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# Hyper-Sums of Powers of Integers and the Akiyama-Tanigawa Matrix

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**Abstract**

In this short essay, we consider hyper-sums of powers of integers, namely sums of power sums. We can obtain easily their formulae as polynomials by using formulae for ordinary sums of powers of integers. The coefficient of the first-degree term in each polynomial coincides with the matrix element of the Akiyama-Tanigawa matrix.

## 1 The Akiyama-Tanigawa matrix

The Akiyama-Tanigawa algorithm which is reformulated by K. W. Chen [1] and M. Kaneko [3] is described by the sequence $\{a_{n,m}\}$;

$$a_{n,m} = (m+1)(a_{n-1,m} - a_{n-1,m+1}), \quad n \geq 1,\ m \geq 0 \tag{1}$$

with an initial sequence $a_{0,m}$ $(m = 0, 1, 2, \ldots)$.
Let $B_n(x)$ be the $n$-th Bernoulli polynomial

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \qquad |t| < 2\pi$$

When the initial sequence is $a_{0,m} = 1/(m+1)$ then this algorithm yields $B_n(1)$ as the leading element $a_{n,0}$. Here $B_n(1)$ are, in fact, the Bernoulli numbers $B_n$, with the sole exception of $n = 1$, for which $B_1(1) = -B_1$ holds. We assume this situation here and the Akiyama-Tanigawa matrix $\{a_{n,m}\}$ is then

| $a_{n,0}$ | $a_{n,1}$ | $a_{n,2}$ | $a_{n,3}$ | $a_{n,4}$ | $a_{n,5}$ | $a_{n,6}$ | $a_{n,7}$ | $a_{n,8}$ | $a_{n,9}$ | $a_{n,10}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $1$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ | $\frac{1}{8}$ | $\frac{1}{9}$ | $\frac{1}{10}$ | $\frac{1}{11}$ | $\cdots$ |
| $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ | $\frac{1}{8}$ | $\frac{1}{9}$ | $\frac{1}{10}$ | $\frac{1}{11}$ | $\cdots$ | |
| $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{3}{20}$ | $\frac{2}{15}$ | $\frac{5}{42}$ | $\frac{3}{28}$ | $\frac{7}{72}$ | $\frac{4}{45}$ | $\frac{9}{110}$ | $\cdots$ | | |
| $0$ | $\frac{1}{30}$ | $\frac{1}{20}$ | $\frac{2}{35}$ | $\frac{5}{84}$ | $\frac{5}{84}$ | $\frac{7}{120}$ | $\frac{28}{495}$ | $\cdots$ | | | |
| $-\frac{1}{30}$ | $-\frac{1}{30}$ | $-\frac{3}{140}$ | $-\frac{1}{105}$ | $0$ | $\frac{1}{140}$ | $\frac{49}{3960}$ | $\cdots$ | | | | |
| $0$ | $-\frac{1}{42}$ | $-\frac{1}{28}$ | $-\frac{4}{105}$ | $-\frac{1}{28}$ | $-\frac{29}{924}$ | $\cdots$ | | | | | |
| $\frac{1}{42}$ | $\frac{1}{42}$ | $\frac{1}{140}$ | $-\frac{1}{105}$ | $-\frac{5}{231}$ | $\cdots$ | | | | | | |
| $0$ | $\frac{1}{30}$ | $\frac{1}{20}$ | $\frac{8}{165}$ | $\cdots$ | | | | | | | |
| $-\frac{1}{30}$ | $-\frac{1}{30}$ | $\frac{1}{220}$ | $\cdots$ | | | | | | | | |
| $0$ | $-\frac{5}{66}$ | $\cdots$ | | | | | | | | | |
| $\frac{5}{66}$ | $\cdots$ | | | | | | | | | | |
| $\cdots$ | | | | | | | | | | | |

Each matrix element $a_{n,m}$ is determined uniquely by the recursive rule (1) and the initial values.

## 2 Explicit formulation

In this section, we determine an explicit expression of $a_{n,m}$.

**Proposition 1.**

$$a_{n,m} = \frac{1}{m!}\sum_{i=0}^{m}(-1)^i \begin{bmatrix} m+1 \\ i+1 \end{bmatrix} B_{n+i}(1), \qquad n \in \mathbb{N}. \tag{2}$$

*where* $\left[{n \atop k}\right]$ *are the Stirling numbers of the first kind (Sloane's A008275, cf. [2], Ch. 6.1).*

*Proof.* First, we shall see that this $a_{n,m}$ satisfies the recursive rule (1). To do this, we use the well known recurrence relation $\left[{n+1 \atop k}\right] = n\left[{n \atop k}\right] + \left[{n \atop k-1}\right]$.

$$\begin{aligned}
&(m+1)(a_{n-1,m} - a_{n-1,m+1}) \\
&= (m+1)\Big(\frac{1}{m!}\sum_{i=0}^{m}(-1)^i \begin{bmatrix} m+1 \\ i+1 \end{bmatrix} B_{n+i-1}(1) - \frac{1}{(m+1)!}\sum_{i=0}^{m+1}(-1)^i \begin{bmatrix} m+2 \\ i+1 \end{bmatrix} B_{n+i-1}(1)\Big) \\
&= \frac{1}{m!}\sum_{i=0}^{m+1}(-1)^i\Big((m+1)\begin{bmatrix} m+1 \\ i+1 \end{bmatrix} - \begin{bmatrix} m+2 \\ i+1 \end{bmatrix}\Big) B_{n+i-1}(1) \\
&= \frac{1}{m!}\sum_{i=0}^{m+1}(-1)^{i+1}\begin{bmatrix} m+1 \\ i \end{bmatrix} B_{n+i-1}(1) \\
&= a_{n,m},
\end{aligned}$$

where we have used the fact that $\left[{m+1 \atop 0}\right] = 0$.

Next, to show that the initial sequence $a_{0,m} = 1/(m+1)$, we use an expression that can be found in Kaneko [3]. Let $\left\{{n \atop k}\right\}$ be the Stirling numbers of the second kind (Sloane's A008277, cf. [2], Ch. 6.1). Then

$$B_n(1) = \sum_{i=0}^{n} \frac{(-1)^i i!}{i+1} \left\{ {n+1 \atop i+1} \right\}, \qquad n \geq 0,$$

By the well known inversion formula (cf. [2], Ch. 6.1),

$$\sum_{k=m}^{n} \left[ {n \atop k} \right] \left\{ {k \atop m} \right\} (-1)^{n-k} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases},$$

we can compute what we want easily.

$$\begin{aligned}
(m+1)a_{0,m} &= \frac{m+1}{m!} \sum_{i=0}^{m} (-1)^i \left[ {m+1 \atop i+1} \right] B_i(1) \\
&= \frac{m+1}{m!} \sum_{k=0}^{m} \frac{k!(-1)^k}{k+1} \sum_{i=k}^{m} (-1)^i \left[ {m+1 \atop i+1} \right] \left\{ {i+1 \atop k+1} \right\} \\
&= \frac{(m+1)(-1)^m}{m!} \sum_{k=0}^{m} \frac{k!(-1)^k}{k+1} \sum_{i=k}^{m} (-1)^{m-i} \left[ {m+1 \atop i+1} \right] \left\{ {i+1 \atop k+1} \right\} \\
&= 1
\end{aligned}$$

From the recursive rule (1) and initial values $a_{0,m}$, the proof of this proposition is done and we have an explicit expression of $a_{n,m}$. $\square$

# 3 Hyper-sum polynomials

Now we shall consider "hyper-sums" of powers of integers, namely sums of power sums. Let $P_k^{(0)}(n) = \sum_{i=1}^{n} i^k$, $P_k^{(1)}(n) = \sum_{i=1}^{n} P_k^{(0)}(i)$, $P_k^{(2)}(n) = \sum_{i=1}^{n} P_k^{(1)}(i), \ldots$, and $P_k^{(m)}(n) = \sum_{i=1}^{n} P_k^{(m-1)}(i)$, with $k$ and $n$ positive integers. $P_k^{(m)}(n)$ is a $(k+m+1)$-th degree polynomial in $n$ and $P_k^{(0)}(n)$ is the ordinary power sum $1^k + 2^k + 3^k + \cdots + n^k$. Let $c_{k,m}$ be the coefficient of the first-degree term in $P_k^{(m)}(n)$.

Since the formulae for $P_k^{(0)}(n)$ have been investigated for a long time and abundant methods of determining them have been developed, we can use some of them. Here we choose the formula including the Stirling numbers of the second kind, which is for instance seen in the paper of Srivastava, Joshi and Bisht [4].

$$P_k^{(0)}(n) = \sum_{i=1}^{k} i! \binom{n+1}{i+1} \left\{ {k \atop i} \right\}, \qquad k \in \mathbb{N}$$

where the binomial coefficient $\binom{n+1}{i+1}$ is taken to be zero for $n < i$. Here we use the well known identity (cf. [2], Ch. 5.1),

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1},$$

where the binomial coefficient $\binom{k}{m}$ is zero for $k < m$. This yields

$$\sum_{n=0}^{N} \binom{n+1}{i+1} = \sum_{n=i}^{N} \binom{n+1}{i+1} = \sum_{n=i+1}^{N+1} \binom{n}{i+1} = \binom{N+2}{i+2}.$$

So we can get easily

$$P_k^{(1)}(n) = \sum_{i=1}^{k} i! \binom{n+2}{i+2} \left\{ {k \atop i} \right\}, \qquad k \in \mathbb{N}.$$

Thus, by means of the successive computations, we arrive at

$$P_k^{(m)}(n) = \sum_{i=1}^{k} i! \binom{n+m+1}{i+m+1} \left\{ {k \atop i} \right\}, \qquad k \in \mathbb{N}.$$

From the fact that $\binom{n+m+1}{i+m+1}$ has the factor $n$ for $i \geq 1$, $P_k^{(m)}(n)$ is divisible by $n$. Therefore, to find $c_{k,m}$, we shall use the following relation.

$$c_{k,m} = \left. \frac{P_k^{(m)}(n)}{n} \right|_{n=0}$$

Since

$$\left. \frac{1}{n} \binom{n+m}{k+m} \right|_{n=0} = \frac{m!(k-1)!(-1)^{k-1}}{(k+m)!}.$$

We obtain

$$c_{k,m} = \sum_{i=1}^{k} \frac{(m+1)!(i-1)!i!(-1)^{i-1}}{(i+m+1)!} \left\{ {k \atop i} \right\}, \qquad k \in \mathbb{N}. \tag{3}$$

Now only $c_{0,m}$ has been left for us. But it is not difficult to see it, because the identity $P_0^{(1)}(n) = P_1^{(0)}(n) = \sum_{i=1}^{n} i$ yields $P_0^{(m)}(n) = P_1^{(m-1)}(n)$ immediately. Thus we have $c_{0,m} = c_{1,m-1} = 1/(m+1)$. By combining these consequences, we can state the following proposition.

**Proposition 2.**

$$c_{k,m} = \begin{cases} \dfrac{1}{m+1}, & k = 0 \\ \displaystyle\sum_{i=1}^{k} \frac{(m+1)!(i-1)!i!(-1)^{i-1}}{(i+m+1)!} \left\{ {k \atop i} \right\}, & k \geq 1 \end{cases}$$

## 4 Equality between $a_{n,m}$ and $c_{n,m}$

As one of the important consequences in this essay, let us see the equality between $a_{n,m}$ and $c_{n,m}$. To do this, we shall see the recurrence relation whom $c_{n,m}$ satisfies.

**Lemma 1.**

$$c_{n,m} = (m+1)(c_{n-1,m} - c_{n-1,m+1}), \qquad n \geq 1$$

*Proof.* For the initial case $n = 1$, by proposition 2 we have $c_{0,m} = 1/(m+1)$ and $c_{1,m} = 1/(m+2)$ without difficulty. So we can confirm that $c_{1,m} = (m+1)(c_{0,m} - c_{0,m+1})$ easily.

For the case $n \geq 2$, we use the well known recurrence relation $\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$.

$$\begin{aligned}
c_{n,m} &= \sum_{i=1}^{n} \frac{(m+1)!(i-1)!i!(-1)^{i-1}}{(i+m+1)!} \left\{ {n \atop i} \right\} \\
&= \sum_{i=1}^{n} \frac{(m+1)!(i-1)!i!(-1)^{i-1}}{(i+m+1)!} (i \left\{ {n-1 \atop i} \right\} + \left\{ {n-1 \atop i-1} \right\}) \\
&= \sum_{i=1}^{n-1} \frac{(m+1)!i!i!(-1)^{i-1}}{(i+m+1)!} \left\{ {n-1 \atop i} \right\} - \sum_{i=1}^{n-1} \frac{(m+1)!i!(i+1)!(-1)^{i-1}}{(i+m+2)!} \left\{ {n-1 \atop i} \right\} \\
&= \sum_{i=1}^{n-1} \frac{(m+1)!i!i!(-1)^{i-1}}{(i+m+1)!} \cdot \frac{m+1}{i+m+2} \left\{ {n-1 \atop i} \right\}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(m+1)(c_{n-1,m} - c_{n-1,m+1}) \\
&= (m+1)\Big(\sum_{i=1}^{n-1} \frac{(m+1)!(i-1)!i!(-1)^{i-1}}{(i+m+1)!} \left\{ {n-1 \atop i} \right\} - \sum_{i=1}^{n-1} \frac{(m+2)!(i-1)!i!(-1)^{i-1}}{(i+m+2)!} \left\{ {n-1 \atop i} \right\}\Big) \\
&= (m+1)\Big(\sum_{i=1}^{n-1} \frac{(m+1)!(i-1)!i!(-1)^{i-1}}{(i+m+1)!} \cdot \frac{i}{i+m+2} \left\{ {n-1 \atop i} \right\}\Big) \\
&= \sum_{i=1}^{n-1} \frac{(m+1)!i!i!(-1)^{i-1}}{(i+m+1)!} \cdot \frac{m+1}{i+m+2} \left\{ {n-1 \atop i} \right\}.
\end{aligned}$$

□

Thus this lemma and the fact $c_{0,m} = 1/(m+1)$ yield the next consequence immediately.

**Proposition 3.**

$$c_{n,m} = a_{n,m}.$$

**Corollary 1.**

$$\frac{1}{m!} \sum_{i=0}^{m} (-1)^i \left[ {m+1 \atop i+1} \right] B_{n+i}(1) = \sum_{i=1}^{n} \frac{(m+1)!(i-1)!i!(-1)^{i-1}}{(i+m+1)!} \left\{ {n \atop i} \right\}, \qquad n \in \mathbb{N}.$$

*Proof.* It is clear from (2) and (3). □

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(Concerned with sequences A008275 and A008277.)

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