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Product of Consecutive Tribonacci Numbers With Only One Distinct Digit

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Abstract

Let $(F_n)_{n\geq 0}$ be the sequence of Fibonacci numbers. Marques and Togbé proved that if the product $F_n \cdots F_{n+\ell-1}$ is a repdigit (i.e., a number with only distinct digit in its decimal expansion), with at least two digits, then $(\ell, n) = (1, 10)$. In this paper, we solve the same problem with Tribonacci numbers instead of Fibonacci numbers.

1 Introduction

A positive integer is called a *repdigit* if it has only one distinct digit in its decimal expansion. The sequence of numbers with repeated digits is included in Sloane's *On-Line Encyclopedia of Integer Sequences* (OEIS) [11] as sequence A010785. In 2000, Luca [6] showed that the largest repdigit Fibonacci number is $F_{10} = 55$ and the largest repdigit Lucas number is $L_5 = 11$.

Motivated by the results of Luca [6], several authors have explored repdigits in generalizations of Fibonacci numbers and Lucas numbers. For instance, Bravo and Luca [1] showed that the only repdigit in the *k-generalized Fibonacci sequence*[1], is $F_8^{(3)} = 44$. On the other hand, Bravo and Luca [2] showed that only repdigit in the *k-generalized Lucas sequence*[2], is $L_5^{(4)} = 22$.

Recently, this problem has been extended to study the product of consecutive Fibonacci or Lucas numbers which are repdigits. Marques and Togbé [9] proved that no product of more of two consecutive Fibonacci numbers can be a repdigit with at least two digits, while Irmak and Togbé [5] proved that 77 is the only repdigit with at least two digits appearing as the product of two or more consecutive Lucas numbers.

In this paper, we investigate the presence of repdigits in the product of consecutive Tribonacci numbers. More precisely, we prove the following result.

Theorem 1. *The only solution of the Diophantine equation*

$$T_n \cdots T_{n+(\ell-1)} = d\left(\frac{10^m - 1}{9}\right), \quad \text{in positive integers } n,\ \ell,\ m,\ d, \tag{1}$$

with $1 \leq d \leq 9$ *and* $m \geq 2$ *is* $(n, \ell, m, d) = (8, 1, 2, 4)$*; i.e.,* $T_8 = 44$.

To solve equation (1), we use the 2-adic order of the Tribonacci numbers in order to bound the number of factors ℓ, then linear forms in logarithms to bound $\max\{m, n\}$ and finally a version of the Baker-Davenport Lemma to reduce such bounds to manageable values and finish off with a few calculations.

[1]The *k-generalized Fibonacci sequence* $F^{(k)}$, for an integer $k \geq 2$, satisfies that its first k terms are $0, \ldots, 0, 1$ and each term afterwards is the sum of the preceding k terms. For $k = 2$, this reduces to the familiar *Fibonacci numbers.*, while for $k = 3$ these are the *Tribonacci numbers*.

[2]The *k-generalized Lucas sequence* $L^{(k)}$ satisfies that its first k terms are $0, \ldots, 0, 2, 1$ and each term afterwards is the sum of the preceding k terms. For $k = 2$, this reduces to the familiar *Lucas numbers*.

2 Preliminary results

2.1 The Tribonacci sequence

We consider $T := (T_n)_{n\geq -1}$ given by $T_{-1} = T_0 = 0$, $T_1 = 1$ and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, for all $n \geq 0$. Its characteristic equation, $z^3 - z^2 - z - 1 = 0$, has one real root α and two complex roots β and $\gamma = \bar{\beta}$. In 1982, Spickerman [12] found the Binet formula for the Tribonacci numbers

$$T_s = a\alpha^s + b\beta^s + c\gamma^s, \quad \text{for all} \quad s \geq 0, \tag{2}$$

where

$$a = \frac{1}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{1}{(\beta-\alpha)(\beta-\gamma)}, \quad c = \frac{1}{(\gamma-\alpha)(\gamma-\beta)} = \bar{b}.$$

It is easy to see that $\alpha \in (1.83, 1.84)$, $|\beta| = |\gamma| \in (0.73, 0.74)$, $a \in (0.18, 0.19)$ and $|b| = |c| \in (0.35, 0.36)$. Since $|\beta| = |\gamma| < 1$, setting $e_s := T_s - a\alpha^s$, we have

$$T_s = a\alpha^s + e_s, \qquad \text{with} \quad |e_s| < \frac{1}{\alpha^{s/2}} \quad \text{for all} \quad s \geq 1. \tag{3}$$

Further,

$$\alpha^{s-2} \leq T_s \leq \alpha^{s-1} \quad \text{for all} \quad s \geq 1 \quad (\text{see [1]}). \tag{4}$$

We recall a result of Marques and Lengyel [8] on the 2-adic order of a Tribonacci number. For a prime number p and a nonzero integer r, the p-adic order $\upsilon_p(r)$ is the exponent of the highest power of a prime p which divides r.

Lemma 2. *For $n \geq 1$, we have*

$$\upsilon_2(T_n) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod 4; \\ 1, & \text{if } n \equiv 3,11 \pmod{16}; \\ 2, & \text{if } n \equiv 4,8 \pmod{16}; \\ 3, & \text{if } n \equiv 7 \pmod{16}; \\ \upsilon_2(n) - 1, & \text{if } n \equiv 0 \pmod{16}; \\ \upsilon_2(n+4) - 1, & \text{if } n \equiv 12 \pmod{16}; \\ \upsilon_2((n+1)(n+17)) - 3, & \text{if } n \equiv 15 \pmod{16}. \end{cases}$$

The Tribonacci sequence is included in the OEIS [11] as sequence A101292.

2.2 Linear forms in logarithms

Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial $f(X) := a_0 \prod_{i=1}^{d}(X - \eta^{(i)}) \in \mathbb{Z}[X]$, where the leading coefficient a_0 is positive. The *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\max\{|\eta^{(i)}|, 1\}\right).$$

In particular, if $\eta = p/q \in \mathbb{Q}$ with $\gcd(p,q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in this paper: $h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2$, $h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma)$ and $h(\eta^s) = |s|h(\eta)$ $(s \in \mathbb{Z})$.

Many Diophantine problems can be solved by reducing them to an instance in which one can apply lower bounds for linear forms in logarithms of algebraic numbers. We will use the following theorem, which is a variation of a result of Matveev [10], proved by Bugeaud, Mignotte and Siksek [3, Theorem 9.1].

Theorem 3. *Let $\mathbb{L}$ be a number field of degree $d_{\mathbb{L}}$ over $\mathbb{Q}$, $\eta_1, \ldots, \eta_t$ nonzero elements of $\mathbb{L}$, and $b_1, \ldots, b_t$ rational integers. Put $\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1$ and $B \geq \max\{|b_1|, \ldots, |b_t|\}$. Let $A_i \geq \max\{d_{\mathbb{L}} h(\eta_i), |\log \eta_i|, 0.16\}$ be real numbers, for $i = 1, \ldots, t$. Then, assuming that $\Lambda \neq 0$, we have*

$$|\Lambda| > \exp(-3 \cdot 30^{t+4} \cdot (t+1)^{5.5} \cdot d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log tB) A_1 \cdots A_t).$$

3 Absolute bounds on the variables

3.1 On the number ℓ of factors

We claim that $\ell \leq 6$. Indeed, from Lemma 2 we have the following table for all possible values of $n \equiv x \pmod{16}$, $x \in \{0, 1, 2, \ldots, 15\}$.

ℓ	x	$\upsilon_2(T_n T_{n+1} \cdots T_{n+(\ell-1)})$
1	15	≥ 5
2	7, 11, 14	$\geq 5, \geq 4, \geq 5$
3	6, 10, 13	$\geq 5, \geq 4, \geq 5$
4	0, 4, 5, 9, 12	$\geq 4, \geq 5, \geq 5, \geq 4, \geq 8$
5	3, 8	$\geq 6, \geq 4$
6	2	≥ 6
7	1	≥ 6

Table 1: 2-adic order of product of consecutive Tribonacci numbers

We analyze a pair of cases to illustrate the above table. The remaining cases are similar.

- $n \equiv 1 \pmod{16}$. Here, $n + 2 \equiv 3 \pmod{16}$, $n + 3 \equiv 4 \pmod{16}$ and $n + 6 \equiv 7 \pmod{16}$. So, by Lemma 2 we have $\upsilon_2(T_{n+2}) = 1$, $\upsilon_2(T_{n+3}) = 2$ and $\upsilon_2(T_{n+6}) = 3$. Hence, with $\ell = 7$, $\upsilon_2(T_n T_{n+1} \cdots T_{n+6}) \geq 6$.

- $n \equiv 12 \pmod{16}$. Then, from Lemma 2 we have $\upsilon_2(T_n) = \upsilon_2(n+4) - 1 \geq 3$, since $n + 4 \equiv 0 \pmod{16}$. On other hand, $n + 3 \equiv 15 \pmod{16}$ and $\upsilon_2(T_{n+3}) = \upsilon_2((n+4)(n+20)) - 3 \geq 5$, where we used Lemma 2 and the fact that $n + 20 \equiv 0 \pmod{16}$. Therefore, with $\ell = 4$, $\upsilon_2(T_n T_{n+1} T_{n+2} T_{n+3}) \geq 8$.

Since $\upsilon_2\left(d\left(\frac{10^m-1}{9}\right)\right)=\upsilon_2(d)\le 3$ for all $1\le d\le 9$, it then follows that $\ell\le 6$.

3.2 An absolute bound for m and n

First, assume that $n\ge 20$. Combining (1) and (4), we get

$$10^{m-1}<\alpha^{\ell(n-1)+\frac{\ell(\ell-1)}{2}}.$$

Thus,

$$m<\ell n+\ell(\ell-1)/2. \tag{5}$$

Now, by (3), we get that

$$\begin{aligned} T_n\cdots T_{n+(\ell-1)} &= (a\alpha^n+e_n)\cdots(a\alpha^{n+(\ell-1)}+e_{n+\ell-1})\\ &= a^\ell\alpha^{\ell n+\ell(\ell-1)/2}+r(a,\alpha,n,\ell)\end{aligned}$$

where $r(a,\alpha,n,\ell)$ involves the part of the expansion of the previous line that contains the product of powers of $a,\ \alpha$ and the errors e_i, for $i=n,\ldots n+(\ell-1)$. Moreover, $r(a,\alpha,n,\ell)$ is the sum of 63 terms with maximum absolute value $a^6\alpha^{(\ell-1)n+\ell(\ell-1)/2}\alpha^{-n/2}$.

Combining the above equality with (1), we obtain

$$a^\ell\alpha^{\ell n+\ell(\ell-1)/2}-\frac{d}{9}10^m=-\frac{d}{9}-r(a,\alpha,n,\ell).$$

Dividing both sides of the above equality by $a^\ell\alpha^{\ell n+\ell(\ell-1)/2}$ and taking the absolute value, we conclude that

$$\begin{aligned}\left|\frac{d}{9a^\ell}\alpha^{-(\ell n+\ell(\ell-1)/2)}10^m-1\right| &\le \left(\frac{d}{9}+|r(a,\alpha,n,\ell)|\right)\cdot a^{-\ell}\alpha^{-(\ell n+\ell(\ell-1)/2)} \\ &< (1+63a^{\ell-1}\alpha^{(\ell-1)n+\ell(\ell-1)/2}\alpha^{-n/2})\cdot a^{-\ell}\alpha^{-(\ell n+\ell(\ell-1)/2)}\\ &\le 64a^{-1}\alpha^{-3n/2},\end{aligned} \tag{6}$$

Below, we use Matveev's theorem to find a lower bound for the left-hand side of (6), with the parameters

$$t:=3,\ \ (\eta_1,b_1):=((d/9)a^{-\ell},1),\ \ (\eta_2,b_2):=(\alpha,-(\ell n+\ell(\ell-1)/2))\quad\text{and}\quad(\eta_3,b_3):=(10,m).$$

The number field containing η_1,η_2,η_3 is $\mathbb{L}:=\mathbb{Q}(\alpha,\beta)$, which has $d_{\mathbb{L}}:=6$. We claim that $\Lambda:=\eta_1^{b_1}\eta_2^{b_2}\eta_3^{b_3}-1\ne 0$. Otherwise, we get

$$a^\ell\alpha^{\ell n+\ell(\ell-1)/2}=d\cdot 10^m/9.$$

Conjugating the above relation by the automorphism $\sigma:\alpha\to\beta,\beta\to\alpha,\gamma\to\gamma$ (here, we use the fact that the Galois group of $\mathbb{L}$ over $\mathbb{Q}$ is isomorphic to S_3), and then taking absolute values on both sides of the resulting equality, we obtain

$$|b|^\ell|\beta|^{\ell n+\ell(\ell-1)/2}=d\cdot 10^m/9,$$

which is not possible because $|b|^\ell|\beta|^{\ell n+\ell(\ell-1)/2} < 1$ and $d\cdot 10^m/9 > 10$. Thus, $\Lambda \neq 0$. Next, $h(\eta_1) \leq h(d)+h(9a^\ell) \leq \log 9 + h(9) + \ell h(a)$, $h(\eta_2) = \frac{1}{3}\log\alpha$ and $h(\eta_3) = \log 10$. Now we need to estimate $h(a)$. For it, the minimal polynomial of a is $44X^3+4X-1$. So, $h(a) = \frac{1}{3}\log 44$ and $h(\eta_1) \leq 2\log 9 + 2\log 44$. Thus, we can take $A_1 := 72$, $A_2 := 2$ and $A_3 := 14$. According to (5), we take $B := \ell n + \ell(\ell-1)/2$. Applying Matveev's theorem we get a lower bound for $|\Lambda|$, which by comparing it to (6), leads to

$$\exp\left(-2.72251 \times 10^{19}(1+\log 3(\ell n + \ell(\ell-1)/2))\right) < \frac{356}{\alpha^{3n/2}}.$$

Taking logarithms in the above inequality, we get

$$\begin{aligned}\frac{3n}{2}\log\alpha - \log 356 &< 2.72251 \times 10^{19}(1+\log 3(6n+15))\\ &< 5.44503 \times 10^{19}\log(6n+15),\end{aligned}$$

where we used the fact that $1+\log 3h < 2\log h$, for all $h \geq 9$. Hence,

$$n < 6.1 \times 10^{19}\log(6n+15).$$

Therefore, we obtain $n < 3.1 \times 10^{21}$ and record what we have proved so far.

Lemma 4. *If (n,ℓ,m,d) is a positive integer solution of (1) with $n \geq 20$, $m \geq 2$ and $1 \leq d \leq 9$, then $1 \leq \ell \leq 6$ and*

$$\max\{m,n\} < 3.1 \times 10^{21}.$$

4 Reducing $\max\{m,n\}$

To lower the bound of n, we will use the following result of diophantine approximation, see [4].

Lemma 5. *Let κ be an irrational number, M be a positive integer, and p/q be a convergent of the continued fraction of κ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution of the inequality*

$$0 < |r\kappa - s + \mu| < AB^{-w}$$

in positive integers r, s and w with $r \leq M$ and $w \geq \log(Aq/\epsilon)/\log B$.

Let $\Gamma := m\log 10 - (\ell n + \ell(\ell-1)/2)\log\alpha + \log(d/9a^\ell)$. Therefore, (6) can be rewritten as $|e^\Gamma - 1| < 356/\alpha^{3n/2}$. Since $|e^\Gamma - 1| < 1/2$ for all $n \geq 20$ (because $365/\alpha^{3n/2} < 1/2$), it follows that $e^{|\Gamma|} < 2$ and so $0 < |\Gamma| \leq e^{|\Gamma|} - 1 = e^{|\Gamma|}|e^\Gamma - 1| < 712/\alpha^{3n/2}$. Hence, $0 < |\Gamma| < 712/\alpha^{3n/2}$ holds for $n \geq 20$.

Replacing Γ in the above inequality and dividing both sides of the resulting inequality by $\log\alpha$, we obtain

$$0 < |m\kappa - (\ell n + \ell(\ell-1)/2) + \mu| < 1180(\alpha^{3/2})^{-n}, \tag{7}$$

with $\kappa := \log 10/\log\alpha$ and $\mu := \log(d/9a^{\ell})/\log\alpha$. Here, we took $M := 1.9 \times 10^{22}$, which is an upper bound on m by Lemma 4, and we applied Lemma 5 to inequality (7) for each $1 \le \ell \le 6$ and $1 \le d \le 9$. With the help of the computer algebra system `Mathematica`, we found that

$$q_{47} = 938425170962281070635339 > 6M$$

is a denominator of a convergent of the continued fraction of κ such that the minimum value of $\epsilon := \|\mu q_{47}\| - M\,\|\kappa q_{47}\|$ is greater than 0.00437116. The conditions of Lemma 5 are fulfilled for $A := 1180$ and $B := \alpha^{3/2}$. Then, there are no solutions of (1) on the interval

$$\left[\left\lfloor \frac{\log(1180 q_{47}/\epsilon)}{\log B} \right\rfloor + 1, M\right) = [75, 1.9 \times 10^{22}).$$

So, $m \le 459$. Now, we start again the entire process using this much smaller bound of m. In this application of Lemma 5 we found that the assumption $n \ge 20$ implies $n \le 26$. Thus, $n \le 26$ holds. Hence, it remains to check equation (1) for $1 \le n \le 26$, $1 \le \ell \le 6$, $2 \le m \le 171$ and $1 \le d \le 9$. For this, we use *Mathematica* and conclude that the quadruple $(n, \ell, m, d) = (8, 1, 2, 4)$ is the only solution of the diophantine equation (1). This completes the proof of Theorem 1.

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