Seminar Sophus Lie
1 (1991) 41–46

A short course on the Lie theory of Semigroups II
Lie Semialgebras

Anselm Eggert

Recall the definition 1.4 of LIE semialgebras. In the present discussion, we will call them simply *semialgebras* for short. We collect some material, which mostly can be found in [4].

Using the Propositions 1.3, 1.5 and 1.7, one observes immediately that every invariant wedge is a semialgebra and that every semialgebra is a LIE wedge. (For a rough orientation the following statement may be helpful: There are "many" LIE wedges that are not semialgebras, but there are only "few" semialgebras that are not invariant.)

There is a classification of invariant wedges due to OLSHANSKIĬ, VINBERG, PANEITZ, KUMARESAN, RANJAN, HILGERT and HOFMANN that can be found in [4]. Furthermore, much in known about Lie algebras that contain pointed invariant wedges with inner points. SPINDLER has given an explicit construction of all these Lie algebras in [6].

Thus, the invariant wedges may serve as a supply of well understood semialgebras. However, not every semialgebra is invariant. There is another class of wellknown semialgebras, namely those that are *special* in the sense of Part I of this "short course". Typically, these are not invariant.

A half space is a semialgebra if and only if its bounding hyperplane $\mathfrak{h}(\mathfrak{w})$ is a subalgebra. Since an arbitrary intersection of semialgebras is again a semialgebra, this gives us again a large supply of semialgebras. For these, a special term is usual:

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra. A semialgebra that is a half space in $\mathfrak{g}$ is called a *half space semialgebra*, an an arbitrary intersection of half space semialgebras is called an *intersection semialgebra* or just *intersection algebra.* The german term is *"Schnitt-Semialgebra"* or *"Schnittalgebra"*.

Typically, intersection semialgebras are far from being invariant. On the other hand, HOFMANN achieved a classification of hyperplane subalgebras in [5], so intersection semialgebras are fully understood, and we may consider them as classified.

One final remark concludes this introduction: Usually, we will restrict our attention to semialgebras that generate the underlying Lie algebra as a vector space. This no loss of generality because of the follwing lemma, which follows easily from Proposition 1.5.

Lemma 2.2. *Let $\mathfrak{w}$ be a semialgebra in $\mathfrak{g}$. Then $\mathfrak{w}-\mathfrak{w}$ ist a subalgebra.* ∎

A wedge $\mathfrak{w}$ in $\mathfrak{g}$ that fulfills $\mathfrak{w} - \mathfrak{w} = \mathfrak{g}$ will be called *generating*, and we note that a wedge is generating if and only if it contains inner points.

Examples

In order to establish a certain intuition about semialgebras, it is appropriate to take a look at some examples. In a Lie algebra of dimension less or equal to 3, the follwing lemma leads easily to a description of the semialgebras contained in it.

Lemma 2.3. *Let* $\mathfrak{g}$ *be a Lie algebra with* $\dim \mathfrak{g} \leq 3$. *Then every generating semialgebra* $\mathfrak{w}$ *in* $\mathfrak{g}$ *is an intersection algebra.*

Proof. Let $\mathfrak{w}$ be a semialgebra. From convex analysis we take the information that

$$\mathfrak{w} = \bigcap \big\{ L_x(\mathfrak{w}) : x \in W \text{ and } \mathfrak{t}_x(\mathfrak{w}) \text{ is a hyperplane} \big\}.$$

Here, $L_x(\mathfrak{w})$ denotes the half space bounded by the tangent space $\mathfrak{t}_x(\mathfrak{w})$ and containing $\mathfrak{w}$.

Next we show that all these half spaces $L_x(\mathfrak{w})$ are in fact semialgebras. Let $x \in \mathfrak{w}$ such that $\mathfrak{t} := \mathfrak{t}_x(\mathfrak{w})$ is a hyperplane. we have to show that $\mathfrak{t}$ is a subalgebra. Certainly, there is an element $y \in \mathfrak{t}$ such that $\mathfrak{t} = \mathbb{R}x + \mathbb{R}y$. A short calculation yields

$$[\mathfrak{t}, \mathfrak{t}] \subseteq [\mathbb{R}x + \mathbb{R}y, \mathbb{R}x + \mathbb{R}y] \subseteq \{0\} + \mathbb{R}[x, y] + \{0\} \subseteq \mathfrak{t},$$

where the last inclusion results form Proposition 1.5. ∎

In a two-dimensional Lie algebra, every line is a subalgebra, so every wedge is a semialgebra. The situation becomes more interesting in the three-dimensional case. Here, we have to distinguish between the simple algebras, namely $\mathfrak{sl}(2) = \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(3)$, and the solvable ones.

Let $\mathfrak{g}$ be a three-dimensional solvable Lie algebra. Then $\mathfrak{g}$ contains a two-dimensional abelian ideal $\mathfrak{j}$. Let $h \in \mathfrak{g} \setminus \mathfrak{j}$. Then the real Jordan normal form of $\operatorname{ad} h|\mathfrak{j}$ is one of the follwing matrices

$$\text{(i): } \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \text{(ii): } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \text{(iii): } \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \text{(iv): } \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

with suitable λ, or λ_1 and λ_2 or α and β in $\mathbb{R}$. If we assume $\lambda_1 \neq \lambda_2$ or $\beta \neq 0$, respectively, every three-dimensional Lie algebra $\mathfrak{g}$ falls into exactly one of the cases (i), (ii), (iii) or (iv).

Now, the description of all subalgebras of $\mathfrak{g}$ that are planes is no longer difficult: A plane E in $\mathfrak{g}$ is a subalgebra if and only if $E \cap \mathfrak{j}$ is an ideal of $\mathfrak{g}$. The one-dimensional subspaces of $\mathfrak{j}$ that are ideals of $\mathfrak{g}$ are exactly the one-dimensional $\operatorname{ad} h$-invariant subspaces of $\mathfrak{j}$. Thus there are infinitely many of them in case (i), there is one of them in case (ii), there are two in case (iii) and there is none in case (iv).

A typical picture of a semialgebra in each of the four cases should clarify the situation. Note that in case (ii) and (iv) there are no pointed semialgebras. In each of these cases, the only invariant wedges are the half-spaces bounded by $\mathfrak{j}$. (Except $\mathfrak{g}$ is abelian!)

Bild 1.1.

In the case of the three-dimensional simple Lie algebras, the situation becomes entirely different. In $\mathfrak{so}(3)$, that is $\mathbb{R}^3$ with the vector product, there are no two-dimensional subalgebras, thus there are no nontrivial semialgebras contained in $\mathfrak{so}(3)$.

Let us turn to $\mathfrak{sl}(2)$! The Cartan-Killing form κ of $\mathfrak{sl}(2)$ has signature $++-$, so the set

$$\mathcal{D} := \{x \in \mathfrak{sl}(2) : \kappa(x,x) \leq 0\}$$

is a double cone. An elementary calculation now shows that the two-dimensional subalgebras of $\mathfrak{sl}(2)$ are exaclty the planes that are tangent to $\mathcal{D}$. Again, we draw a picture of some typical semialgebras in $\mathfrak{sl}(2)$.

Bild 2.2.

The only nontrival invariant wedges in $\mathfrak{sl}(2)$ are the halves of the double cone $\mathcal{D}$.

As final example, we consider the four-dimensional Lie algebra $\mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \mathbb{R} \cdot \mathbf{1}$. Here, Lemma 2.2 does not help us anymore, and in fact there are semialgebras in $\mathfrak{gl}(2)$ that are not intersection algebras. It turns out that these have to be invariant.

Theorem 2.4. *Let $\mathfrak{w}$ be a generating semialgebra in $\mathfrak{g} = \mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus \mathbb{R}$. Then $\mathfrak{w}$ is invariant or it is the sum of two semialgebras in $\mathfrak{sl}(2)$ and $\mathbb{R}$, respectively.*

Proof. [1], Theorem 5.5. ■

The Classification

A semialgebra $\mathfrak{w}$ in $\mathfrak{g}$ may contain large ideals. (e.g. $\mathfrak{g}$ itself is a semialgebra in $\mathfrak{g}$ of course.) We would like to factor out these ideals and this possibility is guaranteed by the follwing fact.

Proposition 2.5. *Let $\mathfrak{w}$ be a semialgebra in $\mathfrak{g}$ and $\mathfrak{i}$ an ideal of $\mathfrak{g}$ with $\mathfrak{i} \subseteq \mathfrak{w}$. Then $\mathfrak{w}/\mathfrak{i}$ is a wedge in $\mathfrak{g}/\mathfrak{i}$ (i.e., it is closed) and a semialgebra.* ■

Using Lemma 2.2 and Proposition 2.5 we usually will restrict our attention to semialgebras that are reduced in the following sense:

Definition 2.6. A semialgebra $\mathfrak{w}$ in $\mathfrak{g}$ is called *reduced,* if it is generating (!) and there is no nontrivial ideal of $\mathfrak{g}$ contained in $\mathfrak{w}$.

It turns out that the existence of a reduced semialgebra in a Lie algebra $\mathfrak{g}$ imposes rather strong restrictions on the structure of $\mathfrak{g}$. Typical results are the follwing, due to HOFMANN and LAWSON.

Theorem 2.7. *Let $\mathfrak{g}$ be a Lie algebra containing a reduced semialgebra. then every Cartan algebra of $\mathfrak{g}$ is abelian.* ∎

Theorem 2.8. *Let $\mathfrak{w}$ be a reduced semialgebra in g. If h is an element in the interior of $\mathfrak{w}$, then* $\operatorname{spec}(\operatorname{ad} h) \subseteq \mathbb{R} \cup i\mathbb{R}$. ∎

The basic ideas that eventually lead to a classification of all reduced semialgebras and the Lie algebras containing them, run along the following line (the details may be found in [3]):

Let $\mathfrak{g}$ be a Lie algebra containing a reduced semialgebra $\mathfrak{w}$. Let h denote some regular element in the interior of $\mathfrak{w}$. Then $\mathfrak{h} := \ker \operatorname{ad} h$ is an abelian Cartan algebra of $\mathfrak{g}$. Using the real root decomposition of $\mathfrak{g}$, and knowledge we have on semialgebras in some particular low dimensional Lie algebras (including the three-dimensional solvable ones mentioned above), one arrives at Theorem 2.9. It does not only give explicit information on semialgebras but also on the Lie algebra containing it. The formulation chosen here is somewhat technical.

Recall that the semialgebras in $\mathfrak{sl}(2)$ are described above.

Theorem 2.9. *Let $\mathfrak{w}$ be a reduced semialgebra in $\mathfrak{g}$. Then there are ideals $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ and and $\mathfrak{g}_2$ in $\mathfrak{g}$ such that*

(i) *Each $\mathfrak{s}_i$ is isomorphic to $\mathfrak{sl}(2)$.*

(ii) $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n \oplus \mathfrak{g}_2$.

(iii) *$\mathfrak{w}$ is a sum of semialgebras in each of these ideal.*

Furthermore the semialgebra $\mathfrak{v} := \mathfrak{w} \cap \mathfrak{g}_2$ may be described as follows: The subalgebra $\mathfrak{g}_2$ is a direct sum of a subalgebra $\mathfrak{g}_0$ and ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_m$, and there are linear forms $\alpha_1, \dots, \alpha_m \colon \mathfrak{g}_0 \to \mathbb{R}$, different from 0, such that:

(iv) $\alpha_i|\mathfrak{g}_0' = 0$

(v) *The Lie bracket on $\mathfrak{g}_2 = \mathfrak{g}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_m$ is given by*

$$\begin{aligned}&[x \oplus y_1 \oplus \cdots \oplus y_m, \quad x' \oplus y_1' \oplus \cdots \oplus y_m'] \\ &= [x, x'] \oplus \big(\alpha_1(x)y_1' - \alpha_1(x')y_1\big) \oplus \cdots \oplus \big(\alpha_m(x)y_m' - \alpha_m(x')y_m\big).\end{aligned}$$

(vi) *$\mathfrak{g}_0$ contains an invariant pointed generating wedge.*

(vii) *The semialgebra $\mathfrak{v}$ is an intersection of semialgebras $\mathfrak{v}_0$, and $\mathfrak{v}_1 \dots \mathfrak{v}_m$, where*

  (a) *$\mathfrak{v}_0$ contains $\mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_m$ and is invariant. (Thus it is uniquely determined by an invariant wedge in $\mathfrak{g}_0$. This wedge may be different from the one mentioned in* (vi).*)*

  (b) *Each $\mathfrak{v}_i$ for $i = 1, \dots, m$ contains the ideal $\mathfrak{g}_0' \oplus \mathfrak{m}_1 \oplus \cdots \oplus \widehat{\mathfrak{m}_i} \oplus \cdots \oplus \mathfrak{m}_m$, and is an intersection semialgebra. (As usual, $\widehat{\mathfrak{m}_i}$ means that $\mathfrak{m}_i$ has to be omitted.)* ∎

From the Theorem one obtains a corollary, which is less explicit, but much easier to state:

Corollary 2.10. *Let $\mathfrak{w}$ be a generating semialgebra in $\mathfrak{g}$. Then $\mathfrak{w}$ is the intersection of an intersection semialgebra and an invariant wedge.* ∎

Theorem 2.11. *Let $\mathfrak{g}_0$ be a Lie algebra that contains an pointed generating invariant wedge. Let $\alpha_i\colon \mathfrak{g}_0 \to \mathbb{R}$ $(i = 1, \ldots, m)$ denote linear forms vanishing on $\mathfrak{g}_0'$ and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ be vector spaces. Define a bracket on $\mathfrak{g}_2 := \mathfrak{g}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_m$ via*

$$\begin{aligned} &[x \oplus y_1 \oplus \cdots \oplus y_m\,, \quad x' \oplus y_1' \oplus \cdots \oplus y_m'] \\ &= [x, x'] \oplus \big(\alpha_1(x)y_1' - \alpha_1(x')y_1\big) \oplus \cdots \oplus \big(\alpha_m(x)y_m' - \alpha_m(x')y_m\big). \end{aligned}$$

Then $\mathfrak{g}_2$ is a Lie algebra with respect to this operation.

Finally, let $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{sl}(2) \oplus \cdots \oplus \mathfrak{sl}(2)$ as a direct sum of Lie algebras.

Then $\mathfrak{g}$ contains a reduced semialgebra and all Lie algebras containing a reduced semialgebra may be obtained by this contruction. ∎

There are at least three things in the Theorems 2.9 and 2.11, to which I'd like to draw the reader's attention: Firstly the Lie algebra $\mathfrak{sl}(2)$ with its various semialgebras is obviously an inevitable ingeredient of any theory of LIE semialgebras. Secondly, the theorems do not say anything about invariant wedges, they just reduce the classification of semialgebras to that of invariant wedges. Finally we may describe explicitly the Lie algebras that contain LIE semialgebras — again up to the description of the ones containing invariant pointed generating wedges.

References

[1] Eggert, A., "Zur Klassifikation von Semialgebren", Diplomarbeit, TH Darmstadt, 1988.

[2] —,*Lie Semialgebras in reductive Lie algebras*, Semigroup Forum **41** (1990), 115–121.

[3] —, "Semialgebren in Lie-Algebren", Dissertation, TH Darmstadt, 1991.

[4] Hilgert, J., K. H. Hofmann and J. D. Lawson, "Lie Groups, Convex Cones and Semigroups", Oxford University Press, 1989.

[5] Hofmann, K. H., *Hyperplane Subalgebras in real Lie Algebras*, Geometriae Dedicata **36** (1990), 207–224.

[6] Spindler, K., "Invariante Kegel in Liealgebren", Dissertation, TH Darmstadt, 1988.

Fachbereich Mathematik
Technische Hochschule Darmstadt
SchloŒgartenstraŒe 7
W-6100 Darmstadt, Germany

Received February 15, 1991
and in final form March 1, 1991