-1}(1)$, i.e. $K$ is compact and $C = \mathbb{R}^+K$. We will construct an invertible linear map $\psi : V \to V$ with the following properties:

(i) $\psi|_H = id_H$,

(ii) $\psi(K) \subseteq \widetilde{C} + \mathbb{R}^+x_0$.

Since $\mathbb{R}^+K = C$ we may then conclude that $\psi(C) = \psi(\mathbb{R}^+K) \subseteq \widetilde{C} + \mathbb{R}^+x_0$, hence

$$C \subseteq \psi^{-1}(\widetilde{C}) + \mathbb{R}^+\psi^{-1}(x_0) = \widetilde{C} + \mathbb{R}^+x$$

with $x = \psi^{-1}(x_0)$ because of (i).

Now we construct the map $\psi$: We denote the projection along $x_0$ onto $H$ with $\mathrm{pr}_H$, i.e. $\mathrm{pr}_H(v) = v - \alpha(v)x_0$ since $\alpha(x_0) = 1$. Now $K$ is compact and so is $\mathrm{pr}_H(K)$. We denote with $K_r$ the slices $K \cap \alpha^{-1}([0,r])$ which are also compact for any $r > 0$. Since $K_r \to K_0 = K \cap H \subseteq int(\widetilde{C})$ for $r \to 0$ in the Vietoris topology, there is an $r_0 > 0$ such that $\mathrm{pr}_H(K_{r_0}) \subseteq \widetilde{C}$. Since $int_H(\widetilde{C}) \neq \emptyset$ and $\mathrm{pr}_H(K)$ is compact, we can find some $w \in int_H(\widetilde{C})$ such that

$$r_0w + \mathrm{pr}_H(K) \subseteq \widetilde{C}.$$

Now let $\psi(v) = \mathrm{pr}_H(v) + \alpha(v)w + \alpha(v)x_0$, then $\psi|_H = id_H$, $\psi(x_0 - w) = -w + w + x_0 = x_0$, hence $\psi$ is surjective and therefore invertible. If $v \in K$ then either $\alpha(v) \leq r_0$ which implies

$$\psi(v) = \underbrace{\mathrm{pr}_H(v)}_{\in\widetilde{C}} + \underbrace{\alpha(v)w}_{\in\widetilde{C}} + \alpha(v)x_0 \in \widetilde{C} + \mathbb{R}^+x_0$$

or $\alpha(v) \geq r_0$ and therefore

$$\psi(v) = \underbrace{\mathrm{pr}_H(v) + r_0w}_{\in\widetilde{C}} + \underbrace{(\alpha(v) - r_0)w}_{\in\widetilde{C}} + \alpha(v)x_0 \in \widetilde{C} + \mathbb{R}^+x_0,$$

thus $\psi(K) \subseteq \widetilde{C} + \mathbb{R}^+x_0$. ∎

## 5. Globality of wedges that are bounded by a hyperplane ideal

Now we have all the tools to prove

***Theorem 5.1.*** *Let $G$ be a Lie group containing a normal subgroup $N$ with $G/N \cong \mathbb{R}$. Suppose $W \subseteq \mathfrak{g}$ is a pointed wedge such that*

*(i) $\mathfrak{n}$ is a supporting hyperplane of $W$,*

*(ii) There exists a pointed wedge $W_1 \subseteq \mathfrak{n}$ such that $W_1$ is surrounding $W \cap \mathfrak{n}$ in $\mathfrak{n}$ and $W_1$ is global in $N$.*

*Then $W$ is global in $G$.*

**Proof.** Since $W_1$ is surrounding $W \cap \mathfrak{n}$ in $\mathfrak{n}$, we may choose a wedge $W_2$ such that $W_1$ is surounding $W_2$ and $W_2$ is surrounding $W \cap \mathfrak{n}$ in $\mathfrak{n}$. According to Lemma 4.3 we can find an $X_0 \in \mathfrak{g} \setminus \mathfrak{n}$ such that $W \subseteq W_2 + \mathbb{R}^+ X_0$, the latter being global in $G$ in view of Corollary 3.5. Now Corollary 3.1 yields the globality of $W$. ■

***Remark 5.2.*** *In view of the counterexample, it is clear that we cannot drop the assumption that there is a global wedge $W'$ surrounding $W \cap \mathfrak{n}$. Nevertheless this condition is not necessary for globality, because if we consider the invariant wedge $W$ in the oscillator-algebra, then this wedge is naturally global, but there is no pointed global wedge $W'$ surrounding $W \cap \mathfrak{n} = \mathbb{R}^+ Z$.*

## 6. The globality problem of the Heisenberg-algebra

Let us consider the Heisenberg-algebra $\mathfrak{g}$ with the standard base $\{P, Q, Z\}$ and brackets $[P, Q] = Z$, $[P, Z] = [Q, Z] = 0$. We identify it with the Heisenberg group $G$ via the Campbell-Hausdorff-multiplication. From [1] we know, that the hyperplane-subalgebras of $\mathfrak{g}$ play an important role. A half-space which is bounded by a subalgebra is called a halfspace-semialgebra. The following is an immediate consequence of [1, V.5.41]:

***Corollary 6.1.*** *If $W$ is Lie generating and not contained in a halfspace-semialgebra, then $W$ is controllable.*

The hyperplane subaglebras of $\mathfrak{g}$ may be described as follows:

***Proposition 6.2.*** *A hyperplane $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra iff $\mathfrak{z} \subseteq \mathfrak{h}$. Any proper subalgebra is contained in a hyperplane subalgebra.*

Now we are ready to solve the globality problem of the Heisenberg-algebra.

***Theorem 6.3.*** *Suppose $W \subseteq \mathfrak{g}$ is a Lie-wedge, then the following are equivalent:*

*(i) $W$ is global in $G$.*

*(ii) $W$ is contained in a halfspace-semialgebra.*

*(iii) $W^* \cap \mathfrak{z}^\perp \neq \{0\}$.*

**Proof.** The equivalence of (ii) and (iii) follows from the classification of hyperplane subalgebras in $\mathfrak{g}$. For proving (i)$\Rightarrow$(ii), we consider a global wedge $W$. If $W$ is Lie generating then Corollary 6.1 proves (ii). If $W$ is not Lie generating, then it is contained in a proper subalgebra. Hence (ii) holds in view of Proposition 6.2. The crucial part of the proof is (iii)$\Rightarrow$(i).
We have to consider the different cases of $\dim H(W)$ seperately:

1. $\dim H(W) = 2$. In this case $W$ is either a hyperplane-subalgebra or a halfspace-semialgebra, hence global.
2. $\dim H(W) = 1$. The invariance of $W$ under $e^{ad\, H(W)}$ implies
$$[H(W), W] \subseteq H(W) \cap \mathfrak{z},$$
thus either $H(W) = \mathfrak{z}$ or $W \subseteq H(W) + \mathfrak{z}$. In the first case, $W$ satisfies the conditions of corollary 3.1, and in the second case $W$ is contained in an abelian subalgebra, so $W$ is global in any case.
3. $\dim H(W) = 0$. Let $\omega \in W^* \cap \mathfrak{z}^\perp$, then $\mathfrak{n} := \ker \omega$ is an abelian subalgebra and $G/N \simeq \mathbb{R}$. Take an arbitrary pointed $W'$ surrounding $W$ in $\mathfrak{g}$, then $W' \cap \mathfrak{n}$ is pointed, surrounds $W \cap \mathfrak{n}$ in $\mathfrak{n}$ and is global, since $\mathfrak{n}$ is abelian. Therefore Theorem 5.1 applies, proving the globality of $W$.

Thus everything is proved. ∎

## 7. Some remarks on Lorentzian Lie groups

In [2] Levichev and Levicheva consider the Lorentzian manifold structure on a Lie group $G$ obtained by choosing a Lorentzian wedge $W \subseteq \mathfrak{g}$. For a closed semigroup $S \subseteq G$ we may define the partial order $\leq_S$ on $G$ by $x \leq_S y$ iff $xS \ni y$. The notion of *future-distinguishability* is then equivalent to the condition that $S$ is pointed, i.e. $S \cap S^{-1} = \{1\}$. Hence the Lorentzian manifold $G$ is future-distinguishing iff the Lie wedge $W$ is global in $G$.

It may happen that the Lorentzian manifold $G$ is not geodesically complete, i.e. if we denote $Exp$ the exponential function of the affine connection on $G$ induced by the choice of a bilinear form on $\mathfrak{g}$, then $Exp_1$ is not defined on all of $\mathfrak{g}$. Nevertheless the Lie semigroup generated by a Lorentzian cone is always well-defined, so the future-distinguishability-property is independent of geodesically completeness. Thus it may happen that the future of an event is well-behaved although some geodesics are not infinitely extendable.

As a final remark, we mention that Theorem 5.1 applies to the cases considered in [2], because there the hyperplane ideal $\mathfrak{n}$ is either abelian or almost-abelian, and therefore every wedge $W' \subseteq \mathfrak{n}$ is global in the corresponding group (cf. [1, V.4.13]).

## References

[1] Hilgert, J., K. H. Hofmann and J. D. Lawson, "Lie groups, Convex Cones and Semigroups," Oxford University Press, 1989.

[2] Levichev A., and V. Levicheva, *Distinguishability Condition and the Future Subsemigroup,* Seminar Sophus Lie **2** 1992, 205–212.

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