Journal of Integer Sequences, Vol. 9 (2006),
Article 06.1.4

A Combinatorial Interpretation of the Eigensequence for Composition

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Abstract

The monic sequence that shifts left under convolution with itself is the Catalan numbers with 130+ combinatorial interpretations. Here we establish a combinatorial interpretation for the monic sequence that shifts left under *composition*: it counts permutations that contain a 3241 pattern only as part of a 35241 pattern. We give two recurrences, the first allowing relatively fast computation, the second similar to one for the Catalan numbers. Among the $4 \times 4! = 96$ similarly restricted patterns involving 4 letters (such as $4\underline{2}31$: a 431 pattern occurs only as part of a 4231), four different counting sequences arise: 64 give the Catalan numbers, 16 give the Bell numbers, 12 give sequence A051295 in OEIS, and 4 give a new sequence with an explicit formula.

1 Introduction

The composition of two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ is $(c_n)_{n\geq 1}$ defined by $C(x) = A(B(x))$ where A, B, C are the respective generating functions, $A(x) = \sum_{n\geq 1} a_n x^n$ and so on. A sequence is monic if its first term is 1. There is a unique monic sequence $(b_n)_{n\geq 1}$ whose composition with itself is equal to its left shift, $(b_2, b_3, \ldots)$. This sequence $(b_n)_{n\geq 1}$ begins $1, 1, 2, 6, 23, 104, 531, \ldots$ and is called the (monic) eigensequence for composition [1] .

Consider a permutation on a set of positive integers as a list (or word of distinct letters). A subpermutation is a subword (letters in same order but not not necessarily contiguous). Thus 253 is a subpermutation of 21534. A factor is a subpermutation in which the letters are contiguous. A standard permutation is one on an initial segment of the positive integers. The reduced form of a permutation π on an arbitrary set of positive integers, denoted reduce(π), is the standard permutation obtained by replacing its smallest entry by 1, next smallest by 2, and so on. Thus reduce(352) = 231. The length $|\pi|$ of a permutation π is simply the number of letters in it. Given a permutation π and a standard permutation ρ of weakly smaller length—a "pattern"—a subpermutation of π whose reduced form is ρ is said to be an instance of the pattern ρ in π. Thus 352 forms a 231 pattern in 35124. A 321-avoiding permutation, for example, is one containing no instance of a 321 pattern. The number of 321-avoiding permutations is the Catalan number C_n [2].

Let $\mathcal{A}_n$ denote the set of permutations on $[n]$ in which the pattern 3241 occurs only as part of a 35241 pattern ($3\underline{5}241$OK permutations for short). This curious pattern restriction arises in connection with 2-stack-sortable permutations, which can be characterized as permutations that are both $3\underline{5}241$OK and 2341-avoiding [3]. With $a_n = |\mathcal{A}_n|$, the first several terms of $(a_n)_{n\ge 0}$ coincide with those of $(b_n)_{n\ge 1}$ above. This suggests that $a_n = b_{n+1}$ and the main objective of this paper is to prove bijectively that this is so (§3 and §4). In §2, we consider the structure of a $3\underline{5}241$OK permutation and deduce two recurrences for a_n. A bonus section (§5) classifies all similarly restricted patterns involving 3 or 4 letters by their counting sequences.

2 Structure and Recurrences for $3\underline{5}241$OK Permutations

Every permutation π on $[n]$ has the form $\pi = m_1 L_1 m_2 L_2 \ldots m_r L_r$ where $m_1 < m_2 < \ldots < m_r = n$ are the left-to-right maxima (LRmax for short) of π. With this notation, the following characterization of $3\underline{5}241$OK permutations is easy to verify.

Theorem 1. *A permutation π on $[n]$ is $3\underline{5}241$OK if and only if*
(i) $L_1 < L_2 < \ldots < L_r$ in the sense that $u \in L_i$, $v \in L_j$ with $i < j$ implies $u < v$, and
(ii) each L_i is $3\underline{5}241$OK. □

Recall $\mathcal{A}_n$ is the set of all $3\underline{5}241$OK permutations on $[n]$. Let $\mathcal{A}_{n,k}$ denote the set of permutations in $\mathcal{A}_n$ whose first entry is k, and let $\mathcal{C}_n$ the denote the set of permutations in $\mathcal{A}_n$ whose first two entries are in increasing order ($n \ge 2$, $\mathcal{C}_1 := \mathcal{A}_1 = \{(1)\}$). Set $a_n = |\mathcal{A}_n|$, $a_{n,k} = |\mathcal{A}_{n,k}|$, $c_n = |\mathcal{C}_n|$.

Theorem 2. *a_n is given by the recurrence relations $a_0 = c_1 = 1$ and*

$$\begin{array}{lll} (i) & a_n = \sum_{i=0}^{n-1} a_i c_{n-i} & n \ge 1 \\ (ii) & c_n = \sum_{i=0}^{n-1} i a_{n-1,i} & n \ge 2 \\ (iii) & a_{n,k} = \begin{cases} \sum_{i=0}^{k-1} a_i \sum_{j=k-i}^{n-1-i} a_{n-1-i,j} & 1 \le k \le n-1 \\ a_{n-1} & k = n \end{cases} & \end{array}$$

Proof (i) counts $\mathcal{A}_n$ by $i = |L_1|$ since $m_1 L_1 K \longrightarrow \big(\text{reduce}(L_1), \text{reduce}(m_1 K)\big)$ is a bijection to $\mathcal{A}_i \times \mathcal{C}_{n-i}$. (ii) counts $\mathcal{C}_n$ by second entry, say $i+1$, because $L_1 = \emptyset$ and, with $m_2 = i+1$, $m_1 m_2 L_2 \ldots m_r L_r \longrightarrow \big(m_1, \text{reduce}(m_2 L_2 \ldots m_r L_r)\big)$ is a bijection to $[i] \times \mathcal{A}_{n-1,i}$.

(iii) counts $\mathcal{A}_{n,k}$ $(1 \le k \le n-1)$ by $i = |L_1|$ and $j =$ first entry of reduce$(m_2L_2\ldots)$ since $j \ge k-i$, $m_1 = k$ and $m_1L_1m_2L_2\ldots \longrightarrow \big(L_1, \text{reduce}(m_2L_2\ldots)\big)$ is a bijection to $\mathcal{A}_i \times \mathcal{A}_{n-1-i,j}$. $\square$

There is a more elegant (but less computationally efficient) recurrence involving a sum over compositions. Recall that a composition $\mathbf{c}$ of n is a list $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ of positive integers whose sum is n and if $\mathbf{c} = (c_i)_{i=1}^r$ and $\mathbf{d} = (d_i)_{i=1}^r$ are same-length compositions of n, then $\mathbf{d}$ dominates $\mathbf{c}$ ($\mathbf{d} \succeq \mathbf{c}$) if $d_1 + \ldots + d_i \ge c_1 + \ldots + c_i$ for $i = 1, 2, \ldots, r-1$ (of course, equality holds for $i = r$). Let $\mathcal{C}_n$ denote the set of all compositions of n (there are 2^{n-1} of them). A permutation $\pi = m_1L_1m_2L_2\ldots m_rL_r$ (recall the m_i are the LRmax entries) determines two same-length compositions of $[n]$: $\mathbf{c} = (c_i)_{i=1}^r$ with $c_i = |m_iL_i| = 1 + |L_i|$ and $\mathbf{d} = (d_i)_{i=1}^r$ with $d_i = m_i - m_{i-1}$ ($m_0 := 0$) and necessarily $\mathbf{d} \succeq \mathbf{c}$ (or else a left-to-right max would occur among the L_i). Summing over $\mathbf{c}$, Theorem 1 yields

Theorem 3.

$$a_n = \sum_{\mathbf{c}\in\mathcal{C}_n} |\{\mathbf{d} \in \mathcal{C}_n \,:\, \mathbf{d} \succeq \mathbf{c}\}| \; a_{c_1-1}a_{c_2-1}\cdots a_{c_r-1} \qquad n \ge 1$$

$\square$

Omitting the first factor in the summand, the recurrence

$$a_0 = 1,\; a_n = \sum_{\mathbf{c}\in\mathcal{C}_n} a_{c_1-1}a_{c_2-1}\cdots a_{c_r-1} \qquad n \ge 1$$

is well known to generate the Catalan numbers. For example, it counts Dyck paths by the locations of their returns to ground level. Since the permutations for which $\mathbf{c} = \mathbf{d}$ are precisely those which avoid the pattern 231, another byproduct is that the Catalan numbers count 231-avoiding permutations.

3 Preparing for the Main Bijection

The generating function $B(x) = \sum_{n\ge1} b_nx^n$ for the "shifts left under composition" sequence of §1 is characterized by $B(B(x)) = \frac{B(x)}{x} - 1$. If (b_n) is the counting sequence for some species, say B-structures, then $[x^n]B(B(x))$ is the number of pairs $(X, \mathbf{Y})$ where X is a B-structure of unspecified size k, $1 \le k \le n$ and $\mathbf{Y}$ is a k-list of B-structures of total size n (a k-list is simply a list with k entries). On the other hand, $[x^n](\frac{B(x)}{x} - 1) = b_{n+1}$. Hence, to show that $b_{n+1} = a_n$, our main objective, we need a bijection from the set $\mathcal{A}_n$ of $3\underline{5}241$OK permutations on $[n]$ to pairs $(\rho, \mathbf{v})$ where $\rho \in \mathcal{A}_{k-1}$ for some $1 \le k \le n$ and $\mathbf{v}$ is a k-list of $3\underline{5}241$OK permutations (possibly empty) of total length $n-k$.

Indeed, given $\pi \in \mathcal{A}_n$, the position of n in π determines k and the right factor of π starting at n determines ρ as follows (it is the extraction of $\mathbf{v}$ that will really make us work). It is convenient to define the LIT (longest increasing terminal) entries of a permutation π on $[n]$ to be $k, k+1, \ldots, n$ where k is the smallest integer such that $k, k+1, \ldots, n$ appear in that order in π. Note that the LIT entries form a terminal segment of the LRmax entries. For example, the LIT entries are double underlined and the remaining LRmax entries are single underlined in $\underline{2}\,1\,\underline{4}\,\underline{\underline{7}}\,6\,5\,\underline{\underline{8}}\,\underline{\underline{9}}\,3$. The LIT entries of a permutation on an arbitrary set of positive integers are defined analogously. Now decompose π as $\sigma n \tau$.

Theorem 4. *A permutation $\sigma n \tau$ on $[n]$ is* $3\underline{5}241$OK *if and only if σ, τ are both* $3\underline{5}241$OK *and each entry of σ that exceeds the smallest entry of τ is an LIT entry of σ.*

Proof $(\Rightarrow)$ Suppose $b \in \sigma$ exceeds $c \in \tau$ yet b is not an LIT entry of σ. Then either (i) there exists $a \in \sigma$ preceding b with $a > b$, and $abnc$ is an offending 3241 pattern, or (ii) there exist u, v following b in σ with $u > v > b$ and this time $uvnc$ is an offending 3241.

$(\Leftarrow)$ routine. □

Set $k = |n\tau|$, the length of the right factor of π starting at n, and set $\rho = \text{reduce}(\tau) \in \mathcal{A}_{k-1}$, capturing the underlying permutation of τ. The support of τ can be captured by placing, for each $c \in \text{support}(\tau)$, an asterisk (or star) right before the smallest entry of σ that exceeds c (necessarily an LIT entry in view of Theorem 4) or after $\max(\sigma)$ if there is no such entry. No information is lost if σ is then reduced (keeping the stars in place) to produce σ^* say, because only the LIT entries of σ will be affected and the stars determine both the original LIT entries of σ and the support of τ. For example, $\pi = (2, 8, 3, 1, 11, 4, 6, 5, 13, 7, 15, 9, 10, 14, 12) \in \mathcal{A}_{15}$ gives $k = 5$, $\tau = (9, 10, 14, 12)$ and yields the pair $\rho = (1, 2, 4, 3)$, $\sigma^* = (2, 8, 3, 1, *, *, 9, 4, 6, 5, *, 10, *, 7)$, and π can be recovered from ρ and σ^*. The length of σ^* (disregarding the stars) coincides with the total length $n - k$ of the desired list of $3\underline{5}241$OK permutations. This reduces the problem to giving a bijection from $3\underline{5}241$OK permutations on $[n]$ with $k-1$ stars distributed arbitrarily, just before LIT entries or just after the maximum entry n, to k-lists of (possibly empty) $3\underline{5}241$OK permutations of total length n.

Let $X_{n,k}$ denote the subset of the preceding "starred" permutations on $[n]$ with (i) no stars immediately following or preceding the max n, and (ii) at most one star preceding each non-max LIT entry. In other words, $X_{n,k}$ is the set of $3\underline{5}241$OK permutations on $[n]$ with $k-1$ distinct LIT entries other than n preceded (or marked) by a star. We now show that the following result suffices to construct the desired bijection.

Theorem 5. *With $X_{n,k}$ as just defined, there is a bijection from $X_{n,k}$ to the set of k-lists of* nonempty $3\underline{5}241$*OK permutations of total length n.*

Consider a permutation with (possibly) multiple stars in the allowed locations, for example, with $k = 8$ and hence 7 stars and with a, b denoting non-max LIT entries,

$$\ldots * * * a \ldots * b \ldots * n * * \ldots$$

We will show how to represent this as an element of $X_{n,j}$ for some j together with a bit sequence of j 0s and $k - j$ 1s. The preceding Theorem then converts the element of $X_{n,j}$ to a j-list of nonempty $3\underline{5}241$OK permutations of total length n, while the 1s in the bit sequence specify the locations for empty permutations and we have the desired k-list of permutations of total length n.

To obtain the element of $X_{n,j}$ associated with the given permutation, collapse all contiguous stars to a single star and delete the stars (if any) surrounding n. The example yields

$$\ldots * * * a \ldots * b \ldots * n * * \ldots \quad \to \quad \ldots * a \ldots * b \ldots n \ldots \quad \in X_{n,j}$$

with $j = 3$ and $j - 1 = 2$ stars. To obtain the bit sequence associated with the given permutation, replace each star immediately preceding a non-max LIT entry and n itself by 1, all other stars by 0, and then suppress the permutation entries to get a bit sequence j 1s and $k - j$ 0s; the example yields

$$\ldots * * * a \ldots * b \ldots * n * * \ldots \quad \to \quad (0\,0\,\overset{*a}{1}\,\overset{*b}{1}\,0\,\overset{n}{1}\,0\,0).$$

Clearly, the original "multiple-starred" permutation can be uniquely recovered from the element of $X_{n,j}$ and the bit sequence, and we are all done as soon as we establish Theorem 5. In fact, we will prove Theorem 5 with a specific type of bijection that permits a further reduction in the problem.

Define the LRmax factors of a permutation $m_1 L_1 m_2 L_2 \ldots m_r L_r$ to be $m_1 L_1, m_2 L_2, \ldots, m_r L_r$. Then there is a refined form of Theorem 5 that goes as follows.

Theorem 6. *Let $X_{n,k}$ denote the set of $3\underline{5}241$OK permutations on $[n]$ with $k-1$ of the non-max LIT entries marked. Let $Y_{n,k}$ denote the set of k-lists of nonempty $3\underline{5}241$OK permutations of total length n. There is a bijection from $X_{n,k}$ to $Y_{n,k}$, $\pi \to \mathbf{v}$, of the following type: it rearranges the LRmax factors of π, then suitably concatenates some of them to form a k-list of permutations and, finally, reduces each one to yield $\mathbf{v}$.*

It suffices to present this bijection for the special case where $L_1, L_2, \ldots, L_r$ are all increasing lists, equivalently, since $L_1 < L_2 < \ldots < L_r$ for a $3\underline{5}241$OK permutation, the entire list $L_1 L_2 \ldots L_r$ is increasing. (In the general case, sort each L_i and then apply the special case bijection except, just before reducing, replace each sorted L_i by the original list L_i.)

Note that permutations $\pi = m_1 L_1 m_2 L_2 \ldots m_r L_r$ for which the full list $L_1 L_2 \ldots L_r$ is increasing are precisely the 321-avoiding permutations (and automatically $3\underline{5}241$OK). In summary, we have shown in this section that the entire problem boils down to

Theorem 7. *There is a bijection of the following specific type from 321-avoiding permutations π on $[n]$ with $k-1$ of the non-max LIT entries marked to k-lists $\mathbf{v}$ of nonempty 321-avoiding permutations of total length n: it rearranges the LRmax factors of π, then suitably concatenates some of them to form a k-list of permutations and, finally, reduces each one to yield $\mathbf{v}$.*

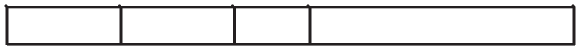
4 The Main Bijection

We now prove Theorem 7 and use

$$\underline{3}\,1\,\underline{5}\,2\,\underline{8}\,4\,6\,\underline{12}\,7\,\underline{15}\,9\,\underline{17}\,10\,11\,\underline{20}\,\underline{\underline{25}}\,\underline{\underline{\overset{*}{26}}}\,13\,\underline{\underline{27}}\,\underline{\underline{\overset{*}{28}}}\,14\,\underline{\underline{\overset{*}{29}}}\,16\,\underline{\underline{30}}\,18\,19\,21\,22\,23\,24$$

with $n = 30$ and $k = 4$ as a working example (as before, the LIT entries are double underlined and the other LRmax entries are single underlined). First, we split the permutation into panes each consisting of one or more LRmax factors. To do so, use a moving window initially covering a right factor of the permutation and consisting of k panes starting, respectively, at the very first LIT entry and the first LIT entry following each starred LIT entry. (Since the window gradually moves left, dropping the last pane and sometimes acquiring a new first pane, it is drawn below the permutation for clarity.)

3 1 5 2 8 4 6 12 7 15 9 17 10 11 20 | 25 26 13 | 27 28 14 | 29 16 | 30 18 19 21 22 23 24 |



The Initial Window

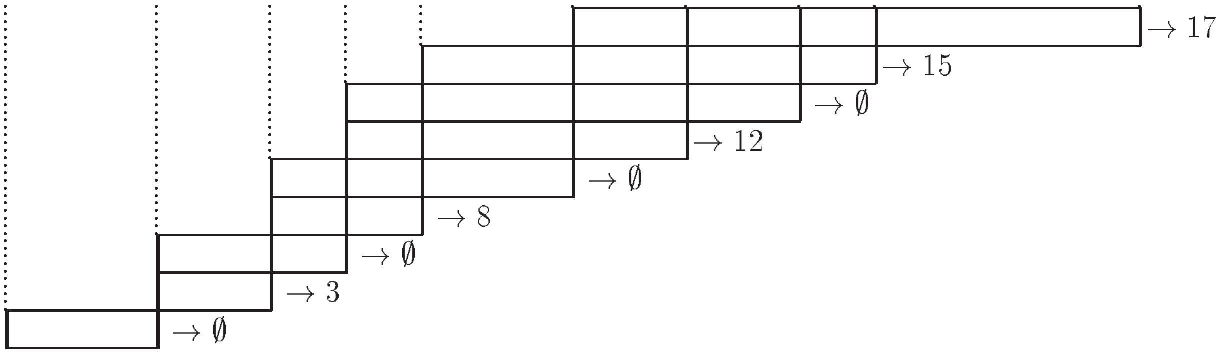
Proceed as follows until the entire permutation is covered by nonoverlapping panes. (It's just possible that the initial window covers the entire permutation but this happens only for the identity permutation. In this case associate the symbol $\emptyset$ with the window and proceed to the second step below.) Find the maximum non-LRmax entry in all but the last pane of the current window (the max of the empty set is considered to be $-\infty$) and denote this number m. Look for LRmax entries not yet empaned that exceed m. There are 2 cases.

Case 1. There is no such LRmax entry. In this case associate $\emptyset$ with the current window, and delete the last pane to obtain the next window.

Case 2. There exist LRmax entries not yet empaned that exceed m. In this case associate the smallest such entry, M say, with the current window and shift the window left by prepending a pane starting at M and deleting the last pane.

The resulting panes and the successive windows, each with an associated LRmax or $\emptyset$ symbol, are shown.

| 3 1 5 2 | 8 4 6 | 12 7 | 15 9 | 17 10 11 20 | 25 26 13 | 27 28 14 | 29 16 | 30 18 19 21 22 23 24 |



The Moving Window

Second, rearrange the panes using a k-row array formed from a list consisting of (i) the windows' associated LRmax entries and $\emptyset$s, and (ii) the first entries of the k panes of the initial window:

$$\emptyset\ 3\ \emptyset\ 8\ \emptyset\ 12\ \emptyset\ 15\ 17\ 25\ 27\ 29\ 30$$

Insert the list entries right to left into array entries lower right to upper left moving up each column in turn EXCEPT, when a $\emptyset$ is inserted, that row is "not accepting any more entries" and is henceforth skipped over. The resulting array is

$$\begin{array}{ccccc} & & \emptyset & 12 & 25 \\ & & & \emptyset & 27 \\ \emptyset & 3 & 8 & 15 & 29 \\ & & \emptyset & 17 & 30 \end{array}$$

Now form a list of k permutations by concatenating the panes initiated by the LRmax entries in each row:

12 7 25 26 13, 27 28 14, 3 1 5 2 8 4 6 15 9 29 16, 17 10 11 20 30 18 19 21 22 23 24

Finally, reduce each one to get the desired k-list of 321-avoiding permutations:

2 1 4 5 3, 2 3 1, 3 1 5 2 7 4 6 9 8 11 10, 3 1 2 6 11 4 5 7 8 9 10.

Note that LIT entries in these reduced permutations correspond to LIT entries in the original as do LRmax entries.

To reverse the process, suppose we are given a list $(\pi_1, \pi_2, \ldots, \pi_k)$ of k 321-avoiding permutations of total length n and let us use the previous example with $k = 4$ and $n = 30$. Double underline the LIT entries in each π_i and single underline the remaining LRmax entries. Form a pane from the first LIT entry to the end of each permutation.

$$\overset{\pi_1}{\underline{2}\ 1\ \boxed{\underline{\underline{4}}\ \underline{\underline{5}}\ 3}}\ ,\ \overset{\pi_2}{\boxed{\underline{\underline{2}}\ \underline{\underline{3}}\ 1}}\ ,\ \overset{\pi_3}{\underline{3}\ 1\ \underline{5}\ 2\ \underline{7}\ 4\ 6\ \underline{9}\ 8\ \boxed{\underline{\underline{11}}\ 10}}\ ,\ \overset{\pi_4}{\underline{3}\ 1\ 2\ \underline{6}\ \boxed{\underline{\underline{11}}\ 4\ 5\ 7\ 8\ 9\ 10}}$$

The Initial Panes

Place $n, n-1, n-2, \ldots$ below the LIT entries working right to left.

$$\underline{2}\ 1\ \begin{array}{|ccc|}\hline \underline{\underline{4}} & \underline{\underline{5}} & 3 \\ \hline 25 & 26 & \\ \hline \end{array}\ ,\ \begin{array}{|ccc|}\hline \underline{\underline{2}} & \underline{\underline{3}} & 1 \\ \hline 27 & 28 & \\ \hline \end{array}\ ,\ \underline{3}\ 1\ \underline{5}\ 2\ \underline{7}\ 4\ 6\ \underline{9}\ 8\ \begin{array}{|cc|}\hline \underline{\underline{11}} & 10 \\ \hline 29 & \\ \hline \end{array}\ ,\ \underline{3}\ 1\ 2\ \underline{6}\ \begin{array}{|ccccccc|}\hline \underline{\underline{11}} & 4 & 5 & 7 & 8 & 9 & 10 \\ \hline 30 & & & & & & \\ \hline \end{array}$$

Place LIT Entries

Set b = largest integer in $[n]$ not yet placed; here $b = 24$. Start with the last permutation in the list. Take the largest entry that is blank below (here, 10), place b in the blank, decrement b and repeat as long as these largest entries are (i) empaned or (ii) underlined. Then add a pane that begins at the leftmost newly added entry unless it is already empaned.

$$\underline{2}\ 1\ \begin{array}{|ccc|}\hline \underline{\underline{4}} & \underline{\underline{5}} & 3 \\ \hline 25 & 26 & \\ \hline \end{array}\ ,\ \begin{array}{|ccc|}\hline \underline{\underline{2}} & \underline{\underline{3}} & 1 \\ \hline 27 & 28 & \\ \hline \end{array}\ ,\ \underline{3}\ 1\ \underline{5}\ 2\ \underline{7}\ 4\ 6\ \underline{9}\ 8\ \begin{array}{|cc|}\hline \underline{\underline{11}} & 10 \\ \hline 29 & \\ \hline \end{array}\ ,\ \begin{array}{|cccc|ccccccc|}\hline \underline{3} & 1 & 2 & \underline{6} & \underline{\underline{11}} & 4 & 5 & 7 & 8 & 9 & 10 \\ \hline 17 & & & 20 & 30 & 18 & 19 & 21 & 22 & 23 & 24 \\ \hline \end{array}$$

Work Last Permutation

Decrement b (now $b = 16$) and proceed to the next permutation to the left in the list (considering the last to be the left neighbor of the first). Repeat until $b = 0$ and no blanks remain. The second step gives

$$\begin{array}{cc|ccc|} \cline{3-5} \underline{2} & 1 & \underline{\underline{4}} & \underline{\underline{5}} & 3 \\ \cline{3-5} & & 25 & 26 & \\ \cline{3-5} \end{array}\ ,\ \begin{array}{|ccc|} \hline \underline{\underline{2}} & \underline{\underline{3}} & 1 \\ \hline 27 & 28 & \\ \hline \end{array}\ ,\ \begin{array}{ccccccc|cc|cc|} \cline{8-11} \underline{3} & 1 & \underline{5} & 2 & \underline{7} & 4 & 6 & \underline{9} & 8 & \underline{\underline{11}} & 10 \\ \cline{8-11} & & & & & & & 15 & & 29 & 16 \\ \cline{8-11} \end{array}\ ,\ \begin{array}{|cccc|ccccccc|} \hline \underline{3} & 1 & 2 & \underline{6} & \underline{\underline{11}} & 4 & 5 & 7 & 8 & 9 & 10 \\ \hline 17 & & & 20 & 30 & 18 & 19 & 21 & 22 & 23 & 24 \\ \hline \end{array}$$

Work Penultimate Permutation

and the final result (omitting the no-longer-needed underlines) is

$$\begin{array}{|cc|ccc|} \hline 2 & 1 & 4 & 5 & 3 \\ \hline 12 & 7 & 25 & 26 & 13 \\ \hline \end{array}\ ,\ \begin{array}{|ccc|} \hline 2 & 3 & 1 \\ \hline 27 & 28 & 14 \\ \hline \end{array}\ ,\ \begin{array}{|cccc|ccc|cc|cc|} \hline 3 & 1 & 5 & 2 & 7 & 4 & 6 & 9 & 8 & 11 & 10 \\ \hline 3 & 1 & 5 & 2 & 8 & 4 & 6 & 15 & 9 & 29 & 16 \\ \hline \end{array}\ ,\ \begin{array}{|cccc|ccccccc|} \hline 3 & 1 & 2 & 6 & 11 & 4 & 5 & 7 & 8 & 9 & 10 \\ \hline 17 & 10 & 11 & 20 & 30 & 18 & 19 & 21 & 22 & 23 & 24 \\ \hline \end{array}$$

Final Result

The marked LIT entries are now retrieved as the max entry in the last pane of all but the last permutation, here $26, 28, 29$. Arrange the panes in increasing order of first entries to retrieve the original starred permutation.

5 Underlined 3- and 4-Patterns

There are $3\times 3! = 18$ similarly restricted patterns involving 3 letters and $4\times 4! = 96$ involving 4 letters. Let's call them underlined patterns; for instance, the underlined 4-pattern $4\underline{2}31$ requires that a 431 pattern occur only as part of a 4231. We say that a permutation meeting this pattern restriction satisfies $4\underline{2}31$ or is $4\underline{2}31$OK.

Only two counting sequences arise for underlined 3-patterns. Seven require the permutation to be increasing, and seven require it to be decreasing; these 14 patterns are counted by the all-1s sequence. The remaining 4 patterns are $\underline{1}\,3\,2$, $2\,1\,\underline{3}$, $2\,3\,\underline{1}$, $\underline{3}\,1\,2$. A permutation π is $\underline{1}32$OK iff $\pi(1) = 1$, $21\underline{3}$OK iff $\pi(n) = n$, and similarly for the other two. Thus these 4 underlined 3-patterns are counted by the shifted factorial sequence $\big((n-1)!\big)_{n\ge1}$.

Four different counting sequences arise for underlined 4-patterns. We will show that 64 give the Catalan numbers, 16 give the Bell numbers, 12 give sequence A051295 in OEIS, and 4 give the sequence with nth term $= (n-1)! + \sum_{k=0}^{n-2} \sum_{\substack{i,j\ge0\\ i+j\le k}} k^{\overline{i}}\,(n-2-k)^{\underline{j}}$ using rising/falling factorial notation.

The reverse and inverse of an underlined pattern are defined in the obvious way so that a permutation satisfies a given underlined pattern if and only if its reverse (resp. inverse) satisfy the reverse (resp. inverse) of the underlined pattern. The operations reverse and inverse on permutations generate a dihedral group of order 8. For example, Inverse $\circ$ Reverse $\circ$ Inverse $=$ Complement where the complement of a permutation π on $[n]$ is $n + 1 - \pi$ (entrywise). Under the action of this group, the 96 underlined 4-patterns are partitioned into equivalence classes, the members of each one all having the same counting sequence. In fact, 64 underlined 4-patterns entail avoidance of the associated 3-pattern altogether. For each of the 6 patterns of length 3, the permutations avoiding it are counted by the Catalan numbers [2]. Hence these 64 are counted by Catalan numbers A000108. (Incidentally, they split into 11 equivalence classes: 5 of size 8 and 6 of size 4.) The remaining 32—the nontrivial underlined 4-patterns—split into 5 classes as shown in the Table (the top row contains convenient class representatives).

$3\,2\,\underline{4}\,1$	$3\,1\,\underline{4}\,2$	$\underline{1}\,3\,4\,2$	$\underline{1}\,3\,2\,4$	$3\,2\,1\,\underline{4}$
$1\,3\,4\,\underline{2}$	$\underline{3}\,1\,4\,2$	$\underline{1}\,4\,2\,3$	$1\,3\,2\,\underline{4}$	$\underline{4}\,1\,2\,3$
$1\,\underline{4}\,2\,3$	$3\,1\,4\,\underline{2}$	$2\,3\,1\,\underline{4}$	$4\,2\,3\,\underline{1}$	$\underline{1}\,4\,3\,2$
$2\,3\,\underline{1}\,4$	$3\,\underline{1}\,4\,2$	$2\,4\,3\,\underline{1}$	$\underline{4}\,2\,3\,1$	$2\,3\,4\,\underline{1}$
$\underline{2}\,4\,3\,1$	$2\,4\,\underline{1}\,3$	$3\,1\,2\,\underline{4}$		
$\underline{3}\,1\,2\,4$	$\underline{2}\,4\,1\,3$	$3\,2\,4\,\underline{1}$		
$4\,\underline{1}\,3\,2$	$2\,4\,1\,\underline{3}$	$\underline{4}\,1\,3\,2$		
$4\,2\,1\,\underline{3}$	$2\,\underline{4}\,1\,3$	$\underline{4}\,2\,1\,3$		
Bell	Bell	A051295	A051295	new

The 5 equivalence classes of nontrivial underlined 4-patterns and their counting sequences

It remains to verify the counting sequences for the 5 classes.

Bell Sequence Consider two canonical ways to represent a set partition of $[n]$.

Canonical Increasing largest entry first in each block, rest of block increasing, blocks arranged in increasing order of first element, as in 412-6-735.

Canonical Decreasing each block decreasing, blocks arranged in increasing order of first element, as in 421-6-753.

Theorem 8. *Splitting a permutation on $[n]$ into its LRmax factors $m_1L_1, \ldots, m_rL_r$ is a bijection from*

(*i*) $32\underline{4}1$OK *permutations on $[n]$ to set partitions of $[n]$ in canonical increasing form.*

(*ii*) $31\underline{4}2$OK *permutations on $[n]$ to set partitions of $[n]$ in canonical decreasing form.*

Proof (i) If π is $32\underline{4}1$OK yet some L_i fails to be increasing, then there exist u preceding v in L_i with $u > v$ and m_iuv is an offending 321 pattern. Conversely, if each L_i is increasing, then a 321 must have its "2" and "1" in different blocks and π is $32\underline{4}1$OK.

(ii) If π is $31\underline{4}2$OK yet some LRmax factor m_iL_i fails to be decreasing, then there exist u preceding v in L_i with $u < v$ and m_iuv is an offending 312 pattern. Conversely, if the LRmax factors of π are all decreasing, then a 312 must have its "1" and "2" in different blocks and π is $31\underline{4}2$OK. □

Since set partitions are counted by the Bell numbers A000110, this verifies the first 2 columns in the Table.

A051295 Sequence Let us count $\underline{1}324$OK permutations on $[n]$. We claim each such permutation π has the form $\pi_1 1 \pi_2$ where π_1 is a $\underline{1}324$OK permutation on $[n-k+2, n]$ with k the position of 1 in π, and π_2 is an arbitrary permutation on $[2, n-k+1]$. To see the claim, observe that each such π is $\underline{1}324$OK. Conversely, if π is $\underline{1}324$OK, all entries preceding 1 in π necessarily exceed all entries following 1 or else there is a subpermutation $u1v$ with $u < v$ forming an offending 324 pattern. While π_1 must be $\underline{1}324$OK, the entry 1 relieves π_2 of any obligation, and the claim follows. This decomposition implies that the number u_n of $\underline{1}324$OK permutations on $[n]$ satisfies $u_n = \sum_{k=1}^{n} u_{k-1}(n-k)!$, and hence (u_n) is sequence A051295 in OEIS.

Next, we count $\underline{1}342$OK permutations on $[n]$ by position k of 1. If 1 occurs in first position, the rest of the permutation is arbitrary. Otherwise, we claim the permutation has the form (with 1 in position $k \geq 2$)

$$a_1 \dots a_{k-1}\, 1\, \pi[2, a_{k-1}-1]\, \pi[a_{k-1}+1, a_{k-2}-1]\, \dots\, \pi[a_2+1, a_1-1]\, \pi[a_1+1, n]$$

where $a_1 > a_2 > \dots > a_{k-1}$ and $\pi[b, c]$ denotes an arbitrary permutation on the interval of integers $[b, c]$ (understood to be the empty permutation if $b > c$).

Such permutations are clearly $\underline{1}342$OK. Conversely, given a $\underline{1}342$OK permutation, the entries $a_1, \dots, a_{k-1}$ preceding 1 are necessarily decreasing left to right for otherwise there is a subpermutation $a\, b\, 1$ with $a < b$. This is a 342 pattern with no hope of the required "1". And for $2 \leq i \leq k-1$, all elements of $[a_i + 1, a_{i-1} - 1]$ precede all elements of $[a_{i-1}+1, a_{i-2}-1] \sqcup \dots \sqcup [a_1+1, n]$ else some $b > a_{i-1}$ precedes some $c < a_{i-1}$ in the permutation and $a_{i-1}\, b\, c$ is an offending 342 pattern because all entries preceding a_{i-1} exceed a_{i-1}. The claim follows.

With $u_{n,k}$ the number of $\underline{1}342$OK permutations on $[n]$ for which 1 occurs in position k, this decomposition implies the generating function identity for each $k \geq 1$

$$\sum_{n\geq 0} u_{n,k} x^n = x^k \Big(\sum_{n \geq 0} n! x^n \Big)^k$$

and so $(u_{n,k})$ forms sequence A084938 in OEIS with row sums A051295. This verifies columns 3 and 4 of the Table.

The fact that $\underline{1}324$ and $\underline{1}342$ are Wilf-equivalent (that is, have the same counting sequence) despite being in different equivalence classes can be explained by a simple bijection from $\underline{1}324$OK permutations to $\underline{1}342$OK permutations. Factor a $\underline{1}324$OK permutation π as

$$m_1\, L_1\, m_2\, L_2\, \dots\, m_r\, L_r$$

where $m_1, m_2, \dots$ are now the left-to-right *minima* of π. Then reassemble as

$$m_1\, m_2\, \dots\, m_r\, L_r\, L_{r-1}\, \dots\, L_1.$$

This works because L_i is a permutation on $[m_i + 1, m_{i-1} - 1]$ for $1 \le i \le r$ $(m_0 := n + 1)$ for otherwise there is an entry b following m_i in π with $b > m_{i-1}$ and $m_{i-1}\, m_i\, b$ is an offending 324 pattern.

New Sequence The 321$\underline{4}$OK permutations on $[n]$ can be counted directly by position of n. If n occurs in last position, the rest of the permutation is unrestricted—$(n-1)!$ choices. If n occurs in position $n-1$ and the last entry is i, then $i+1, i+2, \ldots, n$ must occur in that order else i terminates an offending 321. Thus the permutation is determined by the positions of $1, 2, \ldots, i-1$, for which there are $(n-2)^{\underline{i-1}}$ choices. Here $j^{\underline{i}} = j(j-1)\ldots(j-i+1)$ is the falling factorial.

Now suppose n occurs in position $k \le n-2$. The subpermutation following n must be increasing and hence has the form $1 \le a < y_1 < y_2 < \ldots < y_{n-k-2} < b < n$. Fix a and b and let $(x_i)_{i=1}^r = (a, b) \backslash (y_i)_{i=1}^{n-k-2}$ entailing $r = (b-a-1) - (n-k-2) = b - t$ with $t = a + n - k - 1$. The subpermutation preceding n has no restriction on the entries $< a$ and placing these entries gives $(k-1)^{\underline{a-1}}$ choices. However, its entries $> b$ must be increasing left to right (else there exist $u > v > b$ with u preceding v and $u\, v\, b$ is an offending 321) and must all lie to the right of the x_is (else some $u > b$ precedes some x_i and $u > b > x_i > a$ implies $u\, x_i\, a$ is an offending 321). Thus the subpermutation preceding n and involving entries $> a$ yields only $r!$ choices: arrange the x_is. All told, the desired count is (with $t := a + n - k - 1$ and $n \ge 1$)

$$(n-1)! + \sum_{i=1}^{n-1} (n-2)^{\underline{i-1}} + \sum_{k=1}^{n-2} \sum_{a=1}^{k} \sum_{b=t}^{n-1} (k-1)^{\underline{a-1}} \binom{b-a-1}{n-k-2} (b-t)! =$$

$$(n-1)! + \sum_{k=0}^{n-2} \sum_{\substack{i,j \ge 0 \\ i+j \le k}} k^{\overline{i}}\, (n-2-k)^{\underline{j}}.$$

This sequence begins $(1, 1, 2, 5, 15, 55, 248, 1357, 8809, 66323, 568238, \ldots)_{n \ge 0}$ and is entrywise $\ge$ the next largest counting sequence A051295 $= (1, 1, 2, 5, 15, 54, 235, 1237 \ldots)$; both seem to grow asymptotically as $(n-1)!$. □

A pattern restriction similar to one of our underlined 4-patterns arises in a recent paper [4] on "patience sorting" of a deck of cards into piles, namely, the restriction that a 342 pattern in which the "4" and "2" are contiguous occurs only as part of a 3142 pattern. It is almost immediate, however, that the permutations satisfying this seemingly weaker restriction coincide with our 3$\underline{1}$42OK permutations (and hence are counted by the Bell numbers).

Added in Proof. The main result of this paper is used to obtain another manifestation of the eigensequence for composition A030266 strictly in terms of pattern avoidance in [5].

Acknowledgment

I thank the referee for a careful reading of the paper and several helpful remarks.

References

[1] Mira Bernstein and N. J. A. Sloane, Some canonical sequences of integers, *Linear Algebra and its Applications*, Vol. 226-228 (1995), pp. 57–72; errata Vol. 320 (2000), p. 210, arXiv math.CO/0205301.

[2] Richard P. Stanley, *Enumerative Combinatorics Vol. 2*, Cambridge University Press, 1999. Exercise 6.19 and related material on Catalan numbers are available online at http://www-math.mit.edu/~rstan/ec/ .

[3] S. Dulucq, S. Gire, and O. Guibert, A combinatorial proof of J. West's conjecture, *Discrete Math.* 187 (1998) 71–96.

[4] Alexander Burstein, Combinatorics of Patience Sorting Piles, arXiv math.CO/0506358, 18 Jun 2005, 19 pp.

[5] David Callan, A Wilf Equivalence Related to Two Stack Sortable Permutations, 2005, preprint, arXiv math.CO/0510211 .

2000 *Mathematics Subject Classification*: 05A15.
Keywords: eigensequence, $3\underline{5}241$OK permutation, restricted pattern, underlined pattern.

(Concerned with sequences A000108, A000110, A030266, A051295, A084938, and A110447.)

Received August 4 2005; revised version received December 19 2005. Published in *Journal of Integer Sequences*, January 3 2006.

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