Seminar Sophus Lie
2 (1992) 135–151

Near-Cartan algebras and groups

Karl H. Hofmann

0. Preliminary results on nilpotent algebras

We let $\mathfrak{h}$ denote a real or complex nilpotent Lie algebra and V a finite dimensional complex $\mathfrak{h}$-module. For $X \in \mathfrak{h}$ let $X_V = (v \mapsto X\cdot v): V \to V$. We set $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}$ if the ground field K is $\mathbb{C}$ and let $\mathfrak{h}_{\mathbb{C}}$ denote the complexification of $\mathfrak{h}$ if $K = \mathbb{R}$. Recall that a linear form $\lambda: \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ is a *weight* if there is a nonzero $v \in V$ such that for some natural number n we have $(X_V - \lambda(X)\cdot\mathbf{1}_V)^n v = 0$ for all $X \in \mathfrak{h}$. The element v is called a weight-vector for λ, and V^λ is the set of all weight vectors for λ. We let $V_\lambda = \{v \in V : (\forall X \in \mathfrak{h}) X\cdot v = \lambda(X)\cdot v\}$. Since $V_\lambda \neq \{0\}$ we have a $0 \neq v \in V_\lambda$ so that $\lambda([X,Y])\cdot v = [X,Y]\cdot v = X_V Y_V(v) - Y_V X_V(v) = \lambda(X)\lambda(Y)v - \lambda(Y)\lambda(X)v = 0$. Hence $\lambda([\mathfrak{h},\mathfrak{h}]) = \{0\}$.

We let Λ denote the (finite!) set of nonzero weights and $V = V^0 \oplus \bigoplus_{\lambda\in\Lambda} V^\lambda$ the *weight decomposition* of V. We set $\Lambda^\perp = \{X \in \mathfrak{h} : (\forall\lambda \in \Lambda)\lambda(X) = 0\}$.

Lemma 0.1. $\Lambda^\perp = \{X \in \mathfrak{h} : (\exists n) X_V^n = 0\} = \{X \in \mathfrak{h} : \operatorname{Spec} X_V = \{0\}\} = \bigcap_{\lambda\in\Lambda} \ker \lambda|\mathfrak{h}$. *In particular,* $\Lambda^\perp$ *contains* $[\mathfrak{h},\mathfrak{h}]$ *and thus is an ideal of* $\mathfrak{h}$.

Proof. We have $X \in \Lambda^\perp$ iff for all $\lambda \in \Lambda$ and all $v \in V^\lambda$ we have $X_V^n(v) = 0$ for some n iff $X_V^n = 0$ for some n since $V = \bigoplus_{\lambda\in\Lambda} V^\lambda$. This is the case iff $\operatorname{Spec} X_V = \{0\}$. The remainder is immediate from the definition and the preceding remarks. ∎

Suppose that α is an automorphism of $\mathfrak{h}_{\mathbb{C}}$ and $\varphi \in \operatorname{Hom}(V,V)$ is such that $\varphi(X\cdot v) = \alpha(X)\cdot\varphi(v)$. If λ is a weight, then for $v \in V^\lambda$ we have

$$\begin{aligned} 0 = \varphi\big((X_V - \lambda(X)\cdot\mathbf{1}_V)^n v\big) &= (\alpha(X)_V - \lambda(X)\cdot\mathbf{1}_V)^n\varphi(v) \\ &= (\alpha(X)_V - (\lambda\circ\alpha^{-1})\big(\alpha(X)\big)\cdot\mathbf{1}_V)^n\varphi(v). \end{aligned}$$

Thus $\lambda\circ\alpha \in \Lambda$ and $\varphi(V^\lambda) = V^{\lambda\circ\alpha^{-1}}$. Every such α leaves Λ and thus $\Lambda^\perp$ invariant and therefore induces an automorphism α_Λ of $\mathfrak{h}_{\mathbb{C}}/\Lambda^\perp$ via $\alpha_\Lambda(X + \Lambda^\perp) = \alpha(X) + \Lambda^\perp$.

Lemma 0.2. *If α and φ are as in the preceding paragraph, then the following statements are equivalent:*

(1) $(\forall\lambda \in \Lambda)\,\lambda\circ\alpha = \lambda$.

(2) $\alpha_\Lambda = \mathbf{1}$ (*on* $\mathfrak{h}_{\mathbb{C}}/\Lambda^\perp$).

(3) $\alpha_\Lambda - \mathbf{1}$ *is nilpotent (on* $\mathfrak{h}_\mathbb{C}/\Lambda^\perp$*).*

(4) $\mathfrak{h}_\mathbb{C} = \mathfrak{h}_\mathbb{C}^1(\alpha) + \Lambda^\perp$.

Proof. (1)$\Leftrightarrow$(2): Let Γ denote the subgroup generated by α in $\mathrm{Aut}(\mathfrak{h}_\mathbb{C})$. The Γ-module $\mathfrak{h}_\mathbb{C}/\Lambda^\perp$ is dual to the Γ-module $\mathrm{span}_\mathbb{C}\Lambda$ in $\mathfrak{h}_\mathbb{C}^*$. Now (1) means that $\mathrm{span}_\mathbb{C}\Lambda$ is the trivial Γ-module which is the same as saying that $\mathfrak{h}_\mathbb{C}/\Lambda^\perp$ is the trivial module which is (2).

(2)$\Rightarrow$(3) clear!

(3)$\Rightarrow$(2): We define the permutation α^* of Λ by $\alpha^*(\lambda) = \lambda \circ \alpha^{-1}$. It extends to $E = \mathrm{span}_\mathbb{C}\Lambda$. Since Γ^* is a finite subgroup of $\mathrm{Aut}(E)$, the automorphism α^* is semisimple on E. Since $\alpha_\Lambda - \mathbf{1}$ is nilpotent by (3), then $\alpha^* - \mathbf{1}$ is nilpotent on E. Since $\alpha^* - \mathbf{1}$ is also semisimple, we conclude $\alpha^* - \mathbf{1} = 0$.

(3)$\Leftrightarrow$(4) is basic linear algebra. ∎

Definition 0.3. A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent subalgebra which is its own normalizer. ∎

Let $\mathfrak{g}$ denote a Lie algebra and $\mathfrak{g}_\mathbb{C}$ its complexification if $\mathfrak{g}$ is real. Otherwise let $\mathfrak{g}_\mathbb{C} = \mathfrak{g}$. If $\mathfrak{h}$ is a Cartan algebra of $\mathfrak{g}$, then we can apply the preceding with $V = \mathfrak{g}_\mathbb{C}$ and the adjoint action. In particular, Λ is now the set of all roots, including the zero-root. We observe that $X \in \mathfrak{h} \cap \Lambda^\perp$ iff $\mathrm{ad}\, X$ is nilpotent. If $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$, then $\frac{\mathfrak{h}+\mathfrak{n}}{\mathfrak{n}}$ is a Cartan algebra of the reductive algebra $\mathfrak{g}/\mathfrak{n}$, and since $\mathrm{ad}\,\xi$ for $\xi \in \mathfrak{g}/\mathfrak{n}$ is semisimple on $\mathfrak{g}/\mathfrak{n}$ we conclude $\mathfrak{h} \cap \Lambda^\perp \subseteq \mathfrak{n}$. Thus $\mathfrak{h} \cap \Lambda^\perp \subseteq \mathfrak{h} \cap \mathfrak{n}$. Since $X \in \mathfrak{n}$ implies that $\mathrm{ad}\, X$ is nilpotent we have

$$\mathfrak{h} \cap \Lambda^\perp = \mathfrak{h} \cap \mathfrak{n}. \tag{1}$$

Each automorphism α of $\mathfrak{g}$ extends uniquely to an automorphism of $\mathfrak{g}_\mathbb{C}$ which we shall again denote by α. Thus we have $\mathrm{Aut}\,\mathfrak{g} \subseteq \mathrm{Aut}\,\mathfrak{g}_\mathbb{C}$.

Definition 0.4. We define

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) = \{\alpha \in \mathrm{Aut}\,\mathfrak{g} : \alpha(\mathfrak{h}) = \mathfrak{h}\} \subseteq \mathrm{Aut}\,\mathfrak{g}_\mathbb{C}.$$

The function $\pi: \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) \to S(\Lambda)$ into the group of all permutations of Λ given by $\pi(\alpha)(\lambda) = \lambda \circ \alpha^{-1}$ is a representation. We set

$$\mathrm{CAut}(\mathfrak{g}, \mathfrak{h}) = \ker \pi, \quad B\mathcal{W}(\mathfrak{g}, \mathfrak{h}) = \mathrm{im}\, \pi.$$

We call $B\mathcal{W}(\mathfrak{g}, \mathfrak{h})$ the *big Weyl group of* $\mathfrak{g}$ *w. r. t.* $\mathfrak{h}$. ∎

If V if a complex vector space and α an automorphism then we set $V^\lambda(\alpha) = \{v \in V : (\exists n)\,(\alpha - \lambda \cdot \mathbf{1}_V)^n(v) = 0\}$.

Proposition 0.5. *For an element* $\alpha \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ *the following statements are equivalent:*

(1) $\alpha \in \mathrm{CAut}(\mathfrak{g}, \mathfrak{h})$.

(2) $\alpha_\Lambda = \mathbf{1}$ *(on* $\mathfrak{h}_\mathbb{C}/\Lambda^\perp$*).*

(3) $\alpha_\Lambda - \mathbf{1}$ *is nilpotent (on* $\mathfrak{h}_\mathbb{C}/\Lambda^\perp$ *).*

(4) $\mathfrak{h}_\mathbb{C} \subseteq \Lambda^\perp + \mathfrak{g}^1_\mathbb{C}(\alpha)$.

(5) $\mathfrak{h} \subseteq \mathfrak{g}^1(\alpha) \bmod \mathfrak{n}$ *where* $\mathfrak{n}$ *is the nilradical of* $\mathfrak{g}$.

Proof. The equivalence of (1)–(4) follows from Lemma 0.2.

(4)$\Rightarrow$(5): We know $\mathfrak{h} \cap \Lambda^\perp = \mathfrak{h} \cap \mathfrak{n}$. Thus $\Lambda^\perp \subseteq \mathfrak{n}_\mathbb{C}$. From this and $\mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{g} = \mathfrak{g}^1(\alpha)$ the assertion follows.

(5)$\Rightarrow$(3): If $\mathfrak{h}_\mathbb{C}$ is contained in $\mathfrak{g}^1_\mathbb{C}(\alpha) + \mathfrak{n}_\mathbb{C}$, then $\alpha - \mathbf{1}$ is nilpotent on $(\mathfrak{h}_\mathbb{C} + \mathfrak{n}_\mathbb{C})/\mathfrak{n}_\mathbb{C} \cong \mathfrak{h}_\mathbb{C}/(\mathfrak{h}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C}) = \mathfrak{h}_\mathbb{C}/\Lambda^\perp$. ∎

Lemma 0.6. *The nilpotent group* $e^{\operatorname{ad}\mathfrak{h}}$ *is normal in* $\operatorname{Aut}(\mathfrak{g},\mathfrak{h})$ *and is contained in* $\operatorname{CAut}(\mathfrak{g},\mathfrak{h})$.

Proof. Trivially $\mathfrak{e}^{\operatorname{ad}X}\mathfrak{h} = \mathfrak{h}$ for $X \in \mathfrak{h}$. If $\alpha \in \operatorname{Aut}(\mathfrak{g},\mathfrak{h})$ then $X \in \mathfrak{h}$ implies $\alpha(X) \in \mathfrak{h}$, and thus $\alpha e^{\operatorname{ad}X}\alpha^{-1} = e^{\alpha \operatorname{ad}X\alpha^{-1}} = e^{\operatorname{ad}\alpha(X)}$ is in $e^{\operatorname{ad}\mathfrak{h}}$. Further, $e^{\operatorname{ad}X}|\mathfrak{h}$ is unipotent for $X \in \mathfrak{h}$. Hence $(e^{\operatorname{ad}X})_\Lambda - \mathbf{1}$ is nilpotent on $\mathfrak{h}_C$ and thus (3) of 0.5 is satisfied. Thus $e^{\operatorname{ad}X} \in \operatorname{CAut}(\mathfrak{g},\mathfrak{h})$. ∎

Lemma 0.7. *For* $\beta \in \operatorname{CAut}(\mathfrak{g},\mathfrak{h})$ *the following statements are equivalent:*

(a) $\beta|(\mathfrak{h}\cap\mathfrak{n})$ *is unipotent.*

(b) $\mathfrak{h}\cap\mathfrak{n} \subseteq \mathfrak{g}^1(\beta)$.

(c) $\mathfrak{h} \subseteq \mathfrak{g}^1(\beta)$.

Proof. The equivalence of (a) and (b) follows from the definitions also $\mathfrak{h} \subseteq \mathfrak{g}^1(\beta) + (\mathfrak{h}\cap\mathfrak{n})$. Thus (b) implies (c), and trivially, (c)$\Rightarrow$(b). ∎

For the next lemma we need some preparation.

Lemma 0.8. (KARL-HERMANN NEEB) *Let* α *be a unipotent automorphism of a finite dimensional vector space* V *and* ν *an endomorphism with its additive Jordan decomposition* $\nu = \nu_s + \nu_n$. *Let* $\mathfrak{a}$ *denote a subalgebra of* $\mathfrak{gl}(V)$ *which is nilpotent on* V. *We assume the following hypotheses:*

(i) $\alpha\nu_s = \nu_s\alpha$.

(ii) $\alpha\mathfrak{a}\alpha^{-1} = \mathfrak{a}$.

(iii) $\nu_n \in \mathfrak{a}$. *Write* $\beta = \alpha e^\nu$ *with its multiplicative Jordan decomposition* $\beta = \beta_s\beta_u$.

Then $\beta_s = e^{\nu_s}$ *and* $\beta_u = \alpha e^{\nu_n}$.

Proof. From (i) we know that e^ν_s commutes with α, and since ν_s and ν_n commute anyhow. Hence e^{ν_s} commutes with αe^{ν_n} It therefore remains to show that αe^{ν_n} is unipotent. In the group $\operatorname{Gl}(V)$ we consider the subgroup $U = \langle e^\mathfrak{a}, \alpha\rangle$. The subgroups $e^\mathfrak{a}$ and $\langle\alpha\rangle$ are unipotent, and by (ii) the former is normal in U. Hence U is unipotent by [4], Proposition 2.2 on p. 64. Since $\alpha e^\nu_n \in U$ by (iii), the assertion follows. ∎

Lemma 0.9. *Suppose* $\alpha \in \operatorname{CAut}(\mathfrak{g},\mathfrak{h})$ *and let* $\mathfrak{n}$ *denote the nilradical of* $\mathfrak{g}$. *Then there is a zero neighborhood* U *such that for any regular* $X \in \mathfrak{h}\cap\mathfrak{g}^1(\alpha)$ *we have*

$$\mathfrak{h} = \mathfrak{g}^1(\alpha e^{\operatorname{ad}X}) + (\Lambda^\perp \cap \mathfrak{h}) = \mathfrak{g}^1(\alpha e^{\operatorname{ad}X}) + (\mathfrak{n}\cap\mathfrak{h}). \tag{2}$$

Proof. (KARL-HERMANN NEEB) The second equality in (2) is a consequence of (1). For any $X \in \mathfrak{h}$, we define the automorphism $\beta = \alpha e^{\mathrm{ad}\, X} \in \mathrm{CAut}(\mathfrak{g}, \mathfrak{h})$ and set $\mathfrak{m} = \Lambda^\perp \cap \mathfrak{h}_\mathbb{C} = \mathfrak{n}_\mathbb{C} \cap \mathfrak{h}_\mathbb{C}$. By 0.5(4) and 0.6 we have $\mathfrak{h}_\mathbb{C} \subseteq \mathfrak{g}^1_\mathbb{C}(\beta) + \mathfrak{m}$. In particular, taking $X = 0$ we have $\mathfrak{h}_\mathbb{C} = (\mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C}) + \mathfrak{m}$.

In order to obtain the reverse containment for a suitable X we must show $\mathfrak{g}^1_\mathbb{C}(\beta) \subseteq \mathfrak{h}_\mathbb{C}$.

We consider the generalized eigenspace decomposition

$$\mathfrak{g}_\mathbb{C} = \bigoplus_{\mu \in \mathrm{Spec}(\alpha)} \mathfrak{g}^\mu_\mathbb{C}(\alpha).$$

Since $[\mathfrak{g}^1_\mathbb{C}(\alpha), \mathfrak{g}^\mu_\mathbb{C}(\alpha)] \subseteq \mathfrak{g}^\mu_\mathbb{C}(\alpha)$, all $\mathfrak{g}^\mu_\mathbb{C}(\alpha)$ are $\mathfrak{g}^1_\mathbb{C}(\alpha)$-modules. Thus, if we now take an arbitrary $X \in \mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C}$ and set $\beta = \alpha e^{\mathrm{ad}\, X}$, then

$$\beta\big(\mathfrak{g}^\mu_\mathbb{C}(\alpha)\big) \subseteq \mathfrak{g}^\mu_\mathbb{C}(\alpha).$$

Thus

$$\mathfrak{g}^1_\mathbb{C}(\beta) = \bigoplus_{\mu \in \mathrm{Spec}\, \alpha} \big(\mathfrak{g}^\mu_\mathbb{C}(\alpha)\big)^1(\beta).$$

If $\mu \neq 1$ and if X is small enough, then 1 is not in the spectrum of $\beta|\big(\mathfrak{g}^\mu_\mathbb{C}(\alpha)\big)$ and thus $\big(\mathfrak{g}^\mu_\mathbb{C}(\alpha)\big)^1(\beta) = \{0\}$. Hence there is a zero neighborhood U in $\mathfrak{h}$ such that $X \in U \cap \mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C}$ implies $\mathfrak{g}^1_\mathbb{C}(\beta) \subseteq \mathfrak{g}^1_\mathbb{C}(\alpha)$.

Now we consider the vector space

$$\widetilde{\mathfrak{a}} = \mathbb{C}{\cdot}X + (\mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C}).$$

Since $X \in \mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C}$ and $\mathfrak{n}_\mathbb{C}$ is an ideal, $\widetilde{\mathfrak{a}}$ is a subalgebra of $\mathfrak{h}_\mathbb{C}$. Note that $\mathrm{ad}(\mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C})$ is nilpotent since $\mathfrak{n}_\mathbb{C}$ is the nilradial. The nilpotent part

$$(\mathrm{ad}\, \widetilde{\mathfrak{a}})_n = \mathbb{C}{\cdot}(\mathrm{ad}\, X)_n + (\mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C})$$

acts nilpotently on $\mathfrak{g}_\mathbb{C}$. We define $V = \mathfrak{g}^1_\mathbb{C}(\alpha)$ and set $\mathfrak{a} = (\mathrm{ad}\, \widetilde{\mathfrak{a}})_n|V \subseteq \mathfrak{gl}(V)$. We claim that $\mathfrak{a}$ is an algebra. Now for any derivation D of a Lie algebra and any of its elements x one has $[D, \mathrm{ad}\, x] = \mathrm{ad}(Dx)$. But $(\mathrm{ad}\, X)_n$ is a derivation since $\mathrm{Der}(\mathfrak{g}_\mathbb{C})$ is scindable in $\mathfrak{gl}(\mathfrak{g}_\mathbb{C})$ as the Lie algebra of the algebraic group $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})$. Hence $[(\mathrm{ad}\, X)_n, \mathrm{ad}(\mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C})] = \mathrm{ad}\,\big((\mathrm{ad}\, X)_n(\mathrm{ad}(\mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C}))\big)$. But $(\mathrm{ad}\, X)_s|\mathfrak{h}_\mathbb{C} = 0$ since $\mathfrak{h}_\mathbb{C}$ is nilpotent, and thus $(\mathrm{ad}\, X)_n|\mathfrak{h}_\mathbb{C} = (\mathrm{ad}\, X)|\mathfrak{h}_\mathbb{C}$. Thus $(\mathrm{ad}\, X)_n(\mathrm{ad}(\mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C}) \subseteq \mathrm{ad}\,\big(\mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C}\big) \subseteq \widetilde{\mathfrak{a}}$. This proves that $\mathfrak{a}$ is an algebra.

We shall verify the hypotheses of Lemma 0.8 with $\alpha|V$ in place of α and $\mathrm{ad}\, X|V$ in place of ν. For this purpose we have to check hypotheses (i), (ii) and (iii). Since $X \in \widetilde{\mathfrak{a}}$, condition (iii) is satisfied. The operator $(\mathrm{ad}\, X)_s$ acts by scalar multiplication on each root space $\mathfrak{g}^\lambda_\mathbb{C}$, and α leaves each of them invariant because of $\alpha \in \mathrm{CAut}(\mathfrak{g}, \mathfrak{h})$. Hence $[\alpha, (\mathrm{ad}\, X)_s] = 0$ and thus (i) follows. It remains to verify that $Y = r{\cdot}X + Z$ with $Z \in \mathfrak{g}^1_\mathbb{C}(\alpha) \cap \mathfrak{h}_\mathbb{C} \cap \mathfrak{n}_\mathbb{C}$ implies

$$(*) \qquad (\alpha|V)(\mathrm{ad}\, Y)_n|V(\alpha|V)^{-1} \in \mathfrak{a}.$$

Now $\alpha(\operatorname{ad} Y)_n\alpha^{-1} = (\alpha(\operatorname{ad} Y)\alpha^{-1})_n = \big(\operatorname{ad}\alpha(Y)\big)_n$. The element $\alpha(X) - X$ is contained in $\mathfrak{g}^1_{\mathbb{C}}(\alpha) \cap \mathfrak{h}_{\mathbb{C}}$ since $\mathfrak{g}^1_{\mathbb{C}}(\alpha)$ and $\mathfrak{h}_{\mathbb{C}}$ are invariant under α. Further, $\alpha - \mathrm{id}$ induces on $\mathfrak{h}_{\mathbb{C}}/(\mathfrak{h}_{\mathbb{C}} \cap \mathfrak{n}_{\mathbb{C}})$ the zero map by Proposition 0.5(2). Thus $\alpha(X) - X \in \mathfrak{n}_{\mathbb{C}}$. Thus

$$\alpha(X) - X \in \mathfrak{g}^1_{\mathbb{C}}(\alpha) \cap \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{n}_{\mathbb{C}},$$

whence

$$\big(\operatorname{ad}\alpha(X)\big)_n \in \big(\operatorname{ad} X + \mathfrak{g}^1_{\mathbb{C}}(\alpha) \cap \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{n}_{\mathbb{C}}\big)_n,$$

and thus

$$(\alpha|V)\big(\operatorname{ad}(r{\cdot}X)\big)_n|V(\alpha|V)^{-1} \in \mathfrak{a}.$$

Also $\alpha(Z) \in \mathfrak{g}^1_{\mathbb{C}}(\alpha) \cap \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{n}_{\mathbb{C}}$ since $\mathfrak{g}^1_{\mathbb{C}}(\alpha)$, $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{n}_{\mathbb{C}}$ are all invariant under α. Hence

$$(\alpha|V)(\operatorname{ad} Z)_n|V(\alpha|V)^{-1} = \operatorname{ad}\big(\alpha(Z)\big)|V \in \mathfrak{a},$$

too. Thus $(*)$ is proven and condition (iii) of Lemma 0.8 is satisfied. Lemma 0.8 then applies and shows that on $V = \mathfrak{g}^1_{\mathbb{C}}(\alpha)$ the unipotent factor β_u of β is $\alpha e^{(\operatorname{ad} X)_n}$ and the semisimple factor is $e^{(\operatorname{ad} X)_s}$. Thus the generalized eigenspace decomposition of $\beta|V$ is the eigenspace decomposition of $e^{(\operatorname{ad} X)_s}|V$.

If we finally assume that X is regular, then

$$\mathfrak{g}^1_{\mathbb{C}}(\beta) = \mathfrak{g}^1_{\mathbb{C}}(\alpha) \cap (\mathfrak{g}_{\mathbb{C}})_1(e^{(\operatorname{ad} X)_s}) = \mathfrak{g}^1_{\mathbb{C}}(\alpha) \cap \mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{h}_{\mathbb{C}}$$

which is what we had to show. ∎

1. Definitions

We let $\mathcal{H}(\mathfrak{g})$ denote the set of all Cartan algebras. ∎

The group $\mathrm{Aut}(\mathfrak{g})$ acts on $\mathcal{H}(\mathfrak{g})$. Let $\mathrm{Inn}(\mathfrak{g}) = \langle e^{\operatorname{ad} X}\rangle$ denote the subgroup of inner automorphisms. If the ground field is algebraically closed, then all Cartan algebras are conjugate, i.e., $\mathrm{Inn}(\mathfrak{g})$ is transitive on $\mathcal{H}(\mathfrak{g})$. Over the reals, $\mathcal{H}(\mathfrak{g})$ decomposes into finitely many $\mathrm{Inn}(\mathfrak{g})$-orbits. The Cartan algebras have the same dimension, called the *rank* $\operatorname{rank}\mathfrak{g}$ of $\mathfrak{g}$.

An element $X \in \mathfrak{g}$ is called *regular* if the nilspace $\mathfrak{g}^0(\operatorname{ad} X)$ has the smallest possible dimension. If this is the case then $\mathfrak{g}^0(\operatorname{ad} X)$ is a Cartan algebra, and every Cartan algebra is so obtained. The set $\operatorname{reg}\mathfrak{g}$ of regular elements is open dense in $\mathfrak{g}$, and from what we set it follows that

$$\operatorname{reg}(\mathfrak{g}) \subseteq \bigcup \mathcal{H}(\mathfrak{g})$$

.

Cartan subgroups of a connected real or complex Lie group G are harder to define than Cartan algebras [8]. A necessary condition is that $\mathfrak{h} = L(H)$ is a Cartan algebra. Let $\Lambda \subseteq \mathfrak{h}^*_{\mathbb{C}}$ the set of nonzero roots on the complexification $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{h}$. Let $N(\mathfrak{h}) = \{g \in G : \mathrm{Ad}(g)\mathfrak{h} = \mathfrak{h}\}$ denote the normalizer of H_0 or, equivalently, $\mathfrak{h}$ in G. Then $N(\mathfrak{h})$ acts on Λ on the right via $(\lambda, g) \mapsto \lambda \circ \mathrm{Ad}(g)$.

Definition 1.1.a. We set

$$C(\mathfrak{h}) = \{g \in N(\mathfrak{h}) : \lambda \circ \mathrm{Ad}(g) = \lambda \quad \text{for all } \lambda \in \Lambda\}.$$

We say that a subgroup H of a connected real or complex Lie group G is a *Cartan group* if $L(H)$ is a Cartan algebra and $H = C\big(L(H)\big)$. We let $\mathcal{H}(G)$ denote the set of all Cartan groups. ∎

One notes at once that $C(\mathfrak{h})$ is normal in $N(\mathfrak{h})$. Then $N(\mathfrak{h})/C(\mathfrak{h})$ is a well defined discrete group

Definition 1.1.b. We say that the (discrete) group $\mathcal{W}(G,\mathfrak{h}) \stackrel{\text{def}}{=} N(\mathfrak{h})/C(\mathfrak{h})$ is the *Weyl group of G with respect to $\mathfrak{h}$.* ∎

We note that $\mathrm{Ad}\colon N(H) \to \mathrm{Aut}(\mathfrak{h},\mathfrak{g})$ is a homomorphism with kernel $N(H) \cap Z(G)$. We observe that

$$C(\mathfrak{h}) = \mathrm{Ad}^{-1}\big(\mathrm{CAut}(\mathfrak{g},\mathfrak{h})\big).$$

As a consequence we record:

Proposition 1.2. (i) *Let G be any Lie group and $\mathfrak{h}$ a Cartan algebra of $\mathfrak{g}$. For an element $g \in N(\mathfrak{h})$ the following statments are equivalent:*

(1) $g \in C(\mathfrak{h})$.

(2) $\mathrm{Ad}(g)_\Lambda = \mathbf{1}$ *(on $\mathfrak{h}_{\mathbb{C}}/\Lambda^\perp$).*

(3) $\mathrm{Ad}(g)_\Lambda - \mathbf{1}$ *is nilpotent (on $\mathfrak{h}_{\mathbb{C}}/\Lambda^\perp$).*

(4) $\mathfrak{h}_{\mathbb{C}} \subseteq \Lambda^\perp + \mathfrak{g}^1_{\mathbb{C}}\big(\mathrm{Ad}(g)\big)$.

(5) $\mathfrak{h} \subseteq \mathfrak{g}^1\big(\mathrm{Ad}(g)\big) + (\mathfrak{n} \cap \mathfrak{h})$ *where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$.*

(ii) *For each $a \in G$ we have $aC(\mathfrak{h})a^{-1} = C\big(\mathrm{Ad}(a)\mathfrak{h}\big)$.*

Proof. (i) follows from the preceding remark and from Proposition 0.5.

(ii) If φ, $\psi \in \mathrm{Aut}(\mathfrak{g})$ then $\varphi\big(\mathfrak{g}^1(\psi)\big) = \mathfrak{g}^1(\varphi\psi\varphi^{-1})$. Now let $\varphi = \mathrm{Ad}(a)$ and $\psi = \mathrm{Ad}(g)$. Then $\mathrm{Ad}(a)\mathfrak{g}^1\big(\mathrm{Ad}(g)\big) = \mathfrak{g}^1\big(\mathrm{Ad}(aga^{-1})\big)$. Now (5) above is equivalent to $\mathrm{Ad}(a)\mathfrak{h} \subseteq \mathfrak{g}^1\big(\mathrm{Ad}(aga^{-1})\big) + (\mathfrak{n} \cap \mathrm{Ad}(a)\mathfrak{h})$ since $\mathfrak{n}$ is an ideal. Thus the assertion $aC(\mathfrak{h})a^{-1} = C(\mathrm{Ad}(a)\mathfrak{h})$ follows from (i). ∎

Lemma 1.3. *For each $g \in G$ and each identity neighborhood W of G there is a neighborhood V of g such that for $v \in V$ there is a $w \in W$ such that $\mathfrak{g}^1(v) \subseteq \mathrm{Ad}(w)\mathfrak{g}^1(\mathrm{Ad}\, g) = \mathfrak{g}^1\big(\mathrm{Ad}(wgw^{-1})\big)$.*

Proof. [3], Ch. VII, §4, n$^{\mathrm{O}}$ 2, Prop.5. ∎

In particular, $\dim \mathfrak{g}^1(v) \le \dim \mathfrak{g}^1(g)$. An element $g \in G$ is called *regular* if $v \mapsto \dim \mathfrak{g}^1(\mathrm{Ad}\, v)\colon G \to \mathbb{N}_0$ is constant on a neighborhood of $\mathfrak{g}$. The set $\mathrm{Reg}(G)$ of all regular elements is open and dense in G (see [3], Ch. VII, §1, n$^{\mathrm{O}}$ 4, Prop.1).

Let $\mathrm{reg}\exp$ denote the set of all $X \in \mathfrak{g}$ such that $d\exp(X)$ is invertible. We have observed the following fact:

Lemma 1.4. *Suppose that G is a real Lie group and $X \in \mathfrak{g}$. Then* $\exp X \in \operatorname{Reg}(G)$ *if and only if* $X \in \operatorname{reg}\mathfrak{g} \cap \operatorname{reg}\exp$. *If this holds, then* $\mathfrak{g}^1(\operatorname{Ad}(\exp X)) = \mathfrak{g}^0(\operatorname{ad} X)$.

Proof. See [6], Lemma 3. ∎

In other words, we have

$$\begin{aligned} \exp^{-1}(\operatorname{Reg} G) &= \operatorname{reg}\mathfrak{g} \cap \operatorname{reg}\exp, \\ \exp(\operatorname{reg}\mathfrak{g} \cap \operatorname{reg}\exp) &= \operatorname{Reg} G \cap \exp\mathfrak{g}. \end{aligned} \tag{3}$$

If G is connected, then for every regular g the set $\mathfrak{g}^1(\operatorname{Ad} g)$ is a Cartan algebra, and every Cartan algebra is so obtained (see [3], Ch. VII, §4, n$^{\text{o}}$ 4, Prop.8). In particular, every regular element g is contained in $N\big(\mathfrak{g}^1(\operatorname{Ad}(g))\big)$. By 1.2(5) we have in fact $g \in C\big(\mathfrak{g}^1(\operatorname{Ad}(g))\big)$. Thus in an arbitrary Lie group G, if $g \in G_0 \cap \operatorname{Reg} G$, then $\mathfrak{h} = \mathfrak{g}^1\big(\operatorname{Ad}(g)\big)$ is a Cartan algebra and by the preceding we conclude that $g \in C(\mathfrak{h})$. As a consequence, we record

Proposition 1.5. *If G is a Lie group then*

$$G_0 \cap \operatorname{Reg}(G) \subseteq \bigcup \mathcal{H}(G). \qquad ∎$$

Proposition 1.6. (i) *Let G a Lie group and $\mathfrak{h}$ a Cartan algebra of $\mathfrak{g}$. Then arbitrarily close to any point $g \in C(\mathfrak{h})$ there is an element $g' \in C(\mathfrak{h})$ such that (with the nilradical $\mathfrak{n}$ of $\mathfrak{g}$)*

$$\mathfrak{h} = \mathfrak{g}^1(\operatorname{Ad}(g')) + (\mathfrak{h} \cap \mathfrak{n}).$$

(ii) *If $g \in G_0$, then arbitrarily close to g there are regular points $g' \in C(\mathfrak{h})$ such that*

$$\mathfrak{h} = \mathfrak{g}^1\big(\operatorname{Ad}(g')\big).$$

In particular, $\operatorname{Reg}(G) \cap G_0 \cap C(\mathfrak{h})$ *is open and dense in* $G_0 \cap C(\mathfrak{h})$.

Proof. (i) Let $g \in C(\mathfrak{h})$. Recall that $\operatorname{Ad}(g)e^{\operatorname{ad} X} = \operatorname{Ad}(g \exp X)$. By Lemma 0.9 there are arbitrarily small elements X in $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g}^1(\operatorname{Ad}(g)e^{\operatorname{ad} X}) + (\Lambda^\perp \cap \mathfrak{h}) = \mathfrak{g}^1\big(\operatorname{Ad}(g\exp(X))\big) + (\mathfrak{h} \cap \mathfrak{n})$ with the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ (see also 0.5(5)). Thus $\mathfrak{h} = \mathfrak{g}^1(\operatorname{Ad}(g')) + (\mathfrak{h} \cap \mathfrak{n})$ with $g' = g\exp X \in C(\mathfrak{h})$ as close to g as we wish. In particular, since G_0 is open in G, if $g \in G_0$ then we may also take $g' \in G_0$. Let us simplify notation by setting $g = g'$ and assume that

$$\mathfrak{h} = \mathfrak{g}^1(\operatorname{Ad}(g)) + (\mathfrak{h} \cap \mathfrak{n}). \tag{4}$$

By 1.4, given an identity neighborhood W in G, there is a neighorhood V of g such that for $v \in V$ there is a $w \in W$ such that $\mathfrak{g}^1(wgw^{-1}) \supseteq \mathfrak{g}^1(v)$. Thus (4) implies

$$\operatorname{Ad}(w)\mathfrak{h} = \mathfrak{g}^1(wgw^{-1}) + (\operatorname{Ad}(w)(\mathfrak{h}) \cap \mathfrak{n}) \supseteq \mathfrak{g}^1(v) + (\operatorname{Ad}(w)(\mathfrak{h}) \cap \mathfrak{n})$$

In particular, $\dim \mathfrak{g}^1(v) \leq \operatorname{rank}\mathfrak{g}$.

(ii) If $g \in G_0$, then $\dim \mathfrak{g}^1(\operatorname{Ad}(v)) \geq \operatorname{rank}\mathfrak{g}$ (see [3], Ch. VII, §4, n$^{\text{o}}$ 4, Prop.8(i)). Hence $v \mapsto \dim \mathfrak{g}^1\big(\operatorname{Ad}(v)\big)$ is locally constant around g and thus $g \in \operatorname{Reg}(G)$. We have seen that $\mathfrak{h} = \mathfrak{g}^1\big(\operatorname{Ad}(g)\big)$. ∎

Example E1. (i) Let $G=\mathrm{SO}(3)$ and $\mathfrak{g}=\mathfrak{so}(3)=\operatorname{span}\{e_j : j\in\mathbb{Z}/3\mathbb{Z}\}$ with $[e_j,e_{j+1}]=e_{j+2}$.

$$e_0=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\quad e_1=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},\quad e_2=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}.$$

We consider $\mathfrak{h}=\mathbb{R}{\cdot}e_0$. Then

$$C(\mathfrak{h})=\exp\mathfrak{h}=\left\{\begin{pmatrix}\cos t&\sin t&0\\-\sin t&\cos t&0\\0&0&1\end{pmatrix} : t\in\mathbb{R}\right\}.$$

$$N(\mathfrak{h})=C(\mathfrak{h})\cup C(\mathfrak{h})\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}=C(\mathfrak{h})\cup C(\mathfrak{h})\exp(\pi{\cdot}e_2).$$

Here $N(\mathfrak{h})\subseteq\operatorname{Reg}(G)=G\setminus\{\mathbf{1}\}$. Note $\mathcal{W}(G,\mathfrak{h})\cong\mathbb{Z}(2)$. Also, if $g=\exp(\pi{\cdot}e_2)$, then $\mathfrak{g}^1\big(\operatorname{Ad}(g)\big)=\mathbb{R}{\cdot}e_2$ and $\mathfrak{h}\cap\mathfrak{g}^1\big(\operatorname{Ad}(g)\big)=\{0\}$.

(ii) Let $G=\mathrm{Sl}(2,\mathbb{R})$ and $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$, spanned by

$$h=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad p=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad q=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Then $[h,p]=2p$, $[h,q]=-2q$, $[p,q]=h$. Set $u=p-q=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ and $h'=p+q$. Then $[\frac{1}{2}{\cdot}u,h]=h'$, $[\frac{1}{2}{\cdot}u,h']=-h$. We take $\mathfrak{h}=\mathbb{R}{\cdot}h$. Then

$$C(\mathfrak{h})=\exp\mathfrak{h}\cup-\exp\mathfrak{h}=\left\{\pm\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix} : t\in\mathbb{R}\right\}.$$

$$N(\mathfrak{h})=C(\mathfrak{h})\cup C(\mathfrak{h})\exp\frac{\pi}{2}{\cdot}u.$$

We have $N(\mathfrak{h})\setminus\{\mathbf{1}\}\subseteq\operatorname{Reg}G\cap\exp\mathfrak{g}$ and $\mathcal{W}(G,\mathfrak{h})\cong\mathbb{Z}(2)$. Also, if $g=\exp(\frac{\pi}{2}{\cdot}u)$, then $\mathfrak{g}^1\big(\operatorname{Ad}(g)\big)=\mathbb{R}{\cdot}u$ and $\mathfrak{h}\cap\mathfrak{g}^1\big(\operatorname{Ad}(g)\big)=\{0\}$. ∎

We construct an example which illustrates that the situation we consider is rather general in the absence of connectivity.

Example E2. Let $\mathfrak{h}$ be a nilpotent Lie algebra and $\psi:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{v})$ a representation over $\mathbb{C}$. Then we construct on $\mathfrak{g}\overset{\text{def}}{=}\mathfrak{h}\oplus\mathfrak{v}$ a bracket

$$[(X_1,Y_1),(X_2,Y_2)]=\big([X_1,X_2],\psi(X_1)(Y_2)-\psi(X_2)(Y_1)\big),$$

making $\mathfrak{g}$ into a solvable Lie algebra with the abelian ideal $\mathfrak{v}$.

Let us assume that $\psi(\mathfrak{h})(Y)=\{0\}$ implies $Y=0$. Then $\mathfrak{h}$ is a Cartan subalgebra. We write

$$X{\cdot}Y=\psi(X)(Y),\quad X\in\mathfrak{h},\,Y\in\mathfrak{v}.$$

Then for $X \in \mathfrak{h}$ we have $(\operatorname{ad} X)|\mathfrak{v} = (Y \mapsto X{\cdot}Y)$. In particular, if we set $X_\psi(Y) = X{\cdot}Y = \psi(X)(Y)$, then $\lambda: \mathfrak{h} \to \mathbb{C}$ is a root iff the nilspace $\mathfrak{v}^\lambda$ of $X_\psi - \lambda(X){\cdot}\mathbf{1}$ is nonzero, i.e., if λ is a weight of ψ.

Now any automorphism in $\operatorname{Aut}(\mathfrak{g},\mathfrak{h})$ is of the form $\alpha\times\beta$ with $\alpha \in \operatorname{Aut}(\mathfrak{h})$ and $\beta \in \operatorname{Gl}(\mathfrak{v})$ such that $\psi\big(\alpha(X)\big)\big(\beta(Y)\big) = \beta(\psi(X)Y)$, i.e.,

$$\begin{aligned}\operatorname{Aut}(\mathfrak{g},\mathfrak{h}) &= \{(\alpha,\beta) \in \operatorname{Aut}(\mathfrak{h}) \times \operatorname{Gl}(\mathfrak{v}) : \psi \circ (\alpha \oplus \beta) = \beta \circ \psi\} \\ &= \{(\alpha,\beta) \in \operatorname{Aut}(\mathfrak{h}) \times \operatorname{Gl}(\mathfrak{v}) : (\forall (X,Y) \in \mathfrak{h} \oplus \mathfrak{v})\, \alpha(X){\cdot}\beta(Y) = \beta(X{\cdot}Y)\}\end{aligned}.$$

We suppose now that an arbitrary nilpotent Lie algebra $\mathfrak{h}$ is given and that Λ is any finite set of nonzero vectors in $(\mathfrak{h}')^\perp \subseteq \mathfrak{h}^*$. We set

$$\operatorname{Aut}(\Lambda) = \{\varphi \in \operatorname{Gl}\big(\operatorname{span}(\Lambda)\big) : \varphi(\Lambda) = \Lambda)\}$$

and consider the finite orbit space $J = \Lambda/\operatorname{Aut}(\Lambda)$ and the orbit map $\rho: \Lambda \to J$. For each $j \in J$ we fix an arbitrary nilpotent representation $\pi_j: \mathfrak{h} \to \mathfrak{gl}(\mathfrak{v}_j)$ of $\mathfrak{h}$ (e.g., the zero representation!) and define $\mathfrak{v} = \bigoplus_{\lambda\in\Lambda} \mathfrak{v}_{\rho(\lambda)}$. We set

$$\psi(X)(Y) = X{\cdot}\left(\bigoplus_{\lambda\in\Lambda} Y_\lambda\right) = \left(\bigoplus_{\lambda\in\Lambda} (\lambda(X){\cdot}\mathbf{1} + \pi_{\rho(\lambda)}(X))(Y_\lambda)\right).$$

Let $\alpha \in \operatorname{Aut}(\mathfrak{h})$ with $\alpha^*|\Lambda \in \operatorname{Aut}(\Lambda)$. Notice that for an abelian $\mathfrak{h}$, every element of $\operatorname{Aut}(\Lambda)$ is so obtained by the definition of $\operatorname{Aut}(\Lambda)$, and if $\operatorname{span}(\Lambda) = \mathfrak{h}^*$, then this representation is unique. Now we define $\beta: \mathfrak{v} \to \mathfrak{v}$ by

$$\beta\left(\bigoplus_{\lambda\in\Lambda} Y_\lambda\right) = \bigoplus_{\lambda\in\Lambda} Y_{\alpha^*(\lambda)}.$$

Now

$$\alpha(X){\cdot}\beta(Y) = \bigoplus_{\lambda\in\Lambda} \big(\lambda\big(\alpha(X)\big){\cdot}\mathbf{1} + \pi_{\rho(\lambda)}(X)\big)(Y_{\alpha^*(\lambda)})$$

and

$$\beta(X{\cdot}Y) = \bigoplus_{\lambda\in\Lambda} \big((\alpha^*(\lambda)(X){\cdot}\mathbf{1} + \pi_{\rho(\alpha^*(\lambda))}(X)\big)(Y_{\alpha^*(\lambda)}).$$

We see that $\alpha \oplus \beta \in \operatorname{Aut}(\mathfrak{g},\mathfrak{h})$. In particular, if $\mathfrak{h}$ is abelian, then

$$B\mathcal{W}(\mathfrak{g},\mathfrak{h}) = \operatorname{Aut}(\Lambda).$$

If Λ is any finite generating set of a real Hilbert space $\mathfrak{h}$ with scalar product $(\cdot|\cdot)$, we can take $\mathfrak{h}$ as an abelian Lie algebra with $\mathfrak{h}^* = \mathfrak{h}$ (writing $\lambda(X) = (\lambda|X)$) and for π_j the one dimensional zero representation. Then we see that

every finite group Γ which is isomorphic to a group $\operatorname{Aut}(\Lambda)$ for a finite set Λ of vectors spanning a real vector space, can occur as a big Weyl group of a Lie algebra $\mathfrak{g}$ with respect to some Cartan subgroup $\mathfrak{h}$.

If G_0 is the simply connected Lie group with Lie algebra $\mathfrak{g}$, then we may construct the semidirect product $G_0 \rtimes_\gamma \mathrm{Aut}(\Lambda)$ with the action γ induced by the action of $\mathrm{Aut}(\Lambda)$ on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$. Thus $\mathrm{Aut}(\Lambda)$ is realized as the Weyl group of a Lie group G with respect to a Cartan algebra $\mathfrak{h}$ of $\mathfrak{g}$.

Every family $(\beta_\lambda)_{\lambda\in\Lambda}$ of intertwining operators $\beta_\lambda\colon \mathfrak{v}_{\rho(\lambda)} \to \mathfrak{v}_{\rho(\lambda)}$ for the representation $\pi_{\rho(\lambda)}$ yields an automorphism

$$(X,Y) \mapsto (X, \bigoplus_{\lambda\in\Lambda} \beta_\lambda(Y_\lambda)) \quad \text{in} \quad \mathrm{CAut}(\mathfrak{g},\mathfrak{h}).$$

In particular, if $\mathfrak{h} = \mathbb{C} = \mathfrak{h}^*$, $\Lambda = \{1\}$, $\pi\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{v})$, $\pi(z)(v) = z{\cdot}v$, then $\mathfrak{g}$ is almost abelian, $\mathrm{Aut}(\Lambda) = \{1\}$, and

$$\mathrm{CAut}(\mathfrak{g},\mathfrak{h}) = \{\mathbf{1}\} \times \mathrm{Gl}(\mathfrak{v}) \cong \mathrm{Gl}(\mathfrak{v}).$$

For the definitive information on this issue see NEEB [9].

The space of closed subgroups. In [6] we considered for a real Lie group G with Lie algebra $\mathfrak{g}$ the compact Hausdorff spaces $\Sigma(G)$ of all closed subgroups of G and $\Sigma(\mathfrak{g})$ of all additive closed subgroups of $\mathfrak{g}$. We note $\mathcal{H}(\mathfrak{g}) \subseteq \Sigma(\mathfrak{g})$ and $\mathcal{H}(G) \subseteq \Sigma(G)$.

Definition 1.8. A subalgebra $\mathfrak{h}$ of a real or complex Lie algebra $\mathfrak{g}$ is said to be *a near-Cartan algebra* if $\mathfrak{h} \in \overline{\mathcal{H}(\mathfrak{g})}$. A subgroup H of a real or complex Lie group is called a *near-Cartan group* of G if $H \in \overline{\mathcal{H}(G)}$. ∎

Proposition 1.9. *Let $\mathfrak{g}$ be a a real or complex Lie algebra and $\mathfrak{h}$ a near-Cartan subalgebra. Then*

(i) $\dim \mathfrak{h} = \mathrm{rank}(\mathfrak{g})$

(ii) $\mathfrak{h}$ *is nilpotent and its class of nilpotency is dominated by that of the Cartan algebras of* $\mathfrak{g}$.

Proof. (i) The subspace in $\Sigma(\mathfrak{g})$ of all vector spaces of dimension $\dim(\mathrm{rank}\,\mathfrak{g})$ is a compact manifold containing $\mathcal{H}(\mathfrak{g})$. Then $\overline{\mathcal{H}(\mathfrak{g})}$ is contained in this manifold.

(ii) It was proved in [6], Prop. 24 that the space of closed nilpotent subgroups of class $\leq k$ in $\Sigma(G)$ is closed for all k. ∎

Proposition 1.10. *For each real or complex Lie algebra $\mathfrak{g}$ and each real or complex Lie group G we have*

$$\mathfrak{g} = \bigcup \overline{\mathcal{H}(\mathfrak{g})},$$
$$G = \bigcup \overline{\mathcal{H}(G)}.$$

Proof. Let $X \in \mathfrak{g}$. There is a sequence $X_n \in \mathrm{reg}\,\mathfrak{g}$ with $X = \lim X_n$. The sequence $\mathfrak{g}^0(\mathrm{ad}\,X_n) \in \mathcal{H}(\mathfrak{g})$ has a subsequence in the compact space $\overline{\mathcal{H}}(\mathfrak{g})$ converging to an element $\mathfrak{h}$. Then $X \in \mathfrak{h}$ by the definition of the topology of $\Sigma(\mathfrak{g})$. The proof of the second part is similar. ∎

Proposition 1.11. *Each near-Cartan subgroup H of a Lie group G contains the center Z of G.*

Proof. By definition, $H = \lim H_n$ in $\overline{\mathcal{H}(G)}$ with a sequence of Cartan subgroups H_n. Now $Z \subseteq C(\mathfrak{h}_n) = H_n$ for all $n \in \mathbb{N}$. The assertion $Z \subseteq \lim H_n = H$ follows. ∎

It is very instructive to consider, in this context, the examples of the universal covering group of $\mathrm{Sl}(2, \mathbb{R})$ and that of the group of motions of the euclidean plane. (For the latter example, see e.g. [6], Example 23.)

Lemma 1.12. *Let $f: G \to G/Z$ the quotient homomorphism of a locally compact group G modulo a closed normal subgroup Z. Let Σ_Z denote the subspace in $\Sigma(G)$ of all closed subgroups H of G with $Z \subseteq H$. Then the function*

$$H \mapsto H/Z : \Sigma_Z(G) \to \Sigma(G/Z)$$

is a homeomorphism.

Proof. The function is clearly a bijection, and domain and range are compact Hausdorff spaces. It therefore suffices to establish its continuity. Let A_n be a net of closed subgroups of G converging to A. This means that for every compact subspace C of G and every identity neighborhood U in G eventually we have $A_n \cap C \subseteq A \cap CU$ and $A \cap C \subseteq A_n \cap CU$ (see [1], Chap. VIII, §5, n^o 6, p. 188). Now let K be a compact subspace of G/Z and V an identity neighborhood in G/Z. Then there is a compact subspace C of G such that $K = f(C)$. We claim that $f(A_n) \cap K = f(A_n) \cap f(C) = f(A_n \cap C)$: indeed the last term is contained in the preceding, and if $f(c) = f(a_n)$ with $c \in C$ and $a_n \in A_n$ then there is a $z \in Z$ with $c = a_n z$. But $Z \subseteq H$ implies $a_n z \in A_n$ and so $c \in A_n \cap C$. Let $U = f^{-1}(V)$. Then by the same argument, $f(A) \cap KV = f(A \cap CU)$. Hence $f(A_n) \cap K \subseteq f(A) \cap KV$ iff $f(A_n \cap C) \subseteq f(A \cap CU)$ and this is eventually the case by hypothesis on A_n. In the same vein we see that eventually $f(A) \cap K \subseteq f(A_n) \cap KV$. Thus $\lim f(A_n) = f(A)$ which we had to show. ∎

If $\mathfrak{g}$ is a Lie algebra, then the group $\mathrm{int}(\mathfrak{g}) = \langle e^{\mathrm{ad}\,\mathfrak{g}} \rangle$ of inner automorphisms is an analytic subgroup of $\mathrm{Aut}(\mathfrak{g})$ whose Lie algebra is $\mathrm{ad}\,\mathfrak{g} \subseteq \mathrm{Der}(\mathfrak{g})$. If one endows $\mathrm{int}(\mathfrak{g})$ with its intrinsic Lie group structure then it is isomorphic to G/Z where G is any of the connected Lie groups with $L(G) \cong \mathfrak{g}$ and Z is its center. In a sense, therefore, the following proposition says that the Cartan subgroups of a connected Lie group are determined by its Lie algebra $\mathfrak{g}$ alone.

Proposition 1.13. *The quotient homomorphism $f: G \to G/Z$ maps*

(i) *$\mathcal{H}(G)$ bijectively onto $\mathcal{H}(G/Z)$, and*
(ii) *$\overline{\mathcal{H}(G)}$ bijectively onto $\overline{\mathcal{H}(G/Z)}$.*

Proof. Every Cartan algebra $\mathfrak{h}$ in $\mathfrak{g}$ contains the center $\mathfrak{z}$ and $\mathfrak{h}/\mathfrak{z}$ is a Cartan algebra of $\mathfrak{g}/\mathfrak{z}$. Also, every Cartan algebra of $\mathfrak{g}/\mathfrak{z}$ is of this form (see [3], Chap. VII, §2, n^o 2, p. 21). Let $H = C(\mathfrak{h})$ be a Cartan group of G. This means that $h \in H$ implies $\mathrm{Ad}(h) \in \mathrm{CAut}(\mathfrak{g}, \mathfrak{h})$. Then $\mathrm{Ad}\big(f(h)\big) \in \mathrm{CAut}(\mathfrak{g}/\mathfrak{z}, \mathfrak{h}/\mathfrak{z})$ as is readily verified. Conversely, if $g \in G$ is such that $\mathrm{Ad}\big(f(g)\big) \in \mathrm{CAut}(\mathfrak{g}/\mathfrak{z}, \mathfrak{g}/\mathfrak{z})$

then also $\mathrm{Ad}(g) \in \mathrm{CAut}(\mathfrak{g}, \mathfrak{h})$, i.e., $g \in C(\mathfrak{h}) = H$. This proves the first part of the assertion concerning (i).

Now we turn to part (ii). The quotient map $f: G \to G/Z$ induces a homeomorphism $\varphi: \Sigma_Z(G) \to \Sigma(G/Z)$, $\varphi(H) = H/Z$ by Lemma 1.12. Now $\overline{\mathcal{H}(G)} \subseteq \Sigma_Z$ by Proposition 1.11. By the first part of the proof, φ maps $\overline{\mathcal{H}(G)}$ bijectively onto $\mathcal{H}(G/Z)$. Then φ maps $\overline{\mathcal{H}(G)}$ homeomorphically onto $\overline{\mathcal{H}(G/Z)}$. ∎

2. More on near-Cartan groups

Definition 2.1. Let $\mathfrak{g}$ denote a real Lie algebra. A subalgebra $\mathfrak{m}$ is said to be a *subalgebra of maximal rank* if every Cartan subalgebra of $\mathfrak{m}$ is a Cartan algebra of $\mathfrak{g}$. The set of maximal rank subalgebras of $\mathfrak{g}$ is denoted by $\mathcal{M}(\mathfrak{g}) \subseteq \Sigma(\mathfrak{g})$. A closed subgroup M of a Lie group with Lie algebra $\mathfrak{g}$ is called *a subgroup of maximal rank* if $\mathfrak{m} = L(M)$ is a maximal rank subalgebra of $\mathfrak{g}$. The set of maximal rank subgroups of G is denoted by $\mathcal{M}(G) \subseteq \Sigma(G)$. ∎

Every Cartan algebra is a maximal rank subalgebra of $\mathfrak{g}$ and every Cartan subgroup of G is a maximal rank subgroup.

Lemma 2.2. *Let B be any open neighborhood of 0 in the Lie algebra $\mathfrak{g}$ of a Lie group G such that $\exp|B: B \to U$ is a diffeomorphism onto an open identity neighborhood of G. Let M be a maximal rank subgroup of G. Let $S \overset{\mathrm{def}}{=} (\exp|B)^{-1}(U \cap M)$. Then $S = \mathfrak{m} \cap B$.*

Proof. (i) If $X \in \mathfrak{m} \cap B$, then $\exp X \in M \cap U$ and thus $X \in S$ by the definition of S. Thus $\mathfrak{m} \cap B \subseteq S$.

(ii) In order to show that $S \subseteq B \cap \mathfrak{m}$ we must show that $S \subseteq \mathfrak{m}$. Since M is closed then S is closed in B. It therefore suffices to find a dense subset $T \subseteq S$ with $T \subseteq \mathfrak{m}$. We let $T = (\exp|B)^{-1}(\mathrm{Reg}\, M \cap U)$. Since $\mathrm{Reg}(M)$ is dense in M then T is dense in S. Now let $X \in T$. We must show $X \in \mathfrak{m}$. Now $m = \exp X \in \mathrm{Reg}(M)$ and $\mathfrak{m}^1(\mathrm{Ad}(m)|\mathfrak{m})$ is a Cartan algebra in $\mathfrak{m}$. Since M is a maximal rank subgroup, it is a Cartan algebra of $\mathfrak{g}$. Hence $\mathfrak{g}^1\big(\mathrm{Ad}(m)\big) \supseteq \mathfrak{m}^1(\mathrm{Ad}(m)|\mathfrak{m})$ is a Cartan algebra of $\mathfrak{g}$, equality holds, and m is regular in G. Now $\mathrm{Ad}(m) = \mathrm{Ad}(\exp X) = e^{\mathrm{ad}\, X}$. It follows that $(\mathrm{Ad}(m) - 1)(X) = (\sum_{j=1}^{\infty} \frac{1}{j!}(\mathrm{ad}\, X)^j)X = 0$. Thus $X \in \mathfrak{g}^1(\mathrm{Ad}(m)) = \mathfrak{g}^1(\mathrm{Ad}(m)|\mathfrak{m}) \subseteq \mathfrak{m}$. ∎

Proposition 2.3. *Let $H = \lim H_n$ in $\overline{\mathcal{M}(G)}$ and suppose that $H_n \in \mathcal{M}(G)$. Then $L(H) = \lim L(H_n)$.*

Proof. Let C denote an open, relatively compact convex symmetric Campbell-Hausdorff-neighborhood in $\mathfrak{g}$ contained in a neighborhood B as is Lemma 2.2. Then $S_n \overset{\mathrm{def}}{=} (\exp|C)^{-1}(H_n \cap \exp C)$ converges to $S \overset{\mathrm{def}}{=} (\exp|C)^{-1}(H \cap \exp C)$ in the space of closed subsets of C. If $H_n \in \mathcal{M}(G)$, then $S_n = C \cap L(H_n)$ by Lemma 2.2. Now $\dim H_n = \dim L(H_n)$ and $\dim H = \dim S$. If $(n(j))_{j \in \mathbb{N}}$ is any subsequence such that $L(H_{N(j)})$ converges to E in the compact space $\Sigma_L(\mathfrak{g})$, then $C \cap L(H_{n(j)})$ converges to $C \cap E$ on one hand and to S on the other. Hence $S = C \cap E$. Thus $E = \mathbb{R}\cdot(C \cap E)$ does not depend on the choice of the subsequence and $E = \lim L(H_n)$. Now E is a vector subspace of $\mathfrak{g}$ such that $C \cap E = (\exp|C)^{-1}(H \cap \exp C)$. If we recall that for a closed subgroup H of G we have $L(H) = \{X \in \mathfrak{g} : \exp \mathbb{R}\cdot X \subseteq H\}$ we may conclude $E = L(H)$. ■

We set $\Sigma_m(G) = \{H \in \Sigma(G) : \dim H = m\}$.

Corollary 2.4. $\overline{\mathcal{M}(G) \cap \Sigma_m(G)} \subset \Sigma_m(G)$.

Proof. Consequence of Proposition 2.3. ■

Now we apply this with $m = \operatorname{rank} G$. Since $\mathcal{H}(G) \subseteq \mathcal{M}(G) \cap \Sigma_{\operatorname{rank} G}$, Corollary 2.4 yields at once

Corollary 2.5. *Every near-Cartan group has dimension* $\operatorname{rank}(G)$. ■

More precisely:

Theorem 2.6. *The Lie algebra* $\mathfrak{h} = L(H)$ *of a near-Cartan group* H *is a near-Cartan algebra. The analytic subgroup* $H = \exp \mathfrak{h}$ *generated by a near-Cartan algebra* $\mathfrak{h}$ *is the identity component of a near-Cartan group.*

Proof. (i) Let $H \in \overline{\mathcal{H}(G)}$. Then there is a sequence H_n of Cartan groups with $H = \lim H_n$. Then $L(H) = \lim L(H_n)$ by Proposition 2.3. Since $L(H_n) \in \mathcal{H}(\mathfrak{g})$ we conclude $L(H) \in \overline{\mathcal{H}(\mathfrak{g})}$.

(ii) Let $\mathfrak{h} \in \overline{\mathcal{H}(\mathfrak{g})}$. Then there is a sequence $\mathfrak{h}_n$ of Cartan algebras with $\mathfrak{h} = \lim \mathfrak{h}_n$. The analytic groups $H_n \stackrel{\text{def}}{=} \exp \mathfrak{h}_n$ are closed (since $\mathfrak{h}_n$ is an ideal in $L(\overline{\exp \mathfrak{h}_n})$ and a Cartan algebra is its own normalizer!). Each H_n is the identity component of the Cartan group $C(\mathfrak{h}_n)$ which is a maximal rank group. Because of the compactness of $\overline{\mathcal{H}(G)}$ there is a sequence $(n(j))_{j \in \mathbb{N}}$ such that $H^* = \lim C(\mathfrak{h}_{n(j)})$ exists. By definition, H^* is a near-Cartan group. The relation $L(H^*) = \lim L\big(C(\mathfrak{h}_{n(j)})\big) = \lim \mathfrak{h}_{n(j)} = \mathfrak{h}$ follows from Proposition 2.3. Then $H \stackrel{\text{def}}{=} (H^*)_0 = \exp \mathfrak{h}$ follows. ■

We have seen in [6] that there may be sequences $\mathfrak{h}_n$ of Cartan algebras such that $(H, \mathfrak{h}) = \lim(\exp \mathfrak{h}_n, \mathfrak{h}_n)$ exists in $\overline{\mathcal{H}(G)} \times \overline{\mathcal{H}(\mathfrak{g})}$ such that H is not connected. In that example all Cartan groups were connected, but we have disconnected near-Cartan groups. In constrast with the situation of a Cartan group which is uniquely determined by its identity component, different near-Cartan groups may have the same identity component. This is the case in $\mathrm{Sl}(2, \mathbb{R})$. The algebra $\mathfrak{h} = \mathbb{R}\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the Lie algebra of two different near-Cartan groups $H_1 = \exp \mathfrak{h} = \lim \exp \mathbb{R}\cdot\begin{pmatrix} 0 & 1 \\ \frac{1}{n} & 0 \end{pmatrix}$ and $H_2 = \{\mathbf{1}, -\mathbf{1}\}H_1 = \lim \exp \mathbb{R}\cdot\begin{pmatrix} 0 & 1 \\ -\frac{1}{n} & 0 \end{pmatrix}$.

Further remarks. The function $L\colon \Sigma(G) \to \Sigma_L(\mathfrak{g})$ from the space of closed subgroups into the space of Lie subalgebras of $\mathfrak{g}$ which associates with a closed subgroup K of G its Lie algebra $L(K)$ is not continuous as was observed in [6]. If $G = \mathbb{R}^2/\mathbb{Z}^2$ with $\mathfrak{g} = \mathbb{R}^2$ and the quotient map as exponential function, then $\mathfrak{h}_n = \mathbb{R}\cdot(\frac{1}{n}, 1)$ converges to $\mathfrak{h} = \mathbb{R} \times \{0\}$ in $\Sigma_L(\mathfrak{g})$. However, $H_n = \exp \mathfrak{h}_n$ converges to $G \neq \exp \mathfrak{h}$ in $\Sigma(G)$.

However, in [6] we proved:

Lemma 2.7. *Let G denote a Lie group and set $D = \{(H, \mathfrak{h}) \in \Sigma(G) \times \Sigma_L(\mathfrak{g}) : \mathfrak{h} \subseteq L(H)\}$. Then D is closed in $\Sigma(G) \times \Sigma_L(\mathfrak{g})$.* ∎

Now we set $\mathbb{N}_G = \{0, 1, \ldots, \dim L(G)\}$ and consider the function

$$\dim\colon \Sigma(G) \to \mathbb{N}_G, \quad H \mapsto \dim H = \dim L(H).$$

Lemma 2.8. *The function $\dim\colon \Sigma(G) \to \mathbb{N}_G$ is upper semicontinuous.*

Proof. We claim that the lower graph $\{(H, n) : n \leq \dim L(H), \quad H \in \Sigma(G), n \in \mathbb{N}_G\}$ is closed in $\Sigma(G) \times \mathbb{N}_G$ by Lemma 2.3. Indeed, if $(H, n) = \lim(H_k, n_k)$, then for any subsequence $(k(j))_{j\in\mathbb{N}}$ with $\mathfrak{h} = \lim L(H_{k(j)})$ we have $\mathfrak{h} \subseteq L(H)$, i.e., $\lim \dim L(H_{k(j)}) \leq \dim H$. But $n_{k(j)} \leq \dim L(H_{k(j)})$. Hence $n \leq \lim \dim L(H_{k(j)}) \leq \dim H$. ∎

Lemma 2.9. *Suppose $H = \lim H_n$ in $\Sigma(G)$. If $\dim H = \lim \dim H_n$, then $L(H) = \lim L(H_n)$.*

Proof. This is Corollary 22 in [6]. ∎

Proposition 2.10. (i) *The restrictions*

$$L\colon \Sigma_m(G) \to \Sigma_L(\mathfrak{g}) \quad \textit{and} \quad \dim\colon \Sigma_m(G) \to \mathbb{N}_G$$

are continuous.

(ii) *The function*

$$C\colon \mathcal{H}(\mathfrak{g}) \to \mathcal{H}(G)$$

is continuous.

Proof. (i) follows from Lemma 2.9 and 2.8.

(ii) If $\mathfrak{h} = \lim \mathfrak{h}_n$ in $\mathcal{H}(\mathfrak{g})$, then from [6] we know that there is a sequence $Y_n \to 0$ such that $\mathfrak{h}_n = e^{\operatorname{ad} Y_n}\mathfrak{h}$. If $a_n = \exp Y_n$ then $a_n \to \mathbf{1}$ and $C(\mathfrak{h}_n) = C(\operatorname{Ad}(a_n)\mathfrak{h}) = a_n C(\mathfrak{h}) a_n^{-1}$ by 1.2(ii). Thus $C(\mathfrak{h}) = \lim C(\mathfrak{h}_n)$ as asserted. ∎

Proposition 2.11. (i) *Let $\mathfrak{h}$ be a near-Cartan algebra. Then the following are equivalent*

(1) *$\mathfrak{h}$ is a Cartan algebra*

(2) $\mathfrak{h} \cap \operatorname{reg} \mathfrak{g} \neq \emptyset$.

(3) $\overline{\mathfrak{h} \cap \operatorname{reg} \mathfrak{g}} = \mathfrak{h}$.

(ii) *Suppose that G is connected and let H be a near-Cartan group. Then the following are equivalent:*

(1) *H is a Cartan group.*

(2) $H \cap \mathrm{Reg}(G) \neq \emptyset$.

(3) $H \subseteq \overline{H \cap \mathrm{Reg}\, G}$.

Proof. (i) If $\mathfrak{h}$ is a Cartan algebra, then the set of regular elements of $\mathfrak{g}$ is dense in $\mathfrak{h}$. So (1)⇒(3). Next (3)⇒(2) is trivial. Now we show (2)⇒(1): For any $X \in \mathfrak{h}$ we have $\mathfrak{h} \subseteq \mathfrak{g}^0(\mathrm{ad}\, X)$ since $\mathfrak{h}$ is nilpotent by 1.9(ii). If now X is regular, then $\dim \mathfrak{g}^1(\mathrm{ad}\, X) = \mathrm{rank}\, \mathfrak{g} = \dim \mathfrak{h}$ in view of 1.9(i). Thus $\mathfrak{h} = \mathfrak{g}^1(\mathrm{ad}\, X)$ follows and so $\mathfrak{h}$ is a Cartan algebra.

(ii) If H is a Cartan group, then Proposition 1.6 proves (1)⇒(3). But (3)⇒(2) is trivial. Next we show that (2)⇒(1). By the definition of a near-Cartan group we have $H = \lim H_n$ with $H_n = C(\mathfrak{h}_n)$. If $g \in H$ then $g = \lim g_n$ with $g_n \in C(\mathfrak{h}_n)$. We may assume that g_n is regular by 1.6. Then $\mathfrak{g}^1\big(\mathrm{Ad}(g_n)\big) = \mathfrak{h}_n$. By 1.3 there is a sequence $w_j \to \mathbf{1}$ such that we have $\mathfrak{g}^1\big(\mathrm{Ad}(g_{n_j})\big) \subseteq \mathfrak{g}^1\big(\mathrm{Ad}(w_j g w_j^{-1})\big)$ for a suitable sequence n_j of natural numbers going to infinity. Since the g_n are regular, $\mathfrak{g}^1\big(\mathrm{Ad}(g_{n_j})\big) = \mathfrak{h}_{n_j}$.

Now suppose that g is regular. Then $\dim \mathfrak{g}^1\big(\mathrm{Ad}(w_j g w_j^{-1})\big) = \mathrm{rank}\, \mathfrak{g} = \dim \mathfrak{h}_{n_j}$, and thus we have $\mathfrak{h}_{n_j} = \mathfrak{g}^1\big(\mathrm{Ad}(w_j g w_j^{-1})\big) = w_j \mathfrak{g}^1\big(\mathrm{Ad}(g)\big) w_j^{-1}$. Now $\mathfrak{h} = \lim \mathfrak{h}_{n_j}$ on the one hand and $\mathfrak{g}^1\big(\mathrm{Ad}(g)\big) = \lim w_j \mathfrak{g}^1\big(\mathrm{Ad}(g)\big) w_j^{-1}$ on the other. Thus $\mathfrak{h} = \mathfrak{g}^1(\mathrm{Ad}(g))$ is a Cartan algebra and $g \in C(\mathfrak{h})$ by 1.2(5). But now $C(\mathfrak{h}) = \lim C(\mathfrak{h}_n) = H$ by 2.10(ii). In particular, H is a Cartan group. ∎

3. The exponential function and near-Cartan groups

Let $\exp\colon \mathfrak{g} \to G$ denote the exponential function of a Lie group. For $X \in \mathfrak{g}$ we choose the abbreviation

$$\Omega(X) = \mathrm{Spec}\, \mathrm{ad}\, X \cap 2\pi i \mathbb{N} \subseteq \mathbb{C}.$$

We note that for $X \in \mathfrak{g}$ we have

$$\ker d \exp X = \bigoplus_{\lambda \in \Omega(X)} \ker\big((\mathrm{ad}\, X)^2 + |\lambda|^2\big).$$

(See e.g. [5,6].) Then the set in which the exponential function is singular is

$$S(\mathfrak{g}) = \mathfrak{g} \setminus \mathrm{reg}\exp = \{X \in \mathfrak{g} : \Omega(X) \neq \{0\}\}.$$

This set is invariant under $\mathrm{Aut}(\mathfrak{g})$. As an abbreviation we write $\mathrm{sing}(\mathfrak{g}) = \mathfrak{g} \setminus \mathrm{reg}\, \mathfrak{g}$.

Lemma 3.1. *Let H be a near-Cartan group in a Lie group and assume $X \in$* $\mathrm{reg}\exp$, *$\exp X \in H$. Then $X \in \mathfrak{h}$.*

Proof. By the definition of a near-Cartan group we find a sequence of Cartan groups H_n converging to H in $\overline{\mathcal{H}(G)}$. Since $H_n \cap \mathrm{Reg}\, G$ is dense in $G_0 \cap H_n$ by 2.11, and by the definition of the topology in $\Sigma(G)$, we find a sequence of regular elements $g_n \in G_0 \cap H_n$ converging to $\exp X$. Since the exponential function is regular at X, there is an open neighborhood U of X in $\mathfrak{g}$ and an open neighborhood V of $\exp X$ in G such that the restriction $\exp|U\colon U \to G$ corestricts to a diffeomorphism $\varepsilon\colon U \to V$. We may assume that $g_n \in V$. Set $X_n = \varepsilon^{-1}(g_n)$. Then $\lim X_n = \lim \varepsilon^{-1}(g_n) = \varepsilon^{-1}(\exp X) = \varepsilon^{-1}\big(\varepsilon(X)\big) = X$. Also, $\exp X_n = \varepsilon(X_n) = \varepsilon\big(\varepsilon^{-1}(g_n)\big) = g_n$. Since $g_n \in \operatorname{Reg} G$ we know that $X_n \in \operatorname{reg}\exp \mathfrak{g} \cap \operatorname{reg} \mathfrak{g}$ by 1.4. Then $\mathfrak{g}^0(\operatorname{ad} X_n) = \mathfrak{g}^1(\operatorname{Ad}(g_n)) = \mathfrak{h}_n$ by 1.4. Thus $X_n \in \mathfrak{h}_n$. From 2.10(i) we infer $\mathfrak{h} = \lim \mathfrak{h}_n$. But then $X = \lim X_n$ and $X_n \in \mathfrak{h}_n$ show $X \in \mathfrak{h}$. ∎

It is interesting to observe that, in the preceding proof, the hypothesis "exp *is regular at* X" cannot be replaced by the weaker hypothesis "$\exp \mathfrak{g}$ *is a neighborhood of* $\exp X$". Using this hypothesis we could still find elements $X_n \in \operatorname{reg}\exp \cap \operatorname{reg}\mathfrak{g} \cap \mathfrak{h}_n$ with $\exp X_n = g_n$. But we could not conclude that $X = \lim X_n$. The weaker conclusion that $X' = \lim X_{n(j)}$ for some sequence $n(j) \to \infty$ of natural numbers would suffice for the conclusion that we could find an $X' \in \mathfrak{h}$ with $\exp X = \exp X'$. Thus the modification of the proof of 3.1 which we have just suggested yields the following obervation:

Remark 3.2. Assume that H is a Cartan subgroup and $\exp X \in H$ and assume that there is a bounded subset B in $\mathfrak{g}$ such that $\exp B$ is a neighborhood of $\exp X$. Then there is an $X' \in \mathfrak{h}$ with $\exp X = \exp X'$. ∎

Lemma 3.3. *Let G be a Lie group and H a near-Cartan group with Lie algebra $\mathfrak{h}$. If H is not a Cartan group, then* $\exp^{-1} H \subseteq S(\mathfrak{g}) \cup \operatorname{sing}(\mathfrak{g})$.

Proof. Suppose that $\exp X \in H$. Since H is not a Cartan group, then $\exp X \in H$ cannot be regular by 2.11. Hence $X \notin \operatorname{reg}(\mathfrak{g}) \cap \operatorname{reg}\exp$ by 1.4. ∎

Theorem 3.4. *Let G be a Lie group and H a near-Cartan group with Lie algebra $\mathfrak{h}$. Then*

$$\exp^{-1} H \subseteq \mathfrak{h} \cup S(\mathfrak{g}).$$

If H is not a Cartan subgroup then

$$\exp^{-1} H \subseteq \big(\mathfrak{h} \cup S(\mathfrak{g})\big) \cap \big(\operatorname{sing}(\mathfrak{g}) \cup S(\mathfrak{g})\big) = \big(\mathfrak{h} \cap \operatorname{sing}(\mathfrak{g})\big) \cup S(\mathfrak{g}).$$

Proof. This follows at once from Lemmas 3.1 and 3.3. ∎

References

[1] Bourbaki, N., "Intégration", Chapitres 7 et 8, Hermann, Paris, 1963.

[2] Bourbaki, N., "Groupes et algèbres de Lie", Chapitres 2 et 3, Hermann Paris, 1972.

[3] Bourbaki, N., "Groupes et algèbres de Lie", Chapitres 7 et 8, Hermann Paris, 1975.

[4] Hochschild, G. P., "Basic Theory of Algebraic Groups and Lie Algebras", Springer Verl. New York etc., 1981.

[5] Hofmann, K. H., *A memo on the singularities of the exponential function*, Seminar Notes 1990.

[6] —, *A memo on the exponential function and regular points*, Archiv der Mathematik **92** (1992), 24–37.

[7] Hofmann, K. H., and A. Mukherjea, *On the density of the image of the exponential function*, Math. Ann. **234** (1978), 263–273.

[8] Neeb, K.-H., *A memo on Cartan subgroups and regular elements*, Seminar Report Technische Hochschule Darmstadt 1992.

[9] —, *Weyl groups of disconnected Lie groups*, Seminar Sophus Lie **2** (1992), 153–157.

Fachbereich Mathematik
Technische Hochschule Darmstadt
Schlossgartenstr. 7
W-6101 Darmstadt, Germany
hofmann@mathematik.th-darmstadt.de

Received July 10, 1992
and in final form October 5, 1992