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Nonlinear Inverse Relations for Bell Polynomials via the Lagrange Inversion Formula

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Abstract

In this paper, by using the classical Lagrange inversion formula, we establish a nonlinear inverse relation that involves the Bell polynomials. As applications of this inverse relation, we not only find a short proof of another nonlinear inverse relation due to Birmajer et al., but also deduce some allied combinatorial identities. Finally, we propose the general problem of finding nonlinear inverse relations and give a positive solution to it.

1 Introduction

Throughout this paper, we adopt the same notation of Henrici [8]. For instance, we use $\mathbb{C}[[x]]$ to denote the ring of formal power series (in short, fps) over the complex number field $\mathbb{C}$ and for any $f(x)=\sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]]$, the coefficient functional

$$[x^n]f(x) = a_n, n \geq 0.$$

For convenience, define

$$\mathcal{L}_0 = \left\{\sum_{n=0}^{\infty} a_n x^n | a_0 \neq 0\right\},$$

$$\mathcal{L}_1 = \left\{\sum_{n=0}^{\infty} a_n x^n | a_0 = 0, a_1 \neq 0\right\}.$$

Moreover, for $f(x), g(x) \in \mathbb{C}[[x]]$, $g(x)$ is said to be the composite inverse of $f(x)$ if $f(g(x)) = g(f(x)) = x$. As conventions, we denote the composite inverse $g(x)$ of $f(x)$ by $f^{\langle -1\rangle}(x)$.

Lemma 1. *Given $f(x) \in \mathbb{C}[[x]]$, $f(x)$ has the composite inverse if and only if $f(x) \in \mathcal{L}_1$.*

We also need the concept of the ordinary Bell polynomials (referenced by A263633 in the OEIS [14]).

Definition 2. For integers $n \geq k \geq 0$ and variables $(x_n)_{n\geq 1}$, the sums

$$\sum_{\sigma_k(n)} \frac{k!}{i_1! i_2! \cdots i_{n-k+1}!} x_1^{i_1} x_2^{i_2} \cdots x_{n-k+1}^{i_{n-k+1}} \tag{1}$$

are called the *ordinary Bell polynomials* in $x_1, x_2, \ldots, x_{n-k+1}$, where $\sigma_k(n)$ denotes the set of partitions of n with k parts, namely, all nonnegative integers $i_1, i_2, \ldots, i_{n-k+1}$ subject to

$$\begin{cases} i_1 + i_2 + \cdots + i_{n-k+1} = k \\ i_1 + 2i_2 + \cdots + (n-k+1)i_{n-k+1} = n. \end{cases} \tag{2}$$

We let $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ denote (1).

As of today, the Bell polynomials have played very important roles in analysis, combinatorics, and number theory. It should be pointed out here that the above Bell polynomials are in agreement with the exponential Bell polynomials [5, Def., p. 133] with the specialization $x_n \to x_n/n!$ and multiplied by $n!/k!$. The exponential Bell polynomials are referenced by A111785 in the OEIS [14].

The ordinary generating function of the Bell polynomials $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ will be often used in our discussions.

Lemma 3. *For any fps $f(x) = \sum_{n\geq 1} x_n x^n$, it holds that*

$$f^k(x) = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) x^n. \tag{3}$$

Proof. According to the Cauchy product of the fps, for given $f(x) = \sum_{n\geq 1} x_n x^n$, let

$$f^k(x) = \sum_{n=k}^{\infty} A_{n,k} x^n.$$

Then

$$A_{n,k} = \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ i_j \geq 1}} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

It is easy to see that

$$\max\{i_j \geq 1 | i_1 + i_2 + \cdots + i_k = n\} = n - k + 1.$$

Consequently, after a bit of series rearrangement, we have

$$\begin{aligned} A_{n,k} &= \sum_{\substack{i_1+2i_2+\cdots+(n-k+1)i_{n-k+1}=n \\ i_1+i_2+\cdots+i_{n-k+1}=k}} \frac{k!}{i_1!i_2!\cdots i_{n-k+1}!} x_1^{i_1} x_2^{i_2} \cdots x_{n-k+1}^{i_{n-k+1}} \\ &= B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \end{aligned}$$

as claimed. □

As far as the Bell polynomials are concerned, it is very natural to study inverse relations lurking behind it. Hereafter, the word "inverse" means a pair of equivalent relations expressing $(x_n)_{n\geq 1}$ in terms of the Bell polynomials in variables $(y_n)_{n\geq 1}$ and vice versa. To the best of our knowledge, it is one of the most interesting problems first posed and solved by Riordan [13, Chaps. 2 and 3], and also investigated by Chou et al. [4] and Mihoubi [12]. The reader may consult Riordan [13, Sect. 5.3] for further details and Mihoubi [12] for many of such inverse relations. It is especially noteworthy that in their paper [2], via the establishment of many interesting identities for the Bell polynomials, Birmajer et al. achieved the following somewhat unusual (essentially different from [4, 10, 13]) inverse relation.

Theorem 4 ([2, Thm. 17])**.** *Let $B_{n,k}(x_1, x_2, \ldots, x_{n-k})$ denote the Bell polynomials as above. Then for any integers a, b, m with $m \geq 1$, $a^2 + b^2 \neq 0$, and any sequence $(x_m)_{m\geq 1}$, the system of nonlinear relations*

$$z_m(b) = \sum_{k=1}^{m} \frac{am + bk}{k(am + b)} \binom{-am - b}{k - 1} B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) \tag{4}$$

is equivalent to the system of nonlinear relations

$$x_m = \sum_{k=1}^{m} \frac{1}{k} \binom{am + bk}{k - 1} B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)). \tag{5}$$

In the above theorem and what follows, we use the notation $\binom{x}{n}$ to denote the generalized binomial coefficients $(x)_n/n!$ and $(x)_n$ to the usual falling factorial $x(x-1)\cdots(x-n+1)$.

Motivated by Birmajer et al.'s result, the aim of the present paper is to establish the following nonlinear inverse relation. [1]

[1]As anonymous reviewer pointed out that Theorem 5 has now been generalized to Cor. 2 of [3], which appeared in *Discrete Math.* in January 2019. However, as our argument of Theorem 5 shows, Cor. 2 of [3] is still a direct consequence of the Lagrange inversion formula. See Section 4.

Theorem 5. *Let $B_{n,k}(x_1, x_2, \ldots, x_{n-k})$ denote the Bell polynomials as above. For any integers $m \geq 1$ and $a, b \in \mathbb{C}$, and any sequence $(x_m)_{m\geq 1}$, define*

$$y_m(b) = \frac{1}{am+b} \sum_{k=1}^{m} \binom{-am-b}{k} B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}). \tag{6}$$

Then

$$x_m = \frac{1}{am+1} \sum_{k=1}^{m} \binom{-(am+1)/b}{k} b^k B_{m,k}(y_1(b), y_2(b), \ldots, y_{m-k+1}(b)), \tag{7}$$

and vice versa.

Later as we shall see, Theorem 5 sheds a new light on the mystery of Theorem 4. Furthermore, by Theorem 5, we easily extend Theorem 4 to the following

Corollary 6. *With the same notation as in Theorem 4, for any integers $m \geq j \geq 1$, the system of nonlinear relations*

$$\begin{aligned} &B_{m,j}(z_1(b), z_2(b), \ldots, z_{m-j+1}(b)) \\ &= \sum_{k=j}^{m} \frac{j(am+bk)}{k(am+bj)} \binom{-am-bj}{k-j} B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) \end{aligned} \tag{8}$$

is equivalent to the system of nonlinear relations

$$\begin{aligned} &B_{m,j}(x_1, x_2, \ldots, x_{m-j+1}) \\ &= \sum_{k=j}^{m} \frac{j}{k} \binom{am+bk}{k-j} B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)). \end{aligned} \tag{9}$$

Corollary 6 leads us to a short proof of another combinatorial identity of [2].

Corollary 7 ([2, Thm. 15 and Rem.])**.** *With all assumptions of Theorem 4, for integers $m \geq s \geq 1$ and arbitrary $\lambda \in \mathbb{C}$, it holds that*

$$\begin{aligned} &\sum_{k=s}^{m} \frac{1}{k} \binom{\lambda}{k-s} B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) \\ &= \sum_{k=s}^{m} \frac{1}{k} \binom{\lambda+am+bk}{k-s} B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)). \end{aligned} \tag{10}$$

2 Proof of Theorem 5

Our argument for Theorem 5 is merely based on the classical Lagrange inversion formula [5, Thms. C and D, p. 150]. This celebrated formula is now known to be a basic but useful tool of finding of the composite inverse of fps. We refer the reader to [6, 9, 11] for further details.

Lemma 8 (The Lagrange inversion formula)**.** *Let $\phi(x) \in \mathcal{L}_0$. Then for any fps $F(x)$, it always holds that*

$$F(x) = \sum_{n=0}^{\infty} a_n \left(\frac{x}{\phi(x)}\right)^n, \tag{11}$$

where

$$a_n = \frac{1}{n}[x^{n-1}]F'(x)\phi^n(x). \tag{12}$$

Next is the full proof of Theorem 5.

Proof. We only need to show the theorem for integers $a, b \geq 1$, since both right-hand sides of (6) and (7) are polynomials in a, b, the equalities remain true for $a, b \in \mathbb{C}$ provided that they hold for all integers $a, b \geq 1$. Now we proceed to establish (7) from (6). As such, for integers $a, b \geq 1$, we may consider

$$f(x) = x\big(1 + \sum_{m=1}^{\infty} x_m x^{am}\big). \tag{13}$$

Then identity (6) amounts to

$$\big(f^{\langle -1\rangle}(x)\big)^b = x^b\big(1 + b\sum_{m=1}^{\infty} y_m(b)x^{am}\big). \tag{14}$$

The proof for this goes as follows. At first, according to Lemma 1, it is reasonable to assume that there exists the expansion

$$\big(f^{\langle -1\rangle}(x)\big)^b = \sum_{n=1}^{\infty} \lambda_n x^n. \tag{15}$$

Further replacing x with $f(x)$ in (15), we obtain

$$x^b = \sum_{n=1}^{\infty} \lambda_n f^n(x). \tag{16}$$

Now we are able to apply the Lagrange inversion formula to (16). As a consequence, we arrive at

$$\begin{aligned}\lambda_n &= \frac{b}{n}[x^{n-b}]\left(\frac{f(x)}{x}\right)^{-n}\\ &= \frac{b}{n}[x^{n-b}]\left(1+\sum_{i=1}^{\infty}\binom{-n}{i}\left(\sum_{m=1}^{\infty}x_m x^{am}\right)^i\right)\\ &= \frac{b}{n}[x^{n-b}]\left(1+\sum_{i=1}^{\infty}\binom{-n}{i}\sum_{m=i}^{\infty}B_{m,i}(x_1,x_2,\ldots,x_{m-i+1})x^{am}\right).\end{aligned}$$

Relabeling the corresponding parameters, we conclude that for $n \not\equiv b \pmod{a}$, $\lambda_n = 0$; for $n = b$, $\lambda_n = 1$; and for $n = am + b$,

$$\lambda_n = \frac{b}{am+b}\sum_{k=1}^{m}\binom{-am-b}{k}B_{m,k}(x_1,x_2,\ldots,x_{m-k+1}) = by_m(b).$$

Having (14) been shown, by Lemma 1 again, we now assume that

$$f(x) = \sum_{n=1}^{\infty}\mu_n x^n. \tag{17}$$

Clearly, to show (7) we need to express μ_n in terms of $y_m(b)$. To do this, we solve (14) for $f^{\langle -1\rangle}(x)$ to obtain

$$f^{\langle -1\rangle}(x) = x\big(1+b\sum_{m=1}^{\infty}y_m(b)x^{am}\big)^{1/b}. \tag{18}$$

Note that (17), after the replacement of x with $f^{\langle -1\rangle}(x)$, reduces to

$$x = \sum_{n=1}^{\infty}\mu_n(f^{\langle -1\rangle}(x))^n.$$

Now by making use of the Lagrange inversion formula and substituting (18) for $f^{\langle -1\rangle}(x)$, we easily compute

$$\begin{aligned}\mu_n &= \frac{1}{n}[x^{n-1}]\left(\frac{f^{\langle -1\rangle}(x)}{x}\right)^{-n}\\ &= \frac{1}{n}[x^{n-1}]\left(1+b\sum_{m=1}^{\infty}y_m(b)x^{am}\right)^{-n/b}\\ &= \frac{1}{n}[x^{n-1}]\left(1+\sum_{i=1}^{\infty}\binom{-n/b}{i}\left(b\sum_{m=1}^{\infty}y_m(b)x^{am}\right)^i\right)\\ &= \frac{1}{n}[x^{n-1}]\left(1+\sum_{i=1}^{\infty}\binom{-n/b}{i}b^i\sum_{m=i}^{\infty}B_{m,i}(y_1(b),y_2(b),\ldots,y_{m-i+1}(b))x^{am}\right).\end{aligned}$$

Relabeling the corresponding parameters, we conclude that $\mu_n = 0$ when $n \neq 1 \pmod a$ and $\mu_1 = 1$; for $n = am+1$,

$$\mu_n = \frac{1}{am+1}\sum_{k=1}^{m}\binom{-(am+1)/b}{k} b^k B_{m,k}(y_1(b), y_2(b), \ldots, y_{m-k+1}(b)).$$

Comparing (17) with (13), we thereby obtain

$$x_m = \frac{1}{am+1}\sum_{k=1}^{m}\binom{-(am+1)/b}{k} b^k B_{m,k}(y_1(b), y_2(b), \ldots, y_{m-k+1}(b)).$$

Hence (7) is proved. It is quite clear that the above derivations are valid in reverse direction. The theorem is therefore proved. $\square$

Some necessary comments on Theorem 5 are in order.

Remark 9. A simple comparison between Theorem 4 and Theorem 5 reminds us that the requirement that a, b be integral and $a^2 + b^2 \neq 0$ can be removed by appealing to the fact that all $y_m(b)$ are polynomials in a and b, whereas these conditions are very necessary to our argument. For instance, when $a = b = 0$, Theorem 5 reduces, treated as the limiting case as a and b tend to zero, to a pair of inverse relations

$$\begin{cases} y_m(0) = \displaystyle\sum_{k=1}^{m}\frac{(-1)^k}{k} B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) \\[2ex] x_m = \displaystyle\sum_{k=1}^{m}\frac{(-1)^k}{k!} B_{m,k}(y_1(0), y_2(0), \ldots, y_{m-k+1}(0)). \end{cases} \tag{19}$$

It is obviously equivalent, in terms of generating functions, to the composite inverse relation between the logarithmic and the exponential fps

$$\begin{cases} -\displaystyle\sum_{m=1}^{\infty} y_m(0)x^m = \log\left(1 + \sum_{m=1}^{\infty} x_m x^m\right) \\[2ex] \displaystyle\sum_{m=1}^{\infty} x_m x^m = \exp\left(-\sum_{m=1}^{\infty} y_m(0)x^m\right) - 1. \end{cases} \tag{20}$$

Remark 10. It is noteworthy that x_m's in (6) are independent of b, suggesting that for $y_i = y_i(1)$,

$$x_m = \sum_{k=1}^{m}\frac{(-1)^k}{am+1}\binom{am+k}{k} B_{m,k}(y_1, y_2, \ldots, y_{m-k+1}). \tag{21}$$

To illustrate Theorem 5, we consider the special case $x_m = 1/m!$ as an example. In this case, by making use of the well-known identity for the Stirling numbers of the second kind

$$x^m = \sum_{k=0}^{m} S(m,k)(x)_k, \tag{22}$$

we may set up an interesting combinatorial identity.

Example 11. For arbitrary complex numbers a, b and integers $m \geq 1$, we have

$$\sum_{k=1}^{m} (-1)^{m-k} \binom{m-1}{k-1} \frac{(am+bk)^{m-1}}{am+1+bk} = \frac{(m-1)! b^{m-1}}{\prod_{k=1}^{m}(am+1+bk)}. \tag{23}$$

Proof. As indicated as above, putting $x_m = 1/m!$ in (6) and using (22), it is easily found that $y_m(b) = (-1)^m (am+b)^{m-1}/m!$. Observe that $by_m(b)$'s are just the Abel polynomials [5, (1c), p. 128] in b, whose generating function turns out to be

$$1 + b \sum_{m=1}^{\infty} y_m(b)(xe^{ax})^m = e^{-bx}. \tag{24}$$

Hence, from Lemma 3 it follows that

$$\begin{aligned} & b^k \sum_{m=k}^{\infty} B_{m,k}(y_1(b), y_2(b), \ldots, y_{m-k+1}(b))(xe^{ax})^m \\ =& (e^{-bx} - 1)^k = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} e^{-bix}. \end{aligned}$$

On applying the Lagrange inversion formula to this expansion, we immediately obtain

$$\begin{aligned} & b^k B_{m,k}(y_1(b), y_2(b), \ldots, y_{m-k+1}(b)) \\ &= \frac{1}{m}[x^{m-1}] \left(-b \sum_{i=0}^{k} (-1)^{k-i} i \binom{k}{i} e^{-bix} \right) e^{-amx} \\ &= (-1)^m \frac{b}{m!} \sum_{i=0}^{k} (-1)^{k-i} i \binom{k}{i} (am+bi)^{m-1}. \end{aligned}$$

Next, substituting the last expression for $b^k B_{m,k}(y_1(b), y_2(b), \ldots, y_{m-k+1}(b))$ of (7) and interchanging the order of two sums, we at once obtain

$$1 = \sum_{i=1}^{m} \frac{(-1)^{m-i}}{(i-1)!} (am+bi)^{m-1} H(m,i), \tag{25}$$

where

$$H(m,i) := \sum_{k=i}^{m} \frac{b^{1-k}}{(k-i)!} \prod_{j=1}^{k-1} (am+1+bj)$$
$$= \frac{b^{1-m}}{(m-i)!} \prod_{j=1, j\neq i}^{m} (am+1+bj).$$

This reduces (25) to the claimed identity. $\square$

3 Proof of Theorem 4

Analogous to the proof of Theorem 5, we are able to show the nonlinear inverse relation of Birmajer et al., (i.e., Theorem 4) in a shorter way. To make this point clear, we need

Lemma 12. *Define*

$$\Gamma_m(j,k,a,b) := \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \frac{bk-i}{am+bk-i+j} \binom{am+bk-i+j}{j}. \tag{26}$$

Then, for $1 \leq k \leq j \leq m$,

$$\Gamma_m(j,k,a,b) = (-1)^k \frac{k(am+bj)}{j(am+bk+j-k)} \binom{am+bk+j-k}{j-k}, \tag{27}$$

and for $m \geq k \geq j+1$, $\Gamma_m(j,k,a,b) = 0$.

Proof. It suffices, for $1 \leq j \leq k-1$, to reformulate the term

$$\frac{bk-i}{am+bk-i+j} \binom{am+bk-i+j}{j} := \sum_{s=0}^{j} \eta_s i^s.$$

Note that all η_s are i-free. In this form, by using of the usual forward-difference operator $\Delta : f(x) \to f(x+1) - f(x)$, it is clear that

$$\Gamma_m(j,k,a,b) = \sum_{s=0}^{j} \eta_s \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^s = \sum_{s=0}^{j} \eta_s \Delta^k (x^s) \Big|_{x=0} = 0.$$

Next we proceed to show (27) for $m \geq j \geq k \geq 1$. For the same reason as indicated at beginning of the proof of Theorem 5 or Remark 9, we only need to show this under $a \geq 1, b \geq 2$. For this, we invoke the classical Hagen-Rothe formula [7] as follows:

$$\sum_{i=0}^{n} \frac{p}{ic+p} \binom{ic+p}{i} \frac{q}{(n-i)c+q} \binom{(n-i)c+q}{n-i} = \frac{p+q}{nc+p+q} \binom{nc+p+q}{n}. \tag{28}$$

Evidently, by putting $c = 1, n = am + bk$ in (28) and then letting $p = -k, q = j$, we immediately obtain

$$\begin{aligned}
&\sum_{i=0}^{am+bk} \frac{-k}{i-k}\binom{i-k}{i}\frac{j}{am+bk-i+j}\binom{am+bk-i+j}{am+bk-i} \\
&= \frac{-k+j}{am+bk-k+j}\binom{am+bk-k+j}{am+bk}.
\end{aligned}$$

Multiplying both sides by $(am+j)/j$ and using the basic relation

$$\binom{i-k}{i} = \frac{(-1)^i(k-i)}{k}\binom{k}{i},$$

then we obtain

$$\begin{aligned}
&\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{am+j}{am+bk-i+j}\binom{am+bk-i+j}{am+bk-i} \\
&= \frac{(-k+j)(am+j)}{j(am+bk-k+j)}\binom{am+bk-k+j}{am+bk}.
\end{aligned} \tag{29}$$

On the other hand, we have

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{am+bk-i+j}{am+bk-i} = \binom{am+bk-k+j}{am+bk}. \tag{30}$$

This is in fact the special case of $c = 1$ and $n = am+bk$ in another Hagen-Rothe formula [7]

$$\sum_{i=0}^{n} \frac{p}{ic+p}\binom{ic+p}{i}\binom{(n-i)c+q}{n-i} = \binom{p+q+nc}{n}, \tag{31}$$

while $p = -k, q = j$ at the same time. Upon subtracting (29) from (30), we obtain

$$\begin{aligned}
&\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{bk-i}{am+bk-i+j}\binom{am+bk-i+j}{am+bk-i} \\
&= \left(1 - \frac{(-k+j)(am+j)}{j(am+bk-k+j)}\right)\binom{am+bk-k+j}{am+bk} \\
&= \frac{j(bk-k)+(am+j)k}{j(am+bk-k+j)}\binom{am+bk-k+j}{am+bk} \\
&= \frac{k(am+bj)}{j(am+bk-k+j)}\binom{am+bk-k+j}{am+bk}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}&\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{bk-i}{am+bk-i+j}\binom{am+bk-i+j}{am+bk-i}\\&=\frac{k(am+bj)}{j(am+bk-k+j)}\binom{am+bk-k+j}{am+bk}.\end{aligned}$$

This, after multiplying $(-1)^k$ on both sides, yields (27). $\square$

Now we are ready to show Birmajer et al.'s nonlinear inverse relation, i.e., Theorem 4, via the use of Theorem 5. For the same reason as indicated at beginning of the proof of Theorem 5 or Remark 9, we only need to prove the theorem for integers $a \geq 1, b \geq 2$.

Proof. First, we show (5) from (4). To this end, by taking both (4) and (6) in Theorem 5 into account and noting that x_m's are independent of b, we easily verify that

$$z_m(b) = (b-1)y_m(b-1) - by_m(b). \tag{32}$$

By virtue of this relation, we can show a key fact that

$$B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(bk-i)y_m(bk-i). \tag{33}$$

The proof goes as follows. By the definition of the Bell polynomials, it holds

$$\left(\sum_{m=1}^{\infty} z_m(b)x^{am}\right)^k = \sum_{m=k}^{\infty} B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b))x^{am}.$$

On the other hand, referencing (14) and (32), we may deduce

$$\begin{aligned}\left(\sum_{m=1}^{\infty} z_m(b)x^{am}\right)^k &= \left((b-1)\sum_{m=1}^{\infty} y_m(b-1)x^{am} - b\sum_{m=1}^{\infty} y_m(b)x^{am}\right)^k\\&= \left(\left(f^{\langle -1\rangle}(x)/x\right)^{b-1} - \left(f^{\langle -1\rangle}(x)/x\right)^{b}\right)^k\\&= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\left(f^{\langle -1\rangle}(x)/x\right)^{bk-i}\\&= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\left(1+(bk-i)\sum_{m=1}^{\infty} y_m(bk-i)x^{am}\right)\\&= \delta_{k,0} + \sum_{m=1}^{\infty}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(bk-i)y_m(bk-i)\right)x^{am},\end{aligned}$$

where $\delta_{k,0}$ denotes the usual Kronecker symbol. Upon equating the coefficients of x^{am} on both sides of these expansions, we immediately obtain (33).

Now, with (33) in hand, we commence evaluating the sum on the right side of (5), namely,

$$\begin{aligned}\Omega &:= \sum_{k=1}^{m} \frac{1}{k}\binom{am+bk}{k-1} B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)) \\ &= \sum_{k=1}^{m} \frac{1}{k}\binom{am+bk}{k-1} \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(bk-i)y_m(bk-i).\end{aligned}$$

Upon inserting the expression of $y_m(b)$ given by (6) into this identity and after some rearrangement, it reduces to

$$\Omega = \sum_{j=1}^{m} \chi_m(j,a,b) B_{m,j}(x_1, x_2, \ldots, x_{m-j+1}),$$

where

$$\chi_m(j,a,b) := \sum_{k=1}^{m} \frac{(-1)^j}{k}\binom{am+bk}{k-1}\Gamma_m(j,k,a,b)$$

with $\Gamma_m(j,k,a,b)$ given by Lemma 12. By virtue of the same lemma, we further find

$$\begin{aligned}&\chi_m(j,a,b) \\ &= \left(\sum_{k=1}^{j} + \sum_{k=j+1}^{m}\right) \frac{(-1)^j}{k}\binom{am+bk}{k-1}\Gamma_m(j,k,a,b) \\ &= \frac{1}{j}\sum_{k=1}^{j} \frac{(-1)^{j-k}(am+bj)}{am+bk+j-k}\binom{am+bk}{k-1}\binom{am+bk+j-k}{j-k} \\ &= \frac{1}{j}\sum_{k-1=0}^{j-1} \frac{am+bj}{am+bk}\binom{am+bk}{k-1}\binom{-am-bk}{j-k} = \frac{am+bj}{j(am+b)}\binom{0}{j-1} = \delta_{j,1}.\end{aligned}$$

The penultimate equality is the special case of (31), wherein setting $p = am+b, q = -am-bj$ and $c = b$, after taking $(n,i) \to (j-1,k-1)$. The preceding computation simplifies

$$\Omega = B_{m,1}(x_1, x_2, \ldots, x_m) = x_m.$$

Hence, (5) is confirmed.

Conversely, assume (5) is given. For simplicity, we let w_m denote the right-hand side of (4), namely,

$$w_m := \sum_{k=1}^{m} \frac{am+bk}{k(am+b)}\binom{-am-b}{k-1} B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}). \tag{34}$$

In the sequel, in order to show (4), we only need to show $z_m(b) = w_m$ for all $m \geq 1$. To that end, performing as above, we first solve for x_m in (34) and find that

$$x_m = \sum_{k=1}^{m} \frac{1}{k}\binom{am+bk}{k-1} B_{m,k}(w_1, w_2, \ldots, w_{m-k+1}). \tag{35}$$

In the meantime, (5) states that

$$x_m = \sum_{k=1}^{m} \frac{1}{k}\binom{am+bk}{k-1} B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)). \tag{36}$$

Therefore, we have

$$\begin{aligned} &B_{m,1}(z_m(b)) + \sum_{k=2}^{m} \frac{1}{k}\binom{am+bk}{k-1} B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)) \\ &= B_{m,1}(w_m) + \sum_{k=2}^{m} \frac{1}{k}\binom{am+bk}{k-1} B_{m,k}(w_1, w_2, \ldots, w_{m-k+1}). \end{aligned} \tag{37}$$

All that remains is to show $z_m(b) = w_m$ by induction on m via the use of (37). When $m = 1$, it is easy to see that

$$x_1 = B_{1,1}(z_1(b)) = B_{1,1}(w_1),$$

so $z_1(b) = w_1$. Suppose further $z_k(b) = w_k$ for all $k \leq m-1$. Then we need to prove that $z_m(b) = w_m$. Since $m-k+1 \leq m-1$ for $k \geq 2$, by the hypothesis, we have $z_i(b) = w_i$ for $1 \leq i \leq m-k+1$, thereby

$$B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)) = B_{m,k}(w_1, w_2, \ldots, w_{m-k+1}).$$

Thus we obtain $B_{m,1}(z_m(b)) = B_{m,1}(w_m)$, i.e., $z_m(b) = w_m$. Summing up, for all integers $m \geq 1, z_m(b) = w_m$, so (4) follows from (34). The theorem is proved. □

Next is a sketched proof of Corollary 6 by use of Theorem 5.

Proof. This is a direct consequence of substituting (6) of Theorem 5 into (33), and then applying Lemma 12 again to the obtained one. Identity (9) follows from (8) by using once again the special case of the Hagen-Rothe formula (31)

$$\sum_{k=r}^{j} \frac{am+bj}{am+bk}\binom{am+bk}{k-r}\binom{-am-bk}{j-k} = \delta_{j,r}.$$

All other details are left to the interested reader. □

We end this section by a short proof of Corollary 7.

Proof. It suffices to compute, by making use of (9), in a straightforward manner that

$$\begin{aligned}
&\sum_{j=s}^{m} \frac{1}{j} \binom{\lambda}{j-s} B_{m,j}(x_1, x_2, \ldots, x_{m-j+1}) \\
&= \sum_{k=s}^{m} \frac{1}{k} \left(\sum_{j=s}^{k} \binom{\lambda}{j-s} \binom{am+bk}{k-j} \right) B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)) \\
&= \sum_{k=s}^{m} \frac{1}{k} \binom{\lambda+am+bk}{k-s} B_{m,k}(z_1(b), z_2(b), \ldots, z_{m-k+1}(b)).
\end{aligned}$$

Note that the inner summation on the second equality can be evaluated in closed form by the Vandermonde convolution formula [7, (1)]. □

4 Recent results on nonlinear inverse relations

Thanks to the anonymous reviewer's suggestions, we are led to the latest work [3]. Indeed, Birmajer et al. [3] found a general and more beautiful nonlinear inverse relation for the Bell polynomials. Their result inspires us to consider the following research problem.

Problem 13. *For any integers $m \geq 1$, let p, $(a_k)_{k=1}^m$, and $(q_k)_{k=1}^m$ be $2m+1$ complex numbers subject to*

$$p \neq 0, \quad \sum_{k=1}^{m} a_k = 0, \quad \sum_{k=1}^{m} a_k q_k \neq 0.$$

Assume further that $F(x) = \sum_{n\geq 1} x_n x^n$ and $\phi(x) = 1 + \sum_{n\geq 1} y_n x^n$ satisfy

$$F(x/\phi^p(x)) = \sum_{k=1}^{m} a_k \phi^{q_k}(x). \tag{38}$$

Find any relationship between the sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$.

Using the Lagrange inversion formula as in the proof of Theorem 5, we can find a positive solution to this problem as follows.

Theorem 14. *With the same notation and assumptions as above, the system of nonlinear relations*

$$x_n = \sum_{k=1}^{n} \left(\sum_{i=1}^{m} \frac{a_i q_i}{np+q_i} \binom{np+q_i}{k} \right) B_{n,k}(y_1, y_2, \ldots, y_{n-k+1}) \tag{39}$$

is equivalent to the system of nonlinear relations

$$y_n = \sum_{k=1}^{n} \frac{\lambda_k(-1/p+n)}{1-pn} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \tag{40}$$

where $\lambda_n(s)$ *is defined by*

$$(n+1)\lambda_{n+1}(s)\sum_{k=1}^{m}a_kq_k = -\Big(ps+n\sum_{k=1}^{m}a_kq_k\lambda_1(-q_k/p)\Big)\lambda_n(s) \\ -\sum_{k=1}^{m}a_kq_k\sum_{j=1}^{n-1}\lambda_{n+1-j}(-q_k/p)j\lambda_j(s). \tag{41}$$

Evidently, (39) can be obtained via the direct application of the Lagrange inversion formula to (38), while (40) can be derived by the very similar method as we did for (7) of Theorem 5. For instance, when $m=2$, the nonlinear relation (39)/(40) reduces to

Corollary 15. *Let* $c=r-q\neq 0$. *Then the system of nonlinear relations*

$$x_n=\frac{1}{c}\sum_{k=1}^{n}\left(\frac{q}{q+np}\binom{q+np}{k}-\frac{r}{r+np}\binom{r+np}{k}\right)B_{n,k}(y_1,y_2,\ldots,y_{n-k+1}) \tag{42}$$

is equivalent to the system of nonlinear relations

$$y_n=-\sum_{k=1}^{n}\frac{1}{k!}\prod_{j=1}^{k-1}\big(cj+np+kq-1\big)B_{n,k}(x_1,x_2,\ldots,x_{n-k+1}). \tag{43}$$

We remark that under the parametric replacement $(p,q,r)\to(-p/d,-q/d,-r/d)$ and $(x_n,y_n)\to(-dx_n,dy_n)$, Corollary 15 is in agreement with Cor. 2 of [3]. A full proof of Theorem 14 will be given in a forthcoming paper.

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