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The Fibonacci Number of a Grid Graph and a New Class of Integer Sequences

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Abstract

Given a grid graph G of size mn, we study the number $i(m,n)$ of independent sets in G, as well as $b(m,n)$, the number of maximal such sets. It turns out that the initial cases $b(1,n)$ and $b(2,n)$ lead to a Padovan and a Fibonacci sequence. To determine $b(m,n)$ for $m>2$ we present an adaptation of the transfer matrix method, well known for calculating $i(m,n)$. Finally, we apply our method to obtain explicit values of $b(m,n)$ for $m=3,4,5$ and provide the corresponding generating functions.

1 Introduction

Let $G_{m,n}=(V,E)$ be the *grid graph of size mn* with vertex set

$$V=\{(i,j):1\le i\le m, 1\le j\le n\}$$

and edge set

$$E=\{\{(i,j),(i',j')\}:|i-i'|+|j-j'|=1\}.$$

A set $I\subseteq V$ is called an *independent set* if no two of its elements are joined by an edge. We let $\mathcal{I}_{m,n}$ denote the collection of independent sets in $G_{m,n}$ and $i(m,n)$ the total number of such sets. A *maximal independent set* B in $G_{m,n}$ (maximal with respect to set-inclusion) is called a *basis*. Again, $\mathcal{B}_{m,n}$ represents the collection of bases in $G_{m,n}$ and $b(m,n)$ is the total number of such sets. We are interested in calculating $i(m,n)$ and $b(m,n)$ for any $m,n\in\mathbb{N}$

together with the associated generating functions. The study of $i(m,n)$ is closely related to the "hard-square model" as it is used in statistical physics. Of particular interest is the so-called "hard-square entropy constant" defined to be $\lim_{m,n\to\infty} i(m,n)^{1/mn}$ (see Baxter et al. [1]). Applications also include tiling and, more recently, efficient coding schemes in data storage (see Roth et al. [5]). A basic reference for work on $i(m,n)$ is Calkin and Wilf [3]. More recently, Burstein et al. [2] have enumerated independent sets associated with several classes of (almost) regular graphs and calculated the corresponding generating functions.

Whereas this study of $b(m,n)$ is new, distinguishing bases from independent sets is quite common: just think of matroid theory or weighted independent set problems over graphs with *non-negative* weights on the vertices. The question for the true value of $\lim_{m,n\to\infty} b(m,n)^{1/mn}$, however, is open. We recently learned (see Weigt and Hartmann [9]) that maximal independent sets may be associated with meta-stable liquid states of certain hard-core lattice gas models under compaction.

The main question we will answer in this paper is that of calculating $b(m,n)$ for $m,n > 2$ by a proper adaptation of the "transfer matrix method" as designed by Engel [4] for the calculation of $i(m,n)$. The initial cases $b(1,n)$ and $b(2,n)$ turn out to produce a Padovan and a Fibonacci sequence, respectively. Padovan sequences have little history; their study goes back only to the beginning of the last century. In the 1990's the architect Richard Padovan popularized these sequences and some closely related concepts such as the "plastic constant", the Padovan-analogue to the golden number. We refer the interested reader to Stewart's recreational article [8] for further details. Since $i(m,n)$ is also called the *Fibonacci number* of $G_{m,n}$ we think that, this analogy in mind, it would be convenient to call $b(m,n)$ the *Padovan number* of $G_{m,n}$.

This paper is organised as follows: the next section is devoted to the Fibonacci number $i(m,n)$. We review the transfer matrix method for the calculation of $i(m,n)$ and the results obtained for $m = 1,2,3,4$. Section 3 addresses the combinatorial structure of $\mathcal{B}_{1,n}$ and $\mathcal{B}_{2,n}$ as well as their respective cardinalities. In section 4 we present the new method to calculate $b(m,n)$, and in section 5 we apply our method to explicitly describe $b(m,n)$ for $m = 3,4,5$. Whenever convenient, we indicate the associated generating functions that we calculated using Maple. We conclude with a discussion of open problems.

2 The transfer matrix method for the calculation of $i(m,n)$

Let us start with the notion of "orthogonality". The *j-th column* $G^j = (V^j, E^j)$ of $G_{m,n}$ is the subgraph induced by the vertex set $\{(1,j),(2,j),\ldots,(m,j)\}$ with $\mathcal{I}^j$ denoting the associated collection of independent sets. If now, for $j_1, j_2 \in \{1,\ldots,n\}$, $I = \{(i_1,j_1),\ldots,(i_p,j_1)\}$ and $K = \{(k_1,j_2),\ldots,(k_q,j_2)\}$ are members of $\mathcal{I}^{j_1}$ and $\mathcal{I}^{j_2}$, respectively, we say that *I is orthogonal to K*, $I \perp K$, whenever $\{i_1,\ldots,i_p\}\cap\{k_1,\ldots,k_q\} = \emptyset$. Moreover, for any $I \in \mathcal{I}^j$ we define the set $l(I) = \{(i_1, 1), \ldots, (i_p, 1)\}$, which again is a member of $\mathcal{I}^1$. For any $k \in \{1, \ldots, n\}$, we will also have to count the number of independent sets in $G_{m,k}$, whose intersection with V^k equals a given set I, a number that we represent by $i(m, k, I)$. The basic idea of the *transfer matrix method* (due to Engel [4], see also Stanley [7]) is to start with the values $i(m, 1, I) = 1$ for $I \in \mathcal{I}^1$, and to calculate, for $k = 1, 2, \ldots, n-1$,

$$i(m, k+1, J) = \sum_{I \in \mathcal{I}^k \text{ and } I \perp J} i(m, k, I) \ \forall\ J \in \mathcal{I}^{k+1}, \tag{1}$$

in order to finally obtain

$$i(m, n) = \sum_{I \in \mathcal{I}^n} i(m, n, I). \tag{2}$$

Equations (1) and (2) can be formulated as matrix-vector products by introducing the following matrix $T \in \{0, 1\}^{F_{m+2} \times F_{m+2}}$:

$$\forall\ J, I \in \mathcal{I}^1 \text{ set } T_{JI} = 1 \text{ iff } J \perp I.$$

Clearly, for $k = 1, \ldots, n-1$,

$$i(m, k+1, J) = \sum_{I \in \mathcal{I}^k} T_{JI} \cdot i(m, k, I) \ \forall\ J \in \mathcal{I}^{k+1}$$

and at the end $i(m, n) = \mathbf{1} T^{n-1} \mathbf{1}$.

What really happens with this method is the following: at any stage k, $\mathcal{I}_{m,k}$ is partitioned into a fixed number of F_{m+2} classes, each of which is represented by a unique independent set in G^k. During the following stages only the size of the classes is increasing, and a simple summation at the end yields $i(m, n)$. One may think of proceeding the same way to calculate $b(m, n)$. However, as the example with $m = n = 3$ shows, not only do we lose the integrality of the transfer matrix. After a few steps we get fractional values supposed to represent the cardinalities of the classes. What we really need is to refine our partition, and this will be the subject of section 4.

What are the results we can obtain for small values of m? It is well known that $i(1, n)$ equals F_{n+2}, the $(n+2)$nd Fibonacci number. Other recurrence formulas are available for $m = 2, 3, 4$ (see Sloane [6]). In the following we just resume the related sequences together with their generating functions:

$$(i(1, n))_{n \in \mathbb{N}} = (2, 3, 5, 8, 13, 21, 34, 55, \ldots)$$

with

$$\sum_{n \geq 1} i(1, n) x^n = \frac{2x + x^2}{1 - x - x^2}\ ;$$

$$(i(2,n))_{n\in\mathbb{N}} = (3, 7, 17, 41, 99, 239, 577, 1393, \ldots)$$

with

$$\sum_{n\geq 1} i(2,n)x^n = \frac{3x + x^2}{1 - 2x - x^2} \ ;$$

$$(i(3,n))_{n\in\mathbb{N}} = (5, 17, 63, 227, 827, 2999, 10897, 39561, \ldots)$$

with

$$\sum_{n\geq 1} i(3,n)x^n = \frac{5x + 7x^2 - x^3 - x^4}{1 - 2x - 6x^2 + x^4} \ ;$$

and finally

$$(i(4,n))_{n\in\mathbb{N}} = (8, 41, 227, 1234, 6743, 36787, 200798, 1095851, \ldots)$$

with

$$\sum_{n\geq 1} i(4,n)x^n = \frac{8x + 9x^2 - 9x^3 - 3x^4 + x^5}{1 - 4x - 9x^2 + 5x^3 + 4x^4 - x^5} \ .$$

If we work "backwards", i.e., from column n to column 1 of $G_{m,n}$, we are able to express $i(m,n)$ by a general formula, which for its "sum of products-form" might be interesting in its own. For this and any $I \in \mathcal{I}^1$, we denote with $s(m,I)$ the number of independent sets in $\mathcal{I}^1$, that are orthogonal to I. Using the fact that $i(m,1) = F_{m+2}$ and assuming that $I = \{i_1, \ldots, i_p\}$ with $i_1 < i_2 < \cdots < i_p$, $s(m,I)$ can be written as follows:

$$s(m,I) = F_{i_1+1}F_{i_2-i_1+1}\cdots F_{i_p-i_{p-1}+1}F_{m-i_p+2}.$$

We will not use this explicit form of $s(m,I)$ further but, instead, show how to represent $i(m,n)$ as a function of $s(m,I)$'s. We start with $n = 2$. Clearly,

$$i(m,2) = \sum_{J\in\mathcal{I}^2} i(m,2,J).$$

But $i(m,2,J)$ is just the number of those sets in $\mathcal{I}^1$, that are orthogonal to $l(J)$; therefore,

$$i(m,2) = \sum_{I\in\mathcal{I}^1} s(m,I).$$

For $n = 3$, we have again

$$i(m,3) = \sum_{J\in\mathcal{I}^3} i(m,3,J).$$

But

$$i(m,3,J) = \sum_{K\in\mathcal{I}^2 \text{ and } K\perp J} i(m,2,K);$$

thus,

$$i(m,3)=\sum_{K\in\mathcal{I}^2} s(m,l(K))i(m,2,K),$$

which equals

$$\sum_{I\in\mathcal{I}^1}(s(m,I))^2.$$

For $n=4$, we have

$$i(m,4)=\sum_{J\in\mathcal{I}^4} i(m,4,J)=\sum_{J\in\mathcal{I}^4}\sum_{K\in\mathcal{I}^3\text{ and }K\perp J} i(m,3,K)$$

$$=\sum_{K\in\mathcal{I}^3} s(m,l(K))i(m,3,K)=\sum_{K\in\mathcal{I}^3} s(m,l(K))\left(\sum_{L\in\mathcal{I}^2\text{ and }L\perp K} i(m,2,L)\right),$$

which equals

$$\sum_{I\in\mathcal{I}^1} s(m,I)\left(\sum_{J\in\mathcal{I}^1\text{ and }J\perp I} s(m,J)\right).$$

Finally, for $n=5$ we obtain

$$i(m,5)=\sum_{J\in\mathcal{I}^5} i(m,5,J)=\sum_{J\in\mathcal{I}^5}\sum_{K\in\mathcal{I}^4\text{ and }K\perp J} i(m,4,K)$$

$$=\sum_{K\in\mathcal{I}^4} s(m,l(K))i(m,4,K)=\sum_{K\in\mathcal{I}^4} s(m,l(K))\left(\sum_{L\in\mathcal{I}^3\text{ and }L\perp K}\left(\sum_{M\in\mathcal{I}^2\text{ and }M\perp L} s(m,l(M))\right)\right),$$

which equals

$$\sum_{I\in\mathcal{I}^1} s(m,I)\left(\sum_{K\in\mathcal{I}^1\text{ and }K\perp I}\left(\sum_{J\in\mathcal{I}^1\text{ and }J\perp K} s(m,J)\right)\right).$$

Since in this last expression, any $s(m,J)$ is counted $s(m,I\cup J)$ times,

$$i(m,5)=\sum_{I,J\in\mathcal{I}^1} s(m,I)s(m,I\cup J)s(m,J).$$

We proceed the same way for any $n>5$ to obtain the following result.

Theorem 2.1 *We have*

(i) $i(m,1)=F_{m+2}$, *(ii)* $i(m,2)=\sum_{I\in\mathcal{I}^1} s(m,I)$, *(iii)* $i(m,3)=\sum_{I\in\mathcal{I}^1}(s(m,I))^2$,

(iv) $i(m,4) = \sum_{I\in\mathcal{I}^1} s(m,I)\left(\sum_{J\in\mathcal{I}^1 \text{ and } J\perp I} s(m,J)\right)$,

(v) $i(m,5) = \sum_{I,J\in\mathcal{I}^1} s(m,I)s(m,I\cup J)s(m,J)$.

More generally, for $p \geq 3$,

(vi) $i(m,2p) = \sum_{I_1,\ldots,I_p\in\mathcal{I}^1} s(m,I_1)s(m,I_1\cup I_2)\cdots s(m,I_{p-2}\cup I_{p-1})\left(\sum_{J\in\mathcal{I}^1 \text{ and } J\perp I_{p-1}} s(m,J)\right)$,

(vii) $i(m,2p+1) = \sum_{I_1,\ldots,I_p\in\mathcal{I}^1} s(m,I_1)s(m,I_1\cup I_2)\cdots s(m,I_{p-1}\cup I_p)s(m,I_p)$.

3 $\mathcal{B}_{1,n}$ and $\mathcal{B}_{2,n}$, their structure and cardinality

Let $m = 1$ and $G_{1,n}$ be the corresponding grid graph, a path of length n; also, let B be a maximal independent set in $G_{1,n}$ (an example is depicted in Figure 1).

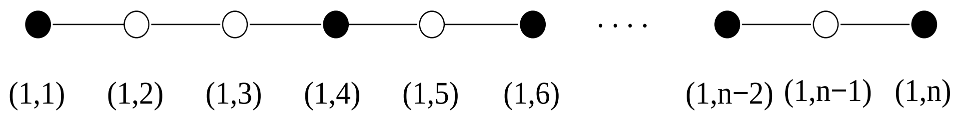

Figure 1: The basis $B = \{(1,1),(1,4),(1,6),\ldots,(1,n-2),(1,n)\}$ for $m = 1$.

Obviously, B cannot contain two consecutive vertices on this path, nor can the complement of such a set contain three such vertices. How can we obtain $b(1,n+1)$? We take a basis B from $\mathcal{B}_{1,n+1}$ and consider the elements it can have in common with the set $\{n-2,n-1,n,n+1\}$. There are four possibilities for this intersection: $\{n-2,n\}$, $\{n-1,n+1\}$, $\{n\}$, and $\{n-2,n+1\}$. The first two possibilities lead to a total number of $b(1,n-1)$, the second to $b(1,n-2)$ bases over $\{1,\ldots,n+1\}$, i.e.,

$$\begin{gathered} b(1,n+1) = b(1,n-1) + b(1,n-2) \text{ for } n \geq 3 \\ (\text{with } b(1,1) = 1, b(1,2) = b(1,3) = 2). \end{gathered} \tag{3}$$

But (3) is nothing but the definition of a Padovan sequence (see Stewart [8]), whose explicit form is

$$(b(1,n))_{n\in\mathbb{N}} = (1,2,2,3,4,5,7,9,12,16,\ldots)\ ,$$

and whose generating function is given by

$$\sum_{n\geq 1} b(1,n)x^n = \frac{x+2x^2+x^3}{1-x^2-x^3}\ .$$

Now let us turn to $m = 2$ and consider a basis $B \in \mathcal{B}_{2,n}$ (see Figure 2 for an example).

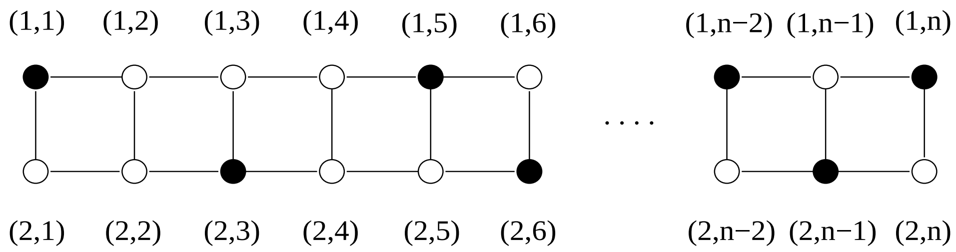

Figure 2: A basis for $m = 2$.

Either $(1,n) \in B$ or $(2,n) \in B$, in which case we can augment B by the vertex $(2,n+1)$ or $(1,n+1)$, respectively, to obtain $b(2,n)$ many bases within $\mathcal{B}_{2,n+1}$. Moreover, if the vertices $(1,n)$ and $(2,n-1)$ are in B, $B \setminus \{(1,n)\} \cup \{(1,n+1)\}$ will be in $\mathcal{B}_{2,n+1}$, and the same holds for $B \setminus \{(2,n)\} \cup \{(2,n+1)\}$ in case that $(2,n)$ and $(1,n-1)$ are elements of B. We hereby obtain the remaining $b(2,n-1)$ many bases of $\mathcal{B}_{2,n+1}$ so that altogether

$$b(2,n+1) = b(2,n) + b(2,n-1) \text{ for } n \geq 2 \qquad (4)$$
$$(\text{with } b(2,1) = b(2,2) = 2),$$

or more explicitly,

$$(b(2,n))_{n \in \mathbb{N}} = (2, 2, 4, 6, 10, 16, 26, 42, 68, 110, \dots) \ ,$$

which is nothing but a Fibonacci sequence with generating function

$$\sum_{n \geq 1} b(2,n) x^n = \frac{2x}{1 - x - x^2} \ .$$

4 $\mathcal{B}_{3,n}$ and the general case

To calculate $b(m,n)$ for $m, n > 2$ we maintain the basic ideas underlying the transfer matrix method for the determination of $i(m,n)$:

1. Creation of a partition of $\mathcal{B}_{m,n}$ that is valid at any stage.
2. Construction of a transfer matrix T reflecting the contribution of a class at stage k to any other one at stage $k+1$.
3. Determination of $b(m,n)$ by means of T and the cardinality of the classes.

To create our partition it will be convenient to analyse the bases with respect to their structure *within the last two columns*. (Recall that in the independent set case $\mathcal{I}_{m,n}$ had been separated into F_{m+2} many classes whose elements had an identical structure within the *last* column). We first need to generate $\mathcal{B}_{m,3}$ from $\mathcal{B}_{m,2}$, for which the following, general result will be useful:

Theorem 4.1 *Given $\mathcal{B}_{m,n}$ and a set of vertices $S = \{(s_1, n), \ldots, (s_p, n)\} \subseteq V^n$ let $r(S)$ denote the set $\{(s_1, n+1), \ldots, (s_p, n+1)\}$. Then $\mathcal{B}_{m,n+1}$ is the collection of all maximal sets within $\{B^{'} : B^{'} = B \setminus I \cup r(I), for\ B \in \mathcal{B}_{m,n}$ and $I \in \mathcal{I}^n\}$.*

Proof: Let $B \in \mathcal{B}_{m,n}$ and $C = V \setminus B$ its complement with respect to V. Then C is a *minimal cover* of E, i.e., C contains at least one element from all the edges and C is minimal with respect to this property. We let $\mathcal{C}_{m,n}$ denote the collection of all these covers. Now consider the set of edges E^+ that we add to $G_{m,n}$ to obtain $G_{m,n+1}$. Obviously,

$$E^+ = \{\{(i,n), (i,n+1)\} : i = 1, \ldots, m\} \cup \{\{(i,n+1), (i+1,n+1)\} : i = 1, \ldots, m-1\}.$$

It is easy to verify that the collection $\mathcal{C}^+$ of minimal covers of E^+ is the following:

$$\mathcal{C}^+ = \{C = I \cup r(\bar{I}) : I \in \mathcal{I}^n, \bar{I} = V^n \setminus I\}.$$

By minimality, such a cover C cannot contain two consecutive vertices $(i,n), (i+1,n) \in V^n$ with $i \in \{1, \ldots, m-1\}$, and any edge $\{(i,n), (i,n+1)\}$ with $i \in \{1, \ldots, m\}$ not covered by (i,n) has to be covered by $(i, n+1)$, these latter vertices covering at the same time all edges $\{(i,n+1), (i+1,n+1)\}$ with $i \in \{1, \ldots, m-1\}$. Now to obtain the collection $\mathcal{C}_{m,n+1}$ of minimal covers of the edges of $G_{m,n+1}$ we simply have to form the sets $C \cup C^{'}$ for any $C \in \mathcal{C}_{m,n}, C^{'} \in \mathcal{C}^+$ and to retain the minimal ones. By complementation we get $\mathcal{B}_{m,n+1}$.

It is clear that generating $\mathcal{B}_{m,n+1}$ this way will produce a large number of non-minimal covers. For $n = 2$ a thorough analysis of the bases in $\mathcal{B}_{m,2}$, however, will allow us to keep this effort at a minimum level.

What is the structure of a basis $B \in \mathcal{B}_{m,2}$? For an answer we consider two elements of B that are consecutive within the first column, i.e., which can be at distance 2, 3, 4, respectively, together with the position of the first or last element of B in that column (Figure 3 illustrates the four cases):

Case i) $(s, 1)$ and $(s+2, 1)$ are in B for $s \in \{1, \ldots, m-2\}$: then, by maximality, $(s+1, 2)$ has to be in B, too.

Case ii) $(s, 1)$ and $(s+3, 1)$ are in B: but then either $(s+1, 2)$ or $(s+2, 2)$ must be in B.

Case iii) $(s, 1)$ and $(s+4, 1)$ are in B: then $(s+2, 2)$ has to be in B.

Case iv) If $(s, 1) \in B$ for $s = 2$ or $m-1$ (or if $(s, 1) \in B$ for $s = 3$ or $s = m-2$, and $(1, 1)$ or $(m, 1) \notin B$, respectively), then $(1, 2)$ or $(m, 2)$, respectively, has to be in B.

Case i) gives rise to the definition of an *alternating path AP* as induced by a maximal alternating sequence in column 1:

$$AP = \{(s, 1), (s+1, 1), \ldots, (s+2t-1, 1), (s+2t, 1)\}$$

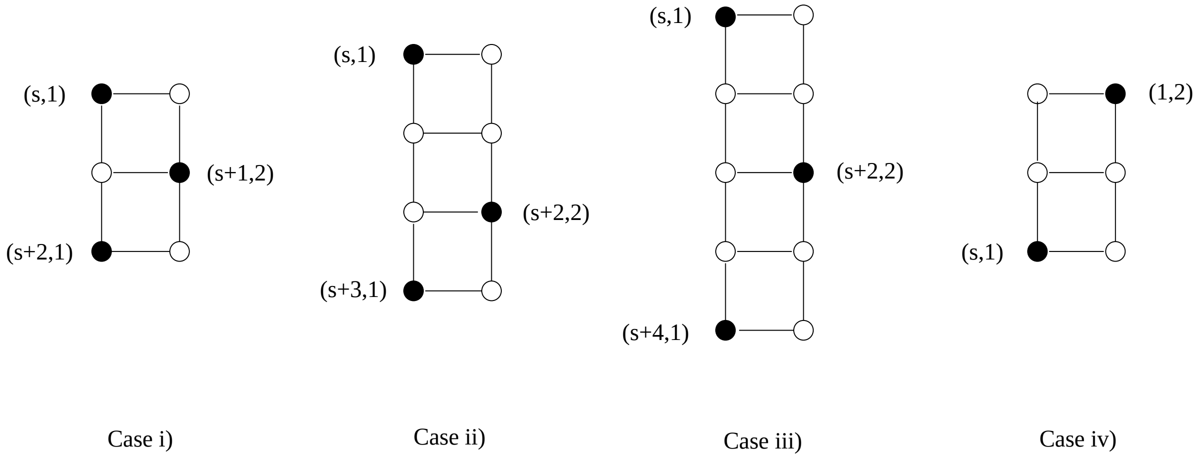

Figure 3: Substructures of a basis $B \in \mathcal{B}_{m,2}$.

with

$$\{(s,1),(s+2,1),\ldots,(s+2t-2,1),(s+2t,1)\} \subseteq B.$$

In such a case, the vertices $(s+1,2),(s+3,2),\ldots,(s+2t-1,2)$ have to be in B, too.

We now come to the proper generation of $\mathcal{B}_{m,3}$. Again, let B be a basis in $\mathcal{B}_{m,2}$. It is our aim to describe all independent sets $I \in \mathcal{I}^2$ for which $B' = B \setminus I \cup r(I)$ (see Theorem 2) is a member of $\mathcal{B}_{m,3}$. To this end we start with $I := \emptyset$ and check *column 2* of B from top to bottom with respect to the substructures studied above within column 1: whenever we find an alternating path AP we modify I according to the following rule:

Rule i) Reduce $AP \cap B$ to those vertices $(v,2)$, for which $v=1$ or $v=m$, or for which both $(v-1,1)$ and $(v+1,1)$ are contained in B; choose among the remaining ones an arbitrary set $BL = \{(b_1,2),\ldots,(b_p,2)\}$ and set $I := I \cup BL$. Within $AP \setminus B$, choose the set

$$W = \{(s+1,2),\ldots,(b_1-3,2),(b_1+3,2),\ldots,(b_2-3,2),(b_2+3,2),\ldots,(s+2t-1,2)\}$$

and set $I := I \cup W$. Also, whenever the vertex $(s,2)$ or $(s+2t,2)$ (with $s>2$, $s+2t<m-1$) has been chosen to be in BL, add vertex $(s-2,2)$ or $(s+2t+2,2)$ to I, respectively.

We are now left with a second check of column 2 with respect to substructures ii), iii), iv): for those not yet treated in rule 1 we have to select 1 or 2 elements according to the following rules:

Rule ii) Choose vertex $(s+1,2)$ or $(s+2,2)$ and add it to I.

Rule iii) Choose vertex $(s+2,2)$ (or vertices $(s+1,2)$ and $(s+3,2)$ if $(s+2,1)$ is in B) and add it (them) to I.

Rule iv) Choose $(1,2)$ or $(m,2)$ if $(2,2)$ or $(m-1,2)$ is in $B \setminus I$, respectively, and add it to I; if $(3,2)$ is the first, or $(m-2,2)$ the last element in B, then choose one of $(1,2), (2,2)$ or $(m-1,2), (m,2)$, respectively, and add it to I.

Set I has been constructed not only to be a member of $\mathcal{I}_2$: the set B' will always be a maximal independent set, i.e., a member of $\mathcal{B}_{m,3}$. On the other hand, any such maximizing I appears as a particular case of our construction so that altogether we generate the complete family $\mathcal{B}_{m,3}$. Also observe that due to the different possibilities to choose a set BL and a white vertex in the other cases, every basis in $\mathcal{B}_{m,2}$ generates a number of bases in $\mathcal{B}_{m,3}$, which equals a power of 2. Finally, our construction allows to partition $\mathcal{B}_{m,3}$ into p equivalence classes, two bases being considered equivalent if they have the same intersection with $V^2 \cup V^3$. Any such intersection can be represented by a *representing graph* $P_{m,2}$ (see Figure 4 for an illustration of the case $m = 3$).

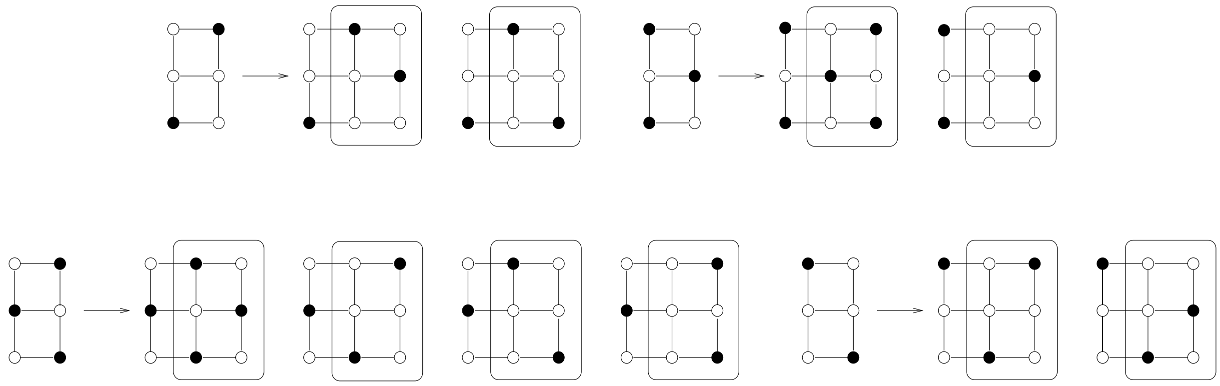

Figure 4: Generation of $\mathcal{B}_{m,3}$ from $\mathcal{B}_{m,2}$ and the 8 representing graphs $P_{m,2}$.

If we now proceed to the generation of $\mathcal{B}_{m,4}$ from $\mathcal{B}_{m,3}$, we can be sure to terminate with the same collection of representing graphs $P_{m,2}$ (and this is true for any stage): the collection of independent sets represented by the $P_{m,2}$*'s* contains $\mathcal{B}_{m,2}$, for any represented J there is a basis B from $\mathcal{B}_{m,2}$ such that $J \subseteq B$ and $B \setminus J$ contains only vertices of the first column.

What is still missing is an answer to the following question: what are the classes of stage k that produce (via Theorem 2) a given class at stage $k+1$? For an answer let us say that *class i contributes to class j* if for some (this is sufficient!) B in class i there is an I in $\mathcal{I}^3$ such that $B \setminus I \cup r(I)$ is an element of ($\mathcal{B}_{m,4}$ and) class j.

We are now ready to introduce the notion of a "transfer matrix": let p be the number of classes obtained from partitioning $\mathcal{B}_{m,3}$, and let $T \in \{0,1\}^{p \times p}$ be defined as follows:

$$T_{ij} = 1 \text{ iff class j contributes to class i .}$$

Then we get the following

Proposition 4.1 *If for k=3,4,5,..., c_i^k denotes the cardinality of class i at stage k, the c_i^3 being given for $i = 1, \ldots, p$, then*

$$c_i^{k+1} = \sum_{j=1}^{p} T_{ij} c_j^k \text{ for } i=1,\ldots,p, \text{ and } b(m, k+1) = \sum_{i=1}^{p} c_i^{k+1}.$$

To conclude this section we just mention that the initial size of matrix T can be reduced as follows: merge two classes into one whenever their representing graphs are (upside down) symmetric (note: T may then contain entries ≥ 2), or if they contribute to the same classes in the same way. These aspects will be further clarified in the next section, where a full solution of the cases $m = 3, 4, 5$ will be presented.

5 The cases $m = 3, 4, 5$

Let us start with $m = 3$. We have already seen how to generate $\mathcal{B}_{m,3}$ from $\mathcal{B}_{m,2}$, as depicted in Figure 4. If we merge "symmetric" classes we come up with 6 classes, whose representing graphs are resumed in Figure 5:

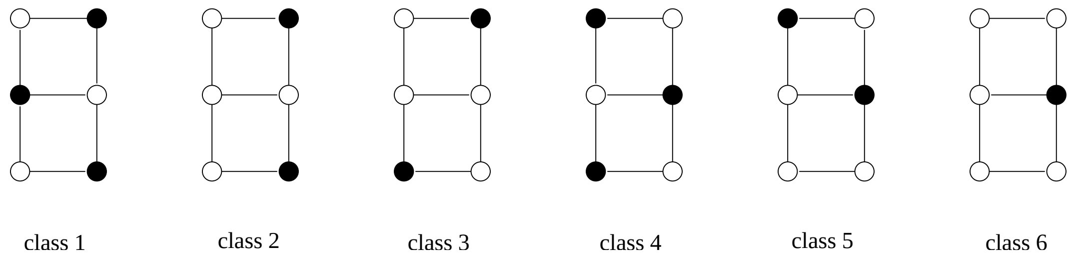

Figure 5: Representing graphs for $m = 3$.

Now observe that classes 5 and 6 contribute to the same classes in the same way, namely class 1, so that we can merge them into one new class 5. The corresponding transfer matrix is

$$T(3) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

which allows us, by application of Proposition 1, to calculate the class sizes c_i^n for $i = 1, \ldots, 5$:

classes	n=3	4	5	6	7	8	9	10
1	1	4	7	13	30	59	117	246
2	1	1	4	7	13	30	59	117
3	4	6	14	28	54	114	232	466
4	1	2	5	11	20	43	89	176
5	3	5	8	19	39	74	157	321

as well as the final values of $b(3,n)$ for $n > 3$, hereby producing the sequence

$$(b(3,n))_{n\in\mathbb{N}} = (2,4,10,18,38,78,156,320,654,1326,\ldots)\ ,$$

whose generating function is

$$\sum_{n\geq 1} b(3,n)x^n = \frac{2x+2x^2+4x^3-2x^4-2x^6}{1-x-x^2-3x^3+x^4+x^5}\ .$$

For $m = 4$ the partition of $\mathcal{B}_{4,n}$ is represented by the following graphs:

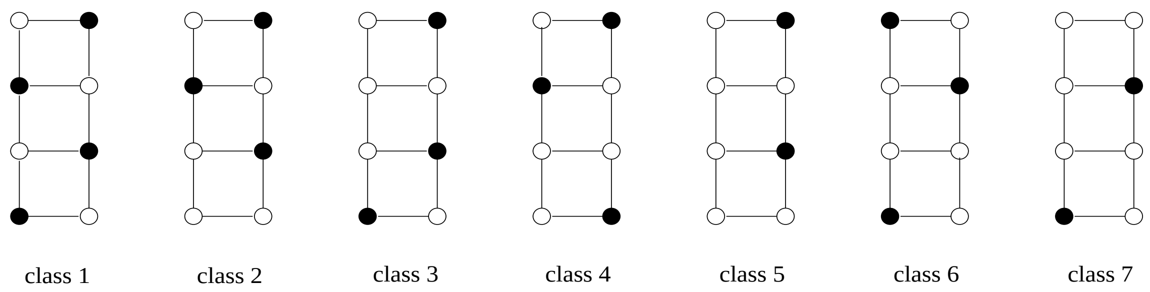

Figure 6: Representing graphs for $m = 4$.

As above, we can merge classes 6 and 7 to obtain the transfer matrix:

$$T(4) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

together with the associated class cardinalities:

classes	n=3	4	5	6	7	8	9	10
1	2	8	20	50	128	324	820	2078
2	2	6	12	32	82	204	520	1316
3	2	4	10	26	64	164	414	1048
4	4	10	26	64	164	414	1048	2656
5	2	2	8	20	50	128	324	820
6	6	12	32	82	204	520	1316	3330

We obtain the sequence:

$$(b(4,n))_{n\in\mathbb{N}} = (3, 6, 18, 42, 108, 274, 692, 1754, 4442, 11248, \ldots)\ ,$$

whose generating function is

$$\sum_{n\geq 1} b(4,n)x^n = \frac{3x+3x^2+3x^3-3x^4-3x^5-2x^6+x^7}{1-x-3x^2-3x^3+x^4+2x^5+x^6}\ .$$

For $m = 5$, finally, we come up with 17 representing graphs:

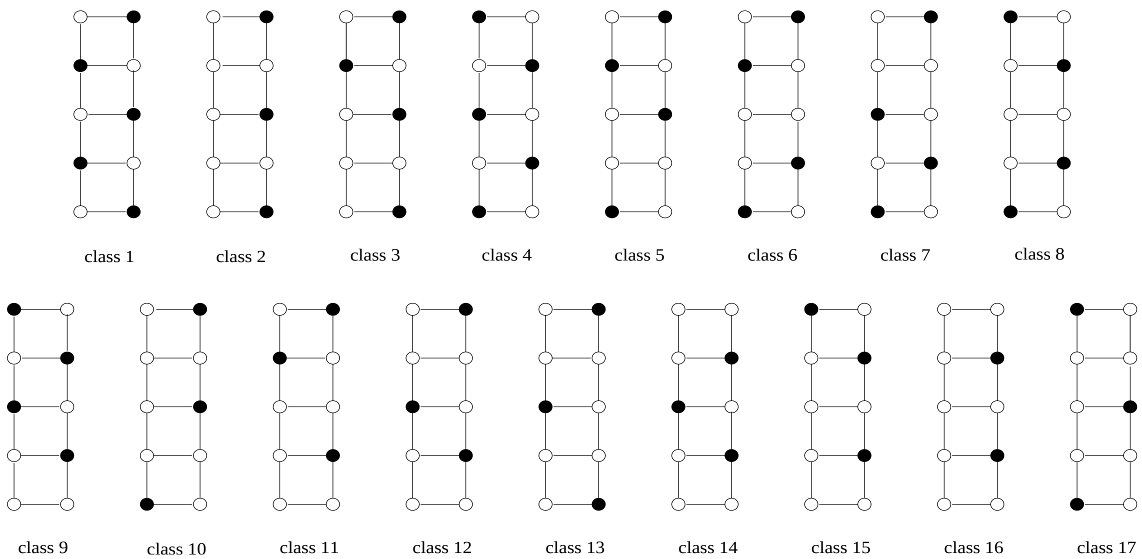

Figure 7: Representing graphs for $m = 5$.

We can merge classes 6 and 11, 9 and 15, 14 and 16 to obtain a 14×14 transfer matrix:

$$T(5) = \begin{bmatrix} 0&0&0&1&0&0&0&1&1&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0&0&1&0&0&0 \\ 0&0&0&2&0&1&1&2&1&0&1&0&0&0 \\ 2&0&1&0&1&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&0&1&0&0&1&0&0&0&0 \\ 2&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&2 \\ 1&0&0&0&1&0&0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&0&0&0&0&1&0&0 \end{bmatrix}$$

and the corresponding class cardinalities:

classes	n=3	4	5	6	7	8	9	10
1	1	8	21	61	218	677	2097	6814
2	1	1	8	21	61	218	677	2097
3	2	6	22	66	206	676	2124	6692
4	1	4	15	51	148	485	1571	4898
5	4	14	40	134	414	1340	4200	13426
6	6	22	66	206	676	2124	6692	21476
7	4	10	40	126	386	1244	4012	12548
8	1	4	7	29	82	267	841	2708
9	4	10	28	102	324	988	3226	10242
10	2	4	22	64	188	642	2030	6318
11	4	8	28	82	278	832	2722	8588
12	4	7	29	82	267	841	2708	8491
13	2	3	11	36	123	357	1176	3765
14	2	7	21	72	209	691	2194	6929

which produce the sequence

$$(b(5,n))_{n\in\mathbb{N}} = (4, 10, 38, 108, 358, 1132, 3580, 11382, 36270, 114992, \ldots)\ ,$$

with generating function

$$\sum_{n\geq 1} b(5,n)x^n =$$

$$\frac{4x+6x^2+12x^3-10x^4-18x^5+2x^6-20x^7+20x^8-50x^9+38x^{10}-18x^{11}+2x^{12}+6x^{13}+2x^{14}}{1-x-4x^2-10x^3-4x^4+20x^5-x^6+2x^7-2x^8+16x^9-4x^{10}-x^{13}}.$$

We may want to compare this sequence with the corresponding one for the independent set case (taken from Sloane [6]):

$$(i(5,n))_{n\in\mathbb{N}} = (13, 99, 827, 6743, 55447, 454385, 3729091, 30584687, 250916131, 2058249165, \ldots).$$

The conclusion with respect to their growth is immediate.

6 Conclusion

In this paper we have introduced a new class of integer sequences over grid graphs generalizing both Padovan and Fibonacci sequences, but still maintaining the essential property of independence. Counting the bases of a grid graph (or other graph families along the same line) and analysing their structure should therefore still be attractive for the applications mentioned in the introduction: tiling (our bases may have a very regular structure), efficient coding schemes (when closely related to the hard-square model) and statistical physics (in particular, the question of the true value of $\lim_{m,n\to\infty} b(m,n)^{1/mn}$).

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