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Subset Parking Functions

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Abstract

A parking function $(c_1,\ldots,c_n)$ can be viewed as having n cars trying to park on a one-way street with n parking spots from left to right, where car i tries to park in spot c_i, and otherwise it parks in the leftmost available spot after c_i. Another way to view this is that each car has a set C_i of "acceptable" parking spots, namely $C_i = [c_i, n]$, and that each car tries to park in the leftmost available spot that it finds acceptable.

Motivated by this, we define a subset parking function $(C_1,\ldots,C_n)$, with each C_i a subset of $\{1,\ldots,n\}$, by having the ith car try to park in the leftmost available element of C_i. We further generalize this idea by restricting our sets to be of size k, intervals, and intervals of length k. In each of these cases we provide formulas for the number of such parking functions.

1 Introduction

Parking functions are well-studied objects in combinatorics, and are often defined in the following way. Imagine that there are n parking spots labeled 1 though n on a one way street. There are n cars, also labeled 1 through n, that wish to park in these spots, and each has a preferred parking spot c_i. When it is car i's turn to park, it goes to its preferred spot c_i and parks there if it is empty. Otherwise, it tries to park in the next available spot that is after c_i. The tuple $(c_1,\ldots,c_n)$ is said to be a parking function if every car succeeds in parking.

For example, $(2,2,2)$ is not a parking function, as car 1 parks in slot 2; car 2 tries to park in slot 2 but can not and goes to the next available slot 3; and car 3 tries to park in slot 2 but can not, and there are no slots after this that are available. On the other hand, $(2,1,1)$ is a parking function as car 1 parks in slot 2; car 2 parks in slot 1; and car 3 tries to park in slot 1 but can not, so it goes to the next available slot 3. We will say that the parking function $(2,1,1)$ has outcome 213, which describes how one would see the cars parked if one were to walk from slot 1 to slot 3.

Parking functions have many interesting combinatorial properties. For example, a necessary and sufficient condition for $(c_1,\ldots,c_n)$ to be a parking function is, after rearranging the c_i in increasing order as $b_1 \le b_2 \le \cdots \le b_n$, we have $b_i \le i$ for all i. In particular, this shows that any permutations of the entries of a parking function is also a parking function. One can also prove that the number of parking functions $\mathrm{PF}(n)$ satisfies

$$\mathrm{PF}(n) = (n+1)^{n-1}. \tag{1}$$

Parking functions have connections to many other areas of combinatorics, such as hyperplane arrangements [6] and the lattice of non-crossing partitions [7]. We refer the reader to the survey of Yan [9] for an elegant proof of (1) and a more in depth study of parking functions.

Many generalizations and variants of parking functions have been studied, such as $\mathbf{x}$-parking functions [8] and G-parking functions [4]. These examples generalize the $b_i \le i$ characterization of parking functions. One can also generalize the parking analogy. An example of this is to allow cars to park a few spaces before their preferred spot if this is already taken, which has been studied recently [1].

In this paper we also consider a variant of parking function that is obtained by modifying the parking rule. To motivate the idea, we observe that a parking function $(c_1,\ldots,c_n)$ can be viewed as each car choosing a set $C_i = [c_i, n]$ of "acceptable" parking spaces, with each car parking in the leftmost available spot which is acceptable to them. One can generalize this idea by allowing each C_i to be an arbitrary set.

To this end, let $[n] := \{1,2,\ldots,n\}$ and let $\mathcal{S}_n$ denote the set of permutations of size n written in one line notation. Given n non-empty subsets $C_i \subseteq [n]$ and a permutation $\pi \in \mathcal{S}_n$, we will say that $\mathcal{C} = (C_1,\ldots,C_n)$ is a subset parking function with outcome π if for all $1 \le i \le n$, having $\pi_j = i$ implies j is the smallest element of $C_i \setminus \{\pi^{-1}_{i'} : i' < i\}$. That is, if car i ends up in spot j, it must find spot j to be acceptable, all the earlier spots which are acceptable are already taken, and no one has taken spot j yet. We let $\mathrm{SPF}(n,\pi)$ denote the number of subset parking functions with outcome π, and we denote the total number of subset parking functions by $\mathrm{SPF}(n) := \sum_{\pi\in\mathcal{S}_n} \mathrm{SPF}(n,\pi)$. Technically we should say that $\mathrm{SPF}(n)$ counts the number of parking functions of size n, but here and throughout we omit explicitly noting this dependency on n whenever it is clear from context.

For example, $(\{2\},\{2,3\},\{1,2,3\})$ is a subset parking function with outcome 312. However, $(\{2,3\},\{1,2,3\},\{2\})$ is not a subset parking function since we require $\pi_2 = 1,\ \pi_1 = 2$, and then no choice from C_3 will work. In particular this shows that for subset parking functions the order of the C_i sets are important, which is not the case in the classical study of parking functions.

Our first goal is to enumerate subset parking functions. To this end, we recall that the inversion number $\mathrm{inv}(\pi)$ of a permutation π is equal to the number of pairs (i,j) such that $i<j$ and $\pi_j<\pi_i$.

Theorem 1. *For any integer $n\ge 1$ and $\pi\in\mathcal{S}_n$,*

$$\mathrm{SPF}(n,\pi)=2^{n(n-1)-\mathrm{inv}(\pi)},$$

$$\mathrm{SPF}(n)=\prod_{i=0}^{n-1}(2^n-2^i).$$

(Classical) parking functions are subset parking functions where each C_i is required to be an interval of the form $[c_i,n]$. We can get other interesting variants by restricting the C_i sets in other ways. For example, we say that $\mathcal{C}=(C_1,\dots,C_n)$ is a k-subset parking function if $\mathcal{C}$ is a subset parking function and $|C_i|=k$ for all i. We let $\mathrm{SPF}_k(n,\pi)$ denote the number of k-subset parking functions with outcome π and $\mathrm{SPF}_k(n)$ the number of k-subset parking functions.

To state our next result, we define the local inversion number $\mathrm{inv}_i(\pi)$ to be the number of pairs (i,j) with $i<j$ and $\pi_j<\pi_i$. Observe that $\mathrm{inv}(\pi)=\sum\mathrm{inv}_i(\pi)$. We adopt the convention that $\binom{0}{0}=1$ and $\binom{0}{x}=0$ for $x>0$.

Theorem 2. *For any integer $n\ge 1$, $\pi\in\mathcal{S}_n$, and $1\le k\le n$,*

$$\mathrm{SPF}_k(n,\pi)=\prod_{i=1}^{n}\binom{n-\mathrm{inv}_i(\pi)-1}{k-1},$$

$$\mathrm{SPF}_k(n)=\prod_{i=0}^{n-1}\left(\binom{n}{k}-\binom{i}{k}\right).$$

We next consider the case that each C_i is an interval, and we call such parking functions interval parking functions. Let $\mathrm{IPF}(n)$ and $\mathrm{IPF}(n,\pi)$ be the total number of interval parking functions and the number of interval parking functions with outcome π, respectively.

To state our full result, given a permutation π we define $a_i(\pi)$ to be the largest j with $1\le j\le i$ such that $\pi_i\ge\{\pi_i,\pi_{i-1},\dots,\pi_{i-j+1}\}$. For example, $a_i(\pi)\ge 2$ if and only if $\pi_i>\pi_{i-1}$. As another example, for $\pi=31524$, we have $a_i(\pi)$ equal to 1, 1, 3, 1, 2 as i ranges from 1 to 5. Finally, define $\mathrm{PF}(n,\pi)$ to be the number of (classical) parking functions with outcome π.

Theorem 3. *For any $n\ge 1$ and $\pi\in\mathcal{S}_n$,*

$$\mathrm{IPF}(n,\pi)=n!\cdot\mathrm{PF}(n,\pi)=n!\prod_{i=1}^{n}a_i(\pi),$$

$$\mathrm{IPF}(n)=n!\cdot\mathrm{PF}(n)=n!\cdot(n+1)^{n-1}.$$

Other properties of interval parking functions are currently being investigated by Christensen, DeMuse, Martin, and Yin [2].

The last variant we consider are k-interval parking functions, which are interval parking functions where each C_i is an interval containing k elements. We let $\mathrm{IPF}_k(n)$ denote the number of k-interval parking functions and $\mathrm{IPF}_k(n, \pi)$ the number of those with outcome π. Define $\mathcal{S}_n^k$ to be the set of permutations π of order n with $\pi_n > \pi_{n-1} > \cdots > \pi_{n-k+1}$.

Theorem 4. *Let k and n be integers with $1 \leq k \leq n$ and let $\pi \in \mathcal{S}_n$. If $\pi \notin \mathcal{S}_n^k$, then $\mathrm{IPF}_k(n, \pi) = 0$. Otherwise,*

$$\mathrm{IPF}_k(n, \pi) = \prod_{i=1}^{n-k} \min\{a_i(\pi), k\} \cdot \prod_{i=n-k+1}^{n} \min\{n - i - k + a_i(\pi) + 1, n - i + 1\}.$$

This formula is rather complicated, but for certain k it is manageable. For example, when $k = 1$ each term in the product is 1. We conclude that $\mathrm{IPF}_1(n, \pi) = 1$ for all π, and hence $\mathrm{IPF}_1(n) = n!$. When $k = n$, we have $\mathrm{IPF}_n(n, \pi) = 1$ when $\pi = 12 \cdots n$ (since in general $a_1(\pi) = 1$ and $a_{n-i}(\pi) \leq n - i$), and otherwise $\mathrm{IPF}_n(n, \pi) = 0$, so $\mathrm{IPF}_n(n) = 1$. Both of these results can also be verified directly. The formulas for $k = n - 1$ and $k = 2$ are also quite nice.

Corollary 5. *If $n \geq 2$ and $\pi \in S_n^{n-1}$ with $\pi_1 = j$, then*

$$\mathrm{IPF}_{n-1}(n, \pi) = \begin{cases} 2^{n-j-1}, & \text{if } j \neq n; \\ 1, & \text{if } j = n. \end{cases}$$

Moreover,

$$\mathrm{IPF}_{n-1}(n) = 2^{n-1}.$$

To state the formula for $k = 2$, we define the ascent number $\mathrm{asc}(\pi)$ of a permutation π to be the number of i with $2 \leq i \leq n$ and $\pi_{i-1} < \pi_i$. Define the Eulerian number $\left\langle {n \atop k} \right\rangle$ to be the number of permutations $\pi \in \mathcal{S}_n$ with $\mathrm{asc}(\pi) = k$.

Corollary 6. *If $n \geq 2$ and $\pi \in S_n^2$, then*

$$\mathrm{IPF}_2(n, \pi) = 2^{\mathrm{asc}(\pi) - 1}.$$

Moreover,

$$\mathrm{IPF}_2(n) = \sum_{k=1}^{n-1} (n - k) \left\langle {n-1 \atop k-1} \right\rangle 2^{k-1}.$$

2 Subset results

We first prove enumeration results for a generalization of subset parking functions where each car is given a list of allowed subset sizes. To this end, given $\mathcal{L} = (L_1, \ldots, L_n)$ with $L_i \subseteq [n]$, we define $\mathrm{SPF}(n, \mathcal{L})$ to be the number of subset parking functions where $|C_i| \in L_i$, and we will call this an $\mathcal{L}$-parking function. Our first goal will be to enumerate $\mathrm{SPF}(n, \mathcal{L})$.

To do this, we define the notion of a partial parking function, which intuitively describes where the first m cars have parked. Let $\mathcal{S}_{m,n}$ denote the set of strings $\pi = \pi_1 \cdots \pi_n$ where for all $1 \le i \le m$ there exists a unique index j with $\pi_j = i$ and such that every other letter is an auxillary letter $*$. Note that $\mathcal{S}_{n,n}$ is simply the set of permutations. For $i \le m$ we let π_i^{-1} denote the unique index j with $\pi_j = i$.

We say that $(C_1, \ldots, C_m)$ with each C_i a non-empty subset of $[n]$ is a partial $\mathcal{L}$-parking function with outcome $\pi \in \mathcal{S}_{m,n}$ if for all $1 \le i \le m$, $|C_i| \in L_i$ and $\pi_i^{-1} = j$ implies j is the smallest element of $C_i \setminus \{\pi_{i'}^{-1} : i' < i\}$. Finally, given a permutation π, we write $\pi^{(m)}$ to denote the string where $\pi_i^{(m)} = \pi_i$ if $i \le m$ and $\pi_i^{(m)} = *$ otherwise. Once one unpacks these definitions, the following is immediate.

Lemma 7. *Let $1 \le m \le n$ and $\pi \in \mathcal{S}_n$. $\mathcal{C}$ is an $\mathcal{L}$-parking function with outcome π if and only if $(C_1, \ldots, C_m)$ is a partial $\mathcal{L}$-parking function with outcome $\pi^{(m)}$ for all m.*

The following lemma shows how to extend partial parking functions.

Lemma 8. *Let $1 \le m \le n$ and $\pi \in \mathcal{S}_n$. If $(C_1, \ldots, C_{m-1})$ is a partial $\mathcal{L}$-parking function with outcome π, then $(C_1, \ldots, C_m)$ is a partial $\mathcal{L}$-parking function if and only if $|C_m| \in L_m$ and $C_m \not\subseteq \{\pi_i^{-1} : i < m\}$.*

Proof. If C_m is such a set, then by assumption $|C_m| \in L_m$ and there exists some minimal j in $C_m \setminus \{\pi_k^{-1} : k < m\}$. Thus by defining π' by $\pi'_i = \pi_i$ for $i \neq j$ and $\pi'_j = m$, we see that $(C_1, \ldots, C_m)$ is a partial $\mathcal{L}$-parking function with outcome π'. Conversely, if $(C_1, \ldots, C_m)$ is a partial $\mathcal{L}$-parking function, then $C_m \setminus \{\pi_k^{-1} : k < m\}$ must be non-empty, so $C_m \not\subseteq \{\pi_k^{-1} : k < m\}$. We also must have $|C_m| \in L_m$ by definition, proving the result. $\square$

Theorem 9. *For any $n \ge 1$ and $\mathcal{L} = (L_1, \ldots, L_n)$,*

$$\mathrm{SPF}(n, \mathcal{L}) = \prod_{i=1}^{n} \left(\sum_{\ell \in L_i} \binom{n}{\ell} - \binom{i-1}{\ell} \right).$$

Proof. Consider the following procedure. We start with an empty list (). Recursively, given a partial $\mathcal{L}$-parking function $(C_1, \ldots, C_{i-1})$, we choose a set C_i such that $(C_1, \ldots, C_i)$ is a partial $\mathcal{L}$-parking function. By Lemma 7, every $\mathcal{L}$-parking function is obtained (uniquely) by this procedure. Thus to obtain our result we need only enumerate how many choices we can make at each stage of the procedure.

Assume one has already chosen $(C_1, \ldots, C_{i-1})$ so now we need to choose C_i. By Lemma 8, for any $\ell \in L_i$, the number of ways to choose an appropriate C_i with $|C_i| = \ell$ is $\binom{n}{\ell} - \binom{i-1}{\ell}$.

Namely, one can choose any ℓ-element subset that is not contained in $\{\pi_j^{-1} : j < i\}$. As we allow $|C_i|$ to be any element of L_i, we conclude that the number of choices for C_i is exactly $\sum_{\ell \in L_i} \binom{n}{\ell} - \binom{i-1}{\ell}$. As the number of choices for C_i is independent of all of the other C_j sets, we conclude that the total number of ways to complete this procedure is the product of all of these sums. This gives the desired result. $\square$

We can prove a similar general theorem when the outcome is specified. To this end, define $\mathrm{SPF}(n, \mathcal{L}, \pi)$ to be the number of $\mathcal{L}$-parking functions with outcome π. Recall that $\mathrm{inv}_i(\pi)$ is defined to be the number of (i, j) with $i < j$ and $\pi_j < \pi_i$.

Theorem 10. *For any $n \geq 1$, $\pi \in \mathcal{S}_n$, and $\mathcal{L} = (L_1, \ldots, L_n)$,*

$$\mathrm{SPF}(n, \mathcal{L}, \pi) = \prod_{i=1}^{n} \left(\sum_{\ell \in L_i} \binom{n - \mathrm{inv}_i(\pi) - 1}{\ell - 1} \right).$$

Proof. Consider the following procedure. We start with an empty list $()$. Recursively, given a partial $\mathcal{L}$-parking function $(C_1, \ldots, C_{i-1})$ with outcome $\pi^{(i-1)}$, we choose a set C_i such that $(C_1, \ldots, C_i)$ is a partial $\mathcal{L}$-parking function with outcome $\pi^{(i)}$. By Lemma 7, every $\mathcal{L}$-parking function with outcome π is obtained (uniquely) by this procedure. Thus to obtain our result we need only enumerate how many choices we can make at each stage of the procedure.

Assume $(C_1, \ldots, C_{i-1})$ is a partial $\mathcal{L}$-parking function with outcome $\pi^{(i-1)}$. By Lemma 8, if we wish to have $|C_i| = \ell \in L_i$, then we must have $C_i \not\subset \{\pi_{i'}^{-1} : i' < i\}$. Moreover, we also must choose this set so that it has outcome $\pi^{(i)}$. If $j = \pi_i^{-1}$, this is equivalent to having C_i be any subset with j the minimal element of $C_i \setminus \{\pi_{i'}^{-1} : i' < i\}$. To summarize, necessary and sufficient conditions for C_i to have $|C_i| = \ell$ are

(a) $|C_i| = \ell$,

(b) $C_i \not\subset \{\pi_{i'}^{-1} : i' < i\}$,

(c) $j \in C_i$, and

(d) $k \notin C_i$ if $k < j$ and $k \notin \{\pi_{i'}^{-1} : i' < i\}$.

Note that (b) is implied by (c), so this is irrelevant. Condition (d) is equivalent to avoiding k with $k < j$ and $i < \pi_k$ (that is, the car that appears in the earlier spot k parks after i). The number of such k is exactly $\mathrm{inv}_i(\pi)$, so we conclude that the number of C_i satisfying these conditions is exactly $\binom{n-1-\mathrm{inv}_i(\pi)}{\ell-1}$. Summing this value over all $\ell \in L_i$ gives the total number of choices for C_i. As this quantity is independent of all the other choices of C_j, we can take their product to arrive at the desired count for $\mathrm{SPF}(n, \mathcal{L}, \pi)$. $\square$

With this we can now prove our results. We start with Theorem 2.

Proof of Theorem 2. Note that k-subset parking functions are precisely $\mathcal{L}$-parking functions where $L_i = \{k\}$ for all i. The result follows from Theorems 9 and 10. $\square$

Proof of Theorem 1. Subset parking functions are precisely $\mathcal{L}$-parking functions where $L_i = [n] \setminus \{0\}$ for all i. By Theorem 9 we have

$$\mathrm{SPF}(n) = \prod_{i=1}^{n} \left((2^n - 1) - (2^{i-1} - 1)\right).$$

Canceling the 1's and reindexing the product gives the first result. For the second result, Theorem 10 implies

$$\mathrm{SPF}(n, \pi) = \prod_{i=1}^{n} 2^{n-\mathrm{inv}_i(\pi)-1} = 2^{n(n-1)-\mathrm{inv}(\pi)},$$

where we used that $\mathrm{inv}(\pi) = \sum \mathrm{inv}_i(\pi)$. $\square$

We note that Theorem 1 implies

$$\sum_{\pi \in \mathcal{S}_n} 2^{n(n-1)-\mathrm{inv}(\pi)} = \prod_{i=0}^{n-1} (2^n - 2^i),$$

which one can verify using the generating function for the inversion statistic. This also provides an alternative way to prove the formula for $\mathrm{SPF}(n)$ given the formulas for each $\mathrm{SPF}(n, \pi)$. Similarly Theorem 2 implies

$$\sum_{\pi \in \mathcal{S}_n} \prod_{i=1}^{n} \binom{n - \mathrm{inv}_i(\pi) - 1}{k - 1} = \prod_{i=1}^{n} \left(\binom{n}{k} - \binom{i-1}{k}\right).$$

We are not aware of a more direct method to prove this.

Before closing this section, we briefly discuss a variant of subset parking functions. Since subset parking functions allow each car to have any set of positions be acceptable, it also makes sense to allow each car to have their own preference order for these spots insteadof always requiring them to park in the left-most available spot.

To formalize this, we say that a list of subsets $\mathcal{C} = (C_1, \ldots, C_n)$, together with a list of bijections $f_i : C_i \to [|C_i|]$, is an ordered parking function with outcome $\pi = \pi_1 \cdots \pi_n$ if for all $1 \leq i \leq n$, $\pi_j = i$ implies $f_i(j) = \min_{j' \in D_i} f_i(j')$, where $D_i := C_i \setminus \{\pi_{i'}^{-1} : i' < i\}$. We let $\mathrm{OPF}(n, \pi)$ denote the number of ordered parking functions with outcome π and $\mathrm{OPF}(n) := \sum \mathrm{OPF}(n, \pi)$ the number of ordered parking functions.

If we define $\mathcal{L}$-ordered parking functions analogous to how we defined $\mathcal{L}$-parking functions, then essentially the same proof used to prove Theorem 9 shows that

$$\mathrm{OPF}(n, \mathcal{L}) = \prod_{i=1}^{n} \left(\sum_{\ell \in L_i} \binom{n}{\ell} \ell! - \binom{i-1}{\ell} \ell!\right).$$

With this established, one can prove a nice analog of Theorem 1. Namely, define $\mathcal{O}(n) = \sum_{\ell=0}^{n} \binom{n}{\ell} \ell!$ to be the number of ordered subset of $[n]$. Then

$$\mathrm{OPF}(n) = \prod_{i=0}^{n-1} (\mathcal{O}(n) - \mathcal{O}(i)).$$

In the ordered setting, every π is equally likely to be the outcome of an ordered parking function, so $\mathrm{OPF}(n, \pi) = \mathrm{OPF}(n)/n!$ for all π.

3 Interval results

As before we first prove a more general theorem. Let $\mathcal{K} = (K_1, \ldots, K_n)$ be such that $K_i \subseteq [n] \setminus \{0\}$ for all i. We say that $(C_1, \ldots, C_n)$ is a $\mathcal{K}$-interval parking function if each C_i is an interval with $|C_i| \in K_i$. We define partial $\mathcal{K}$-interval parking functions analogous to how we defined partial $\mathcal{L}$-parking functions in the previous section, and as before we immediately have the following.

Lemma 11. *Let $1 \le m \le n$ and $\pi \in \mathcal{S}_n$. $\mathcal{C}$ is a $\mathcal{K}$-interval parking function with outcome π if and only if $(C_1, \ldots, C_m)$ is a partial $\mathcal{K}$-interval parking function with outcome $\pi^{(m)}$ for all m.*

We also have an analog of Lemma 8. Recall that we define $a_i(\pi)$ to be the largest $j \le i$ such that $\pi_i \ge \{\pi_i, \pi_{i-1}, \ldots, \pi_{i-j+1}\}$.

Lemma 12. *Let $1 \le m \le n$ and $\pi \in \mathcal{S}_n$. Let $(C_1, \ldots, C_{m-1})$ be a partial $\mathcal{K}$-interval parking function with outcome $\pi^{(m-1)}$ and let $p = \pi_m^{-1}$. Then $(C_1, \ldots, C_m)$ is a partial $\mathcal{K}$-interval parking function with outcome $\pi^{(m)}$ if and only if $C_m = [r, r+k-1]$ with $k \in K_i$ and*

$$\max\{p - a_p(\pi) + 1,\ p - k + 1\} \le r \le \min\{p,\ n - k + 1\}.$$

Proof. Assume $(C_1, \ldots, C_m)$ is such a partial $\mathcal{K}$-interval parking function with $C_m = [r, r + k - 1]$ for some r and k. Because $|C_m| = k$ we require $k \in K_m$, and because $C_m \subseteq [n]$ we must have $r + k - 1 \le n$. We also need $p - k + 1 \le r \le p$ so that this set contains p. Further, we require every $x \in [r, p]$ to satisfy $\pi_x < m$, otherwise p will not be the smallest element of $C_m \setminus \{\pi_i^{-1} : i < m\}$, which would contradict $(C_1, \ldots, C_m)$ having outcome $\pi^{(m)}$. By definition this will not be the case if $r < p - a_p(\pi) + 1$, so $r \ge p - a_p(\pi) + 1$. We conclude that r satisfies the desired inequalities.

Conversely, assume $C_m = [r, r + k]$ has r and k satisfying these conditions. Because $p - a_p(\pi) + 1 \ge 1$ we have $C_m \subseteq [n]$, and we also have $|C_m| = k \in K_i$. Again by definition of $a_p(\pi)$ these inequalities imply that p is the smallest element of $C_m \setminus \{\pi_i^{-1} : i < m\}$, so this gives the desired partial $\mathcal{K}$-interval parking function. $\square$

Let $\mathrm{IPF}(n, \mathcal{K}, \pi)$ denote the number of $\mathcal{K}$-interval parking functions with outcome π and define

$$b_i(\pi, k) := \begin{cases} \min\{a_i(\pi),\ k\}, & \text{if } i \leq n-k; \\ 0, & \text{if } a_i(\pi) < k+i-n; \\ \min\{n-i-k+a_i(\pi)+1, n-i+1\}, & \text{otherwise.} \end{cases}$$

Theorem 13. *For any $n \geq 1$, $\pi \in \mathcal{S}_n$, and $\mathcal{K} = (K_1, \ldots, K_n)$,*

$$\mathrm{IPF}(n, \mathcal{K}, \pi) = \prod_{i=1}^{n} \left(\sum_{k \in K_{\pi_i}} b_i(\pi, k) \right).$$

Proof. We consider the number of ways to iteratively build partial $\mathcal{K}$-interval parking functions with outcomes $\pi^{(m)}$. If one has already chosen $(C_1, \ldots, C_{i-1})$ and $p = \pi_i^{-1}$, then by Lemma 12 the number of ways to choose an appropriate C_i with $|C_i| = k \in K_i$ is the number of r in the range

$$\max\{p - a_p(\pi) + 1, p - k + 1\} \leq r \leq \min\{p, n-k+1\}.$$

If $p \leq n-k$ this number is exactly $\min\{a_p(\pi), k\}$. Otherwise it is

$$\max\{0, \min\{n-k-p+a_p(\pi)+1, n-p+1\}\}.$$

Because $n-p+1 \geq 1$, this quantity is 0 if and only if $a_p(\pi)+1 \leq k+p-n$. Thus the number of choices for C_i with $|C_i| = k$ is exactly $b_p(\pi, k)$. Summing this over all $k \in K_i$ gives a quantity independent of all the other C_j, so we can take the product of these values and conclude

$$\mathrm{IPF}(n, \mathcal{K}, \pi) = \prod_{i=1}^{n} \left(\sum_{k \in K_i} b_{\pi_i^{-1}}(\pi, k) \right).$$

By reindexing this product, we get the stated result. $\square$

Proof of Theorem 4. Recall that we wish to prove

$$\mathrm{IPF}_k(n, \pi) = \prod_{i=1}^{n-k} \min\{a_i(\pi), k\} \cdot \prod_{i=n-k+1}^{n} \min\{n-i-k+a_i(\pi)+1, n-i+1\}$$

whenever $\pi \in \mathcal{S}_n^k$. That is, whenever $\pi_n > \cdots > \pi_{n-k+1}$. Observe that k-interval parking functions are exactly $\mathcal{K}$-interval parking functions with $K_i = \{k\}$ for all i, so a formula for $\mathrm{IPF}_k(n, \pi)$ is given by Theorem 13. It remains to rewrite this formula into the desired form.

If $\pi \notin \mathcal{S}_n^k$, then there exists some i with $0 \leq i \leq k-2$ and $\pi_{n-i} < \pi_{n-i-1}$. This implies $a_{n-i}(\pi) = 1 < k-i$, and hence $b_{n-i}(\pi, k) = 0$. Thus $\mathrm{IPF}_k(n, \pi) = 0$.

From now on we assume $\pi \in \mathcal{S}_n^k$. This implies $a_{n-i}(\pi) \geq k-i$ for all $0 \leq i \leq k-1$, and hence for these i we have $b_{n-i}(\pi, k) = \min\{i-k+a_{n-i}(\pi)+1, i+1\}$. This gives $b_j(\pi, k)$ for all $j \geq n-k+1$, and otherwise we have $b_j(\pi, k) = \min\{a_j(\pi), k\}$. Taking the products of these terms gives the desired result. $\square$

Proof of Corollary 5. The statement can be verified for $n = 2$, so assume $n \geq 3$. By Theorem 2 we have for $\pi \in S_n^{n-1}$ that $\mathrm{IPF}_{n-1}(n, \pi)$ equals

$$\begin{aligned} &\min\{a_1(\pi), n-1\} \cdot \min\{a_2(\pi), n-1\} \cdot \prod_{i=3}^{n} \min\{a_i(\pi) - i + 2, n - i + 1\} \\ &= a_2(\pi) \cdot \prod_{i=3}^{n-1} \min\{a_i(\pi) - i + 2, n - i + 1\}, \end{aligned}$$

where we used $a_1(\pi) \leq 1$ and $a_2(\pi) \leq 2 \leq n - 1$. We claim that this is equal to

$$\prod_{i=2}^{n-1} (a_i(\pi) - i + 2).$$

Indeed this follows from the fact that $a_i(\pi) - i + 2 \leq 2 \leq n - i + 1$ for all $i \leq n - 1$.

Assume $\pi_1 = j$, and recall that $\pi \in \mathcal{S}_n^{n-1}$ implies that $\pi_2 < \cdots < \pi_n$. Thus for all $i > 1$ we have $a_i(\pi) = i$ if $\pi_i > j$ and $a_i(\pi) = i-1$ otherwise. Thus $j = n$ implies that $a_i(\pi) = i-1$ for all $i \geq 2$, and otherwise there are exactly $n - 1 - j$ different i with $2 \leq i \leq n - 1$ and $a_i(\pi) = i$. We conclude the first result. For the second result,

$$\mathrm{IPF}_{n-1}(n) = \sum_{\pi \in \mathcal{S}_n^{n-1}} \mathrm{IPF}_{n-1}(n, \pi) = 1 + \sum_{j=1}^{n-1} 2^{n-j-1} = 2^{n-1}.$$

□

In principle this same technique can be used to compute $\mathrm{IPF}_{n-k}(n, \pi)$ and $\mathrm{IPF}_{n-k}(n)$ for any fixed k, though the case analysis and computations become rather complicated. We note that one can prove $\mathrm{IPF}_{n-1}(n) = 2^{n-1}$ more directly by observing that $(C_1, \ldots, C_n)$ will be an $(n-1)$-interval parking function if and only if $C_n = [2, n]$ and C_i is $[1, n-1]$ or $[2, n]$ for all other i.

Before proving Corollary 6, we give an enumeration result for permutations in $\mathcal{S}_n^2$ with a given number of ascents. We adopt the convention $\left\langle {0 \atop k} \right\rangle = 0$ for $k > 0$, $\left\langle {0 \atop 0} \right\rangle = 1$, and $\left\langle {n \atop -1} \right\rangle = 0$.

Lemma 14. *For all n and k with $n \geq 1$ and $0 \leq k \leq n-1$, let $\mathcal{S}_{n,k}^{+}$ be the set of permutations of size n which have $\pi_{n-1} < \pi_n$ and which have exactly k ascents. If $P(n, k) := |\mathcal{S}_{n,k}^{+}|$, then*

$$P(n, k) = (n - k) \left\langle {n-1 \atop k-1} \right\rangle.$$

We note that this result is implicitly proven in [5], but for completeness we include the full proof here. To prove this, we recall the following recurrence for Eulerian numbers, which is valid for all $n, k \geq 1$ [3].

$$\left\langle {n \atop k} \right\rangle = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle. \tag{2}$$

Proof. The result is true for $k = 0$, so assume that we have proven the result up to $k \geq 1$. For any fixed k the result is true for $n = 1$, so assume the result has been proven up to $n \geq 2$. To help us prove the result, we define $\mathcal{S}^-_{n,k}$ to be the set of permutations which have $\pi_{n-1} > \pi_n$ and which have exactly k ascents. Define $M(n,k) := |\mathcal{S}^-_{n,k}|$. By construction we have

$$P(n,k) + M(n,k) = \left\langle {n \atop k} \right\rangle. \tag{3}$$

Define the map $\phi : \mathcal{S}^+_{n,k} \to \mathcal{S}_{n-1}$ by sending $\pi \in \mathcal{S}^+_{n,k}$ to the word obtained by removing the letter 1 from π and then decreasing the value of each letter by 1. For example, $\phi(32514) = 2143$. We wish to determine the image of ϕ. Let π be a permutation in $\mathcal{S}^+_{n,k}$, and let i denote the position of 1 in π. Note that $i \neq n$ since π ends with an ascent. If $\pi_{i-1} < \pi_{i+1}$ with $1 < i < n$, then $\phi(\pi)$ will continue to have k ascents and end with an ascent, so $\phi(\pi) \in \mathcal{S}^+_{n-1,k}$. If $i = 1$ or $\pi_{i-1} > \pi_{i+1}$ with $1 < i < n-1$, then $\phi(\pi) \in \mathcal{S}^+_{n-1,k-1}$. If $i = n-1$ and $\pi_{n-2} > \pi_n$, then $\phi(\pi) \in \mathcal{S}^-_{n-1,k-1}$.

It remains to show how many times each element of the image is mapped to by ϕ. If $\pi \in \mathcal{S}^+_{n-1,k}$, then 1 can be inserted into π in k ways to obtain an element of $\mathcal{S}^+_{n,k}$ (it can be placed between any of the k ascents $\pi_i < \pi_{i+1}$). If $\pi \in \mathcal{S}^+_{n-1,k-1}$, then 1 can be inserted into π in $n-k$ ways (it can be placed at the beginning of π or between any of the $n-1-k$ descents $\pi_i > \pi_{i+1}$). If $\pi \in \mathcal{S}^-_{n-1,k}$, then 1 must be inserted in between $\pi_{n-1} > \pi_n$ in order to have the word end with an ascent. With this and the inductive hypothesis, we conclude that

$$\begin{aligned} P(n,k) &= kP(n-1,k) + (n-k)P(n-1,k-1) + M(n-1,k-1) \\ &= k(n-k-1)\left\langle {n-2 \atop k-1} \right\rangle + (n-k)^2 \left\langle {n-2 \atop k-2} \right\rangle + M(n-1,k-1). \end{aligned} \tag{4}$$

By using (3), the inductive hypothesis, and (2); we find

$$\begin{aligned} M(n-1,k-1) &= \left\langle {n-1 \atop k-1} \right\rangle - P(n-1,k-1) \\ &= \left\langle {n-1 \atop k-1} \right\rangle - (n-k)\left\langle {n-2 \atop k-2} \right\rangle \\ &= k\left\langle {n-2 \atop k-1} \right\rangle. \end{aligned}$$

Plugging this into (4) and using (2) gives

$$P(n,k) = (n-k)\left(k\left\langle {n-2 \atop k-1} \right\rangle + (n-k)\left\langle {n-2 \atop k-2} \right\rangle \right) = (n-k)\left\langle {n-1 \atop k-1} \right\rangle,$$

as desired. □

Proof of Corollary 6. By Theorem 2 we have, after evaluating terms which are automatically 1,

$$\mathrm{IPF}_2(n,\pi)=\prod_{i=2}^{n-1}\min\{a_i(\pi),2\}.$$

Note that $a_i(\pi)\geq 2$ if and only if $\pi_{i-1}<\pi_i$. There are exactly $\mathrm{asc}(\pi)-1$ different i with $2\leq i<n$ satisfying this, where we subtract 1 since $\pi\in\mathcal{S}_n^2$ implies that there is always an ascent at position $n-1$. We conclude the first result.

For the second result, we sum $\mathrm{IPF}_2(n,\pi)$ over all $\pi\in\mathcal{S}_n^2$. Each term contributes $2^{\mathrm{asc}(\pi)-1}$, so we conclude the result by Lemma 14 after noting that $\left\langle {n-1 \atop -1} \right\rangle=0$. ◻

Corollary 6 shows that, for $n\geq 2$, $\mathrm{IPF}_2(n)$ is equal to A053525, the number of connected threshold graphs on n vertices. This can be proven bijectively from essentially the same proof as in [5], but for brevity we omit the details. The formulas for $k=3$ A327761 and $k=n-2$ A327794 seem complicated (though in principle the latter can be put into a closed form).

Before proving our enumeration results for interval parking functions, we first directly enumerate the number of parking functions with a given outcome. In what follows we treat parking functions as subset parking functions $(C_1,\ldots,C_n)$ where $C_i=[c_i,n]$ for some $1\leq c_i\leq n$. Define a partial parking function $(C_1,\ldots,C_m)$ analogous to how we defined $\mathcal{L}$-partial parking functions. We immediately have the following.

Lemma 15. *Let $1\leq m\leq n$ and $\pi\in\mathcal{S}_n$. $\mathcal{C}$ is a parking function with outcome π if and only if $(C_1,\ldots,C_m)$ is a partial interval parking function with outcome $\pi^{(m)}$ for all m.*

We also have an analog of Lemma 12.

Lemma 16. *Let $1\leq m\leq n$ and $\pi\in\mathcal{S}_n$. Let $(C_1,\ldots,C_{m-1})$ be a partial parking function with outcome $\pi^{(m-1)}$ and let $p=\pi_m^{-1}$. Then $(C_1,\ldots,C_m)$ is a partial parking function with outcome $\pi^{(m)}$ if and only if $C_m=[r,n]$ with $p-a_p(\pi)+1\leq r\leq p$.*

Proof. Assume $(C_1,\ldots,C_m)$ is such a partial parking function with $C_m=[r,n]$ for some r. We need $r\leq p$ so that this set contains j. Further, we require every $x\in[r,p]$ to satisfy $\pi_x<m$, otherwise p will not be the smallest element of $C_m\setminus\{\pi_i^{-1}:i<m\}$, which would contradict $(C_1,\ldots,C_m)$ having outcome $\pi^{(m)}$. By definition this will not be the case if $r<p-a_p(\pi)+1$, so $r\geq p-a_p(\pi)+1$. We conclude that r satisfies the desired inequalities.

Conversely, assume $C_m=[r,n]$ has r satisfying these inequalities. Because $p-a_p(\pi)+1\geq 1$ we have $C_m\subseteq[n]$. Again by definition of $a_p(\pi)$ these inequalities imply that p is the smallest element of $C_m\setminus\{\pi_i^{-1}:i<m\}$, so this gives the desired partial parking function. ◻

Proposition 17.

$$\mathrm{PF}(n,\pi)=\prod_{i=1}^{n}a_i(\pi).$$

Proof. Assume one has chosen $C_1, \ldots, C_{i-1}$ so that $(C_1, \ldots, C_{i-1})$ is a partial parking function with outcome $\pi^{(i-1)}$. There are $a_{\pi_i^{-1}}(\pi)$ choices for C_i to make $(C_1, \ldots, C_i)$ a partial parking function with outcome $\pi^{(i)}$ by Lemma 16. Every parking is obtained this way by Lemma 15, so taking the product over all these values and reindexing gives the desired result. □

We use Theorem 13 to prove Theorem 3, and to do so we require the following lemma.

Lemma 18. *For any $n \geq 1$ and $\pi \in \mathcal{S}_n$,*

$$\sum_{k=1}^{n} b_i(\pi, k) = a_i(\pi)(n - i + 1).$$

Proof. Throughout this proof we use that the "otherwise" case in the definition of $b_i(\pi, k)$ can be written as $n - i + 1 + a_i(\pi) - \max\{a_i(\pi), k\}$.

We first consider the case $a_i(\pi) \leq n-i$ and split the sum into two parts. For $a_i(\pi) < n-i$ we have

$$\begin{aligned} \sum_{k=1}^{n-i} b_i(\pi, k) = \sum_{k=1}^{n-i} \min(a_i(\pi), k) &= \sum_{k=1}^{a_i(\pi)} k + \sum_{k=a_i(\pi)+1}^{n-i} a_i(\pi) \\ &= \binom{a_i(\pi)+1}{2} + (n - i - a_i(\pi))a_i(\pi). \end{aligned} \tag{5}$$

If $a_i(\pi) = n - i$ the same formula holds by essentially the same reasoning.

If $k > n - i + a_i(\pi)$ we have $b_i(\pi, k) = 0$, so the rest of the sum is

$$\begin{aligned} \sum_{k=n-i+1}^{n-i+a_i(\pi)} b_i(\pi, k) &= (n - i + a_i(\pi) + 1)a_i(\pi) - \sum_{k=n-i+1}^{n-i+a_i(\pi)} \max\{a_i(\pi), k\} \\ &= (n - i + a_i(\pi) + 1)a_i(\pi) - \sum_{k=n-i+1}^{n-i+a_i(\pi)} k, \end{aligned} \tag{6}$$

where we used our assumption $a_i(\pi) \leq n - i < k$ in this last equality. Using the identity $\sum_{k=x+1}^{x+y} k = xy + \binom{y+1}{2}$, we conclude that (6) equals

$$(a_i(\pi) + 1)a_i(\pi) - \binom{a_i(\pi)+1}{2}.$$

Adding this to (5) gives the desired result.

Now assume $a_i(\pi) \geq n - i + 1$. In this case we have

$$\begin{aligned} \sum_{k=1}^{n-i} b_i(\pi, k) = \sum_{k=1}^{n-i} \min(a_i(\pi), k) &= \sum_{k=1}^{n-i} k \\ &= \binom{n-i+1}{2}. \end{aligned} \tag{7}$$

The rest of the sum is

$$\begin{aligned}\sum_{k=n-i+1}^{n-i+a_i(\pi)} b_i(\pi,k) &= (n-i+a_i(\pi)+1)a_i(\pi) - \sum_{k=n-i+1}^{n-i+a_i(\pi)} \max\{a_i(\pi),k\} \\ &= (n-i+a_i(\pi)+1)a_i(\pi) - a_i(\pi)(a_i(\pi)-n+i) - \sum_{k=1+a_i(\pi)}^{n-i+a_i(\pi)} k \\ &= (2n-2i+1)a_i(\pi) - \binom{n-i+1}{2} - a_i(\pi)(n-i). \end{aligned} \tag{8}$$

Adding (7) and (8) gives the desired result. □

Proof of Theorem 3. Observe that interval parking functions are exactly $\mathcal{K}$-interval parking functions with $K_i = [n]$ for all i, so by Theorem 13, Lemma 18, and Proposition 17; we have

$$\mathrm{IPF}(n,\pi) = \prod_{i=1}^{n} a_i(\pi)(n-i+1) = n! \prod_{i=1}^{n} a_i(\pi) = n! \cdot \mathrm{PF}(n,\pi).$$

Using (1), we find

$$\mathrm{IPF}(n) = \sum_{\pi\in\mathcal{S}_n} \mathrm{IPF}(n,\pi) = n! \sum_{\pi\in\mathcal{S}_n} \mathrm{PF}(n,\pi) = n!\cdot \mathrm{PF}(n) = n!\cdot(n+1)^{n-1}.$$

□

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