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Symmetric spaces and convex cones

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1. Introduction

We recall some notation and basic facts concerning convex cones in Lie algebras. An excellent source of reference is the monograph [5], which tells the story of convex cones and their relation to Lie semigroups.

A *wedge* W in a finite dimensional real vector space is a topologically closed set, which is invariant under multiplication with non-negative scalars and under addition. In particular, a wedge in this sense is convex. The smallest vector subspace $H(W) \stackrel{\rm def}{=} W \cap -W$ contained in W is called the *edge* of the wedge. A wedge is called *pointed*, if $H(W) = \{0\}$. Sometimes we call a pointed wedge a pointed *cone*. A wedge W is said to be *generating* if it linearly spans the underlying vector space.

The *subtangent wedge* of a point $x \in W$ may be defined as

$$L_x(W) \stackrel{\rm def}{=} \overline{W - \mathbb{R}^+ \cdot x},$$

and the *tangent space* as $T_x(W) \stackrel{\rm def}{=} L_x(W) \cap -L_x(W)$.

Let W be a wedge in a finite dimensional Lie algebra. We distinguish between several invariance properties. A wedge W is called a *Lie wedge* if $e^{\operatorname{ad} H(W)} W = W$, and an *invariant wedge* if $e^{\operatorname{ad} \mathfrak{g}} W = W$. A wedge is a *Lie semi-algebra* if there exists a CH-neigborhood B, such that $(W \cap B) * (W \cap B) \subseteq W$ holds. Here $*$ denotes the Campbell-Hausdorff multiplication. We have the following hierarchy of wedges and their geometric characterization: a wedge W in a finite dimensional Lie algebra $\mathfrak{g}$ is

$$\begin{array}{lcr} \text{a Lie wedge} & \Longleftrightarrow & [H(W), x] \subseteq T_x(W) \\ \text{a Lie semialgebra} & \Longleftrightarrow & [T_x(W), x] \subseteq T_x(W) \\ \text{an invariant wedge} & \Longleftrightarrow & [\mathfrak{g}, x] \subseteq T_x(W) \end{array}$$

for all $x \in W$ (resp., $x \in C^1(W)$, if W is generating). We have the implications

$$\text{invariant wedge} \Rightarrow \text{Lie semialgebra} \Rightarrow \text{Lie wedge}.$$

For further details we refer to [5].

2. Symmetric Spaces and symmetric Lie algebras

Definition 2.1. If $\widehat{\tau}: G \to G$ is an involutive Lie group automorphism we denote by $G_{\widehat{\tau}}$ the set of fixed points under $\widehat{\tau}$. A *symmetric space* is a triple $(G, H, \widehat{\tau})$ where G is a connected Lie group and H is a closed subgroup satisfying $(G_{\widehat{\tau}})_0 \subseteq H \subseteq G_{\widehat{\tau}}$.

If G is simply connected, then H is connected and the homogeneous space $M \stackrel{\text{def}}{=} G/H$ is simply connected. The group G acts on M via $\mu_g(xH) = gxH$, $g \in G, xH \in M$. As usual, we denote by $\pi: G \to M$ the canonical projection. The involution $\widehat{\tau}$ induces in M a *symmetry at* $\xi = xH$ as follows. Define s_ε, $\varepsilon = H$ by $s_\varepsilon \circ \pi = \pi \circ \widehat{\tau}$ and $s_\xi = \mu_g \circ s_\varepsilon \circ \mu_{g^{-1}}$, where $\mu_g(\varepsilon) = \xi$. Beside other properties, ξ is an isolated fixed point of the involution s_ξ. The assignment $\xi \to Q(\xi) \stackrel{\text{def}}{=} s_\xi \circ s_\varepsilon$ is called the *quadratic representation* of M.

Example 2.2. (i) Let G be a connected Lie group and define $\widehat{\tau}: G \times G \to G \times G$ by $\widehat{\tau}(g, g') = (g', g)$. The set of fixed points is the diagonal $\Delta(G) = \{(g, g) \mid g \in G\}$, which is a closed subgroup of $G \times G$. Hence $(G \times G, \Delta(G), \widehat{\tau})$ is a symmetric space.

(ii) Let G be a connected Lie group. Assume that $\widehat{\tau}$ denotes the complex conjugation of the complexification $G_{\mathbb{C}} = G \otimes \mathbb{C}$. Then $(G_{\mathbb{C}}, G, \widehat{\tau})$ is a symmetric space.

Definition 2.3. A *symmetric Lie algebra* is a triple $(\mathfrak{g}, \mathfrak{h}, \tau)$ where $\mathfrak{g}$ is a Lie algebra, $\tau: \mathfrak{g} \to \mathfrak{g}$ is an involutive Lie algebra automorphism, and $\mathfrak{h}$ is the set of fixed elements.

In this paper we prefer an equivalent notation. A Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is a symmetric Lie algebra if and only if $\mathfrak{h}$ is a subalgebra and $\mathfrak{q}$ is an $\mathfrak{h}$-module with $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$. In fact $\mathfrak{h}$, resp., $\mathfrak{q}$ are just the eigenspaces of τ for the eigenvalue $+1$, resp., -1.

Every symmetric space $(G, H, \widehat{\tau})$ determines a unique symmetric Lie algebra via the functor which assigns to a Lie group its Lie algebra. That is, $\mathfrak{g} = L(G)$, $\mathfrak{h} = L(H)$ and $\tau = d\widehat{\tau}(\mathbf{1})$. Conversely, if G is a simply connected Lie group with Lie algebra $L(G) = \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is a symmetric Lie algebra, then the analytic subgroup H of G with Lie algebra $L(H) = \mathfrak{h}$, and the lift $\widehat{\tau}$ of τ, define on G the structure of a symmetric space $(G, H, \widehat{\tau})$ determing $(\mathfrak{g}, \mathfrak{h}, \tau)$.

Lemma 2.4. (i) *The map*

$$\mathfrak{g}/\mathfrak{h} \xrightarrow{\cong} T(M)_{gH}, \quad X + \mathfrak{h} \mapsto d\mu_g(\varepsilon) \circ d\pi(\mathbf{1})(X)$$

defines an isomorphism of vectorspaces of $\mathfrak{g}/\mathfrak{h}$ *onto the tangent space of* M *at* gH.

(ii) *The map*

$$\mathfrak{q} \xrightarrow{\cong} T(M)_{gH}, \quad X \mapsto d\mu_g(\varepsilon) \circ d\pi(\mathbf{1})(X)$$

defines an isomorphism of vectorspaces of $\mathfrak{q}$ *onto the tangent space of* M *at* gH.

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Definition 2.5. The *exponential function* of M is defined by $\mathrm{Exp}\colon \mathfrak{q} \to M$, $\mathrm{Exp} = \pi \circ \exp$.

3. Ordered symmetric spaces

In all that follows let $(G, H, \widehat{\tau})$ with $M = G/H$ be a symmetric space and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the associated symmetric Lie algebra.

Definition 3.1. A *causal structure* on M is a G-invariant cone field Θ, i. e., for every $\xi \in M$ there exists a cone $\Theta(\xi) \subseteq T_\xi(M)$ such that $\Theta(\mu_g(xH)) = d\mu_g(x)\Theta(xH)$ holds for all $g \in G$.

If we identify $T_\varepsilon(M)$ with $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{q}$ via Lemma 2.4, the relation $\mu_h(\varepsilon) = \varepsilon$ for every $h \in H$ implies $d\mu_h(\varepsilon)(X + \mathfrak{h}) = (\mathrm{Ad}(h)X) + \mathfrak{h}$. In particular, the cone $\Theta(\varepsilon)$ is of the form $d\pi(\mathbf{1})(W)$ with a wedge $W = \mathfrak{h} \oplus C$ in $\mathfrak{g}$ satisfying $\mathrm{Ad}(H)C = C$. Conversely, every wedge $W = \mathfrak{h} \oplus C$ with $e^{\mathrm{ad}\,\mathfrak{h}}C = C$ defines a causal structure on M.

Definition 3.2. On M there is defined a *causal order* as follows: $\xi \prec \eta$ if and only if there exists an absulutely continuous curve $\alpha\colon [t_0, t_1] \to M$ with $\alpha(t_0) = \xi$, $\alpha(t_1) = \eta$ and $\alpha'(t) \in \Theta(\alpha(t))$. A curve with these properties is called a *causal curve.*

With $\prec$ also its closure $\preceq$ is an order on M, and we have $\xi \preceq \eta$ if and only if $\eta \in \overline{\{\zeta \mid \xi \prec \zeta\}}$. The manifold M is called *globally causal* if $\preceq$ is a partial order.

A G-invariant partial order on M defines an order on G given by $x \leq y$ if and only if $xH \preceq yH$, and a *semigroup of positivity* $S_\leq = \{x \in G \mid 1 \leq x\}$. In [6] it is shown that the closure of the conal order on M given by the wedge field $\Theta(xH) = d\mu_x(\varepsilon) \circ d\pi(\mathbf{1})W$, where $W = \mathfrak{h} \oplus C$ with C pointed and generating, is a partial order if and only if the wedge W is *global* in the sense of [5], i. e., if and only if $W = L(\overline{\langle \exp W \rangle})$. For the details and the definition of the tangent wedge $L(S)$ of a Lie semigroup S we refer again to [5].

Of particular interest is the case where the set $S = \exp C{\cdot}H$ is a closed semigroup of G.

Example 3.3. Consider the symmetric space $(G_{\mathbb{C}}, G, \widehat{\tau})$ of Example 2.2 (ii). The corresponding symmetric Lie algebra is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i{\cdot}\mathfrak{g}$. Set $W \overset{\mathrm{def}}{=} \mathfrak{g} \oplus i{\cdot}C$ where C is a pointed, generating invariant cone of $\mathfrak{g}$. Then W has the desired properties. In [7], Ol'shanskii showed that for semisimple $\mathfrak{g}$, the wedge W is global and that $S = (\exp iC){\cdot}G$ is a closed semigroup of $G_{\mathbb{C}}$, which plays an important role in representation theory. In [1], the analogous result is proved for solvable $\mathfrak{g}$ and for a more general case, in which the wedge W, however, is supposed to satisfy an additional condition.

In general two main questions arise. Does there exist a classification of wedges $W = \mathfrak{h} \oplus C$ with $\mathrm{Ad}(H)C = C$? When does such a wedge happen to be global? The following is a first step in achieving a partial answer of the first problem.

4. Ol'shanskiĭ wedges in symmetric Lie algebras

In view of the preceding remarks we have the following definition.

Definition 4.1. A wedge W in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is an *Ol'shanskiĭ wedge*, if there is a wedge $C \subseteq \mathfrak{q}$ such that $W = \mathfrak{h} \oplus C$ and the following holds:

(1) $e^{\operatorname{ad}\mathfrak{h}}C = C$.

(2) There is a Campbell-Hausdorff-neighborhood B in $\mathfrak{g}$ such that

$$g(\operatorname{ad} x)L_x(W) = L_x(W)$$

for all $x \in C \cap B$.

Condition (1) implies that $C-C$ and $H(C)$ are $\mathfrak{h}$-modules. Then $W-W = \mathfrak{h} \oplus (C-C)$ is a symmetric subalgebra of $\mathfrak{g}$, in which W obviously is generating. Therefore, we may often restrict our attention to the case where W is generating.

Let $W = \mathfrak{h} \oplus C$ be an Ol'shanskiĭ wedge in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ and $w = h + c \in W$. Then for the subtangent wedges and tangent spaces of W, resp., C we have

$$L_w(W) = \mathfrak{h} \oplus L_c(C) \quad \text{und} \quad T_w(W) = \mathfrak{h} \oplus T_c(C).$$

Furthermore, if W is generating, then $w \in C^1(W)$ if and only if $c \in C^1(C)$.

Theorem 4.2. (Characterization Theorem for Ol'shanskiĭ Wedges)*For a wedge $W = \mathfrak{h} \oplus C$ in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the following conditions are equivalent:*

(1) *W is an Ol'shanskiĭ wedge.*

(2) *W is a Lie wedge.*

(3) $e^{\operatorname{ad}\mathfrak{h}}C = C$.

(4) *$g(\operatorname{ad} c)L_c(W) \subseteq L_c(W)$ for all $c \in C \cap B$ with a suitable* CH*-neighborhood $B \subseteq \mathfrak{g}$.*

(5) *$[\mathfrak{h}, c] \subseteq T_c(C)$ for all $c \in C$.*

If C is generating in $\mathfrak{q}$, then these conditions are equivalent to

(5′) *$[\mathfrak{h}, c] \subseteq T_c(C)$ for all $c \in C$.*

∎

In particular, this shows that conditions (1) and (2) of Definition 4.1 are equivalent.

In order to establish a classification of Ol'shanskiĭ wedges, we proceed as in the classification of Lie semialgebras. The idea is, first to look at low dimensional examples, and afterwards to find some restriction of the structure in the general case by use of these *test subalgebras.* In [3] this is done for three-dimensional symmetric Lie algebras, almost abelian Lie algebras and for the oscillator algebra. In the case of a nilpotent symmetric Lie algebra we can say even more.

Lemma 4.3. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra and $W = \mathfrak{h} \oplus C$ an Ol'shanskiĭ wedge. Let $\mathfrak{i}$ denote the largest ideal of $\mathfrak{g}$ contained in W. Then $\mathfrak{i}$ is a symmetric ideal, i. e., $\mathfrak{i} = \mathfrak{i}_{\mathfrak{h}} \oplus \mathfrak{i}_{\mathfrak{q}}$. In particular, if W contains any ideal of $\mathfrak{g}$, then it contains also a symmetric one.* ∎

Definition 4.4. An Ol'shanskiĭ wedge $W = \mathfrak{h} \oplus C$ is called *reduced*, if W is generating and does not contain any non-zero ideals of $\mathfrak{g}$. If W is an Ol'shanskiĭ wedge, we denote by $\mathfrak{i}_W$ the largest symmetric ideal of $W - W$ that is contained in W.

If $\mathfrak{i} = \mathfrak{i}_{\mathfrak{h}} \oplus \mathfrak{i}_{\mathfrak{q}}$ is a symmetric ideal in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, we endow the factor space $\overline{\mathfrak{g}} = \mathfrak{g}/\mathfrak{i}$ with a structure of a symmetric Lie algebra $\overline{\mathfrak{g}} = \overline{\mathfrak{h}} \oplus \overline{\mathfrak{q}}$ in a canonical way. The quotient map sends $\mathfrak{h}$ onto $\overline{\mathfrak{h}}$, which is isomorphic to $\mathfrak{h}/\mathfrak{i}_{\mathfrak{h}}$, and it maps $\mathfrak{q}$ onto $\overline{\mathfrak{q}}$, which can be identified with $\mathfrak{q}/\mathfrak{i}_{\mathfrak{q}}$. With this preparations we can define the *reduction* $W/\mathfrak{i}_W$ of an Ol'shanskiĭ wedge W, since by Lemma 4.3 the ideal $\mathfrak{i}_W$ is symmetric.

Lemma 4.5. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra and W an Ol'shanskiĭ wedge. Then $W/\mathfrak{i}_W$ is a reduced Ol'shanskiĭ wedge in $(W - W)/\mathfrak{i}_W$.* ∎

Definition 4.6. An Ol'shanskiĭ wedge $W = \mathfrak{h} \oplus C$ in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is called $\mathfrak{q}$-*trivial*, if $[[\mathfrak{q}, \mathfrak{q}], \mathfrak{q}] \subseteq C$ holds.

Proposition 4.7. *Let $W = \mathfrak{h} \oplus C$ be a generating Ol'shanskiĭ wedge in a symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Then the following conditions are equivalent:*

(1) *W is $\mathfrak{q}$-trivial in $\mathfrak{g}$.*

(2) *The reduction $W/\mathfrak{i}_W$ is $\mathfrak{q}/\mathfrak{i}_W$-trivial in $\mathfrak{g}/\mathfrak{i}_W$.*

(3) *$[\mathfrak{q}/\mathfrak{i}_W, \mathfrak{q}/\mathfrak{i}_W] = \{0\}$ and $W/\mathfrak{i}_W$ is $\mathfrak{q}/\mathfrak{i}_W$-pointed.*

(4) *$[\mathfrak{q}/\mathfrak{i}_W, \mathfrak{q}/\mathfrak{i}_W] = \{0\}$.*

∎

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Lie algebra. We define

$$\mathfrak{q}^{[0]} = \mathfrak{q}, \ldots, \mathfrak{q}^{[m+1]} = [\mathfrak{q}, \mathfrak{q}^{[m]}], \quad m \in \mathbb{N}.$$

If n is even, then $\mathfrak{q}^{[n]} \subseteq \mathfrak{q}$ is an $\mathfrak{h}$-submodule, and if n is odd, $\mathfrak{q}^{[n]}$ is an ideal of $\mathfrak{h}$.

Definition 4.8. A symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is called $\mathfrak{q}$-*nilpotent*, if $\mathfrak{q}^{[n]} = \{0\}$ for a suitable $n \in \mathbb{N}$.

With this preparations we have the following result, which gives a also complete classification in nilpotent symmetric Lie algebras.

Theorem 4.9. ($\mathfrak{q}$-Triviality Theorem — The $\mathfrak{q}$-Nilpotency Theorem) *Let* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *be a* $\mathfrak{q}$*-nilpotent symmetric Lie algebra. Then every generating Ol'shanskiĭ wedge in* $\mathfrak{g}$ *is* $\mathfrak{q}$*-trivial.* ∎

Corollary 4.10. *Let* $\mathfrak{n} = \mathfrak{n}_{\mathfrak{h}} \oplus \mathfrak{n}_{\mathfrak{q}}$ *be a nilpotent symmetric subalgebra of a symmetric Lie algebra* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *and* $W = \mathfrak{h} \oplus C$ *a generating Ol'shanskiĭ wedge with* $\mathfrak{n} \cap \operatorname{int} W \neq \emptyset$*. Then* $[[\mathfrak{n}_{\mathfrak{q}}, \mathfrak{n}_{\mathfrak{q}}], \mathfrak{n}_{\mathfrak{q}}] \subseteq H(C)$ *holds.* ∎

The idea is to apply Corollary 4.10 to symmetric Cartan algebras which intersect $\operatorname{int} W$. In [4] and [8] it is proved, that symmetric Cartan algebras in arbitrary symmetric Lie algebras do exist. The classification of invariant cones shows, what important tool Cartan algebras may be. Perhaps, they happen to be of nearly the same importance in our situation.

Invariant Ol'shanskiĭ wedges and Ol'shanskiĭ semialgebras

If $W = \mathfrak{h} \oplus C$ is not only an Ol'shanskiĭ wedge, but also satisfies further invariance properties, such as being invariant or being a Lie semialgebra, we can give a complete desription of the structure of the underlying Lie algebra.

Definition 4.11. An Ol'shanskiĭ wedge W in a symmetric Lie algebra $\mathfrak{g}$ is an *Ol'shanskiĭ semialgebra*, resp., an *invariant Ol'shanskiĭ wedge* if and only if W is a Lie semialgebra, resp., an invariant wedge.

Recall that a wedge W in a Lie algebra $\mathfrak{g}$ is called *trivial* if and only if the commutator algebra is contained in W.

Proposition 4.12. *For a wedge* $W = \mathfrak{h} \oplus C$ *in a symmetric Lie algebra* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *the following conditions are equivalent:*

(1) $T_x(W)$ *is an ideal for all* $x \in W$.

(2) W *is an invariant Ol'shanskiĭ wedge.*

(3) $[\mathfrak{q}, \mathfrak{h}] \subseteq H(C)$.

(4) $H(W)$ *is an ideal in* $\mathfrak{g}$.

(5) W *is trivial.*

If C *is pointed, then these conditions are equivalent to*

(3′) $[\mathfrak{q}, \mathfrak{h}] = \{0\}$.

(4′) $\mathfrak{h}$ *is an ideal of* $\mathfrak{g}$.

∎

This proposition tells us that invariant Ol'shanskiĭ wedges occur very rarely.

Theorem 4.13. (Classification of symmetric Lie algebras with invariant Ol'shanskiĭ wedges) *Let* $\mathfrak{q}$ *be a vectorspace and* $\mathfrak{h}$ *a Lie algebra with centre* $\mathfrak{z}(\mathfrak{h})$. *Further, let* $\kappa\colon \mathfrak{q} \times \mathfrak{q} \to \mathfrak{z}(\mathfrak{h})$ *be a skew-symmetric, bilinear map. Then* $\mathfrak{g}_\kappa \stackrel{\mathrm{def}}{=} \mathfrak{h} \oplus \mathfrak{q}$ *with the bracket*

$$[(h,q),(h',q')] = (\kappa(q,q') + [h,h'], 0)$$

for $q, q' \in \mathfrak{q}$, $h, h' \in \mathfrak{h}$ *is a Lie algebra for which the following holds:*

(1) $\mathfrak{h}$ *is an ideal in* $\mathfrak{g}_\kappa$.

(2) $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{z}(\mathfrak{h})$.

For any pointed *wedge* $C \subseteq \mathfrak{q}$ *the wedge* $W = \mathfrak{h} \oplus C$ *is an invariant Ol'shanskiĭ wedge in* $\mathfrak{g}_\kappa$.

Conversely, if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *is a symmetric Lie algebra supporting an* invariant *Ol'shanskiĭ wedge* $W = \mathfrak{h} \oplus C$ *with* C *pointed, then* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{g}_\kappa$ *with* $\kappa(q,q') = [q,q']$ *for* $q, q' \in \mathfrak{q}$. ∎

This gives a complete classification of symmetric Lie algebras supporting an invariant Ol'shankiĭ wedge $W = \mathfrak{h} \oplus C$ with C pointed.

Proposition 4.14. *For a wedge* $W = \mathfrak{h} \oplus C$ *in a symmetric Lie algebra* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *the following conditions are equivalent:*

(1) $T_x(W)$ *is a subalgebra for all* $x \in W$.

(2) W *is an Ol'shanskiĭ semialgebra.*

(3) $T_c(C)$ *is an* $\mathfrak{h}$*-module for all* $c \in C$.

If the wedge C *is generating, these conditions are equivalent to*

(3') $T_c(C)$ *is an* $\mathfrak{h}$*-module for all* $c \in C^1(C)$.

∎

Theorem 4.15. (The Decomposition Theorem for Ol'shanskiĭ Semialgebras) *Let* $W = \mathfrak{h} \oplus C$ *be an Ol'shanskiĭ semialgebra in a symmetric Lie algebra* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ *and assume* C *is pointed and generating. Then the* $\mathfrak{h}$*-module* $\mathfrak{q}$ *has a decomposition*

$$\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}^+ \quad \textit{with} \quad \mathfrak{q}^+ = \sum_{\alpha \in \Omega} \mathfrak{q}^\alpha,$$

where $\Omega \subseteq \widehat{\mathfrak{h}} \setminus \{0\}$. *Further, the following holds:*

(i) *The weight spaces are given by*

$$\begin{aligned} \mathfrak{q}_0 &= \{x \in \mathfrak{q} \mid [h,x] = 0 \textit{ for } h \in \mathfrak{h}\} \\ &= \bigcap \{T_c(C) \mid c \in C^1(C),\, \alpha_c \neq 0\}. \end{aligned}$$

$$\begin{aligned} \mathfrak{q}^\alpha &= \{x \in \mathfrak{q} \mid [h,x] = \alpha(h)x \textit{ for } h \in \mathfrak{h}\} \\ &= \bigcap \{T_c(C) \mid c \in C^1(C),\, \alpha_c \neq \alpha\}. \end{aligned}$$

(ii) *W is the intersection of hyperplane Ol'shanskiĭ semialgebras and an invariant Ol'shanskiĭ wedge. More precisely,*

$$W = W_0 \cap \bigcap_{\alpha \in \Omega} W_\alpha,$$

where $W_0 = \mathfrak{h} \oplus \widetilde{C}_0$ is invariant and $W_\alpha = \mathfrak{h} \oplus \widetilde{C}_\alpha$ are semialgebras.

(iii) *The wedge W is adapted to the root decomposition in* (i). *There exist pointed generating cones C_0 in $\mathfrak{q}_0$ and C_α in $\mathfrak{q}_\alpha$ such that*

$$W = \mathfrak{h} \oplus C_0 \oplus \sum_{\alpha \in \Omega} C_\alpha.$$

■

With this result, we are able to classify all symmetric Lie algebras supporting generating Ol'shanskiĭ semialgebras in the same way as we classified Lie algebras supporting invariant Ol'shanskiĭ wedges in Theorem 4.13.

A generating Lie semialgebra W is called *reduced*, if the edge $H(W)$ contains no non-trivial ideal of $\mathfrak{g}$. In the case of symmetric Lie algebras which contain reduced Ol'shanskiĭ semialgebras, we can say even more.

Theorem 4.16. *If the symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ supports a reduced Ol'shanskiĭ semialgebra $W = \mathfrak{h} \oplus C$, then $\mathfrak{g} = \mathrm{sl}(2, \mathbb{R})^m \oplus \mathfrak{r}$ with $\mathfrak{r}'' = \{0\}$ and $[\mathfrak{r}_\mathfrak{q}, \mathfrak{r}_\mathfrak{q}] = \{0\}$ where $\mathfrak{r}_\mathfrak{q} = \mathfrak{r} \cap \mathfrak{q}$. There exist generating Ol'shanskiĭ semialgebras $W_\mathfrak{s} \subseteq \mathrm{sl}(2, \mathbb{R})^m$ and $W_\mathfrak{r} \subseteq \mathfrak{r}$ such that $W = W_\mathfrak{s} \oplus W_\mathfrak{r}$.* ■

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