Journal of Integer Sequences, Vol. 22 (2019),
Article 19.2.7

On Properties of the General Bow Sequence

Melissa Dennison
Department of Mathematics
Baldwin Wallace University
Berea, OH 44017
USA
mdenniso@bw.edu

Abstract

In this paper, we define and study a two-parameter family of recursive sequences, which we call the *bow sequences*. The general bow sequence is defined similarly to the Stern sequence, and many of the properties of the bow sequences are related to known properties of the Stern sequence. In particular, we focus on common divisors of consecutive terms, maximum values, sums of terms, and generating functions.

1 Introduction

In 1858, Stern [19] studied the so-called "diatomic" array of integers, which was motivated by a function studied by Eisenstein. The diatomic array can be completely understood by consideration of the Stern sequence A002487. The Stern sequence is defined as follows:

$$s(0) = 0, \quad s(1) = 1; \tag{1}$$

$$s(2n) = s(n), \quad \text{for } n \geq 1; \tag{2}$$

$$s(2n+1) = s(n) + s(n+1), \quad \text{for } n \geq 1. \tag{3}$$

The terms in the Stern sequence can be written in an array, where the r^{th} row consists of $s(n)$ for $2^r \leq n \leq 2^{r+1}$, for $r \geq 0$. The even entries in each row are copied from the previous row, and the odd entries are found by adding adjacent entries in the row above. The first five rows of the array are given in Table 1, with the enumeration of terms as in Sloane's *On-Line Encyclopedia of Integer Sequences* [18].

De Rham [8] was the first to consider the Stern sequence as defined above. He attributed the name to Bachmann [4], who considered only the diatomic array. The related *Stern-Brocot array* [10] was used in defining Minkowski's ?-function [15], and the Stern sequence

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1

Table 1: The first five rows of the diatomic array for the Stern sequence

has recently been used to understand 2-regular sequences [1] and the Tower of Hanoi graph [12].

In this paper we define a new two-parameter family of recursive sequences called the *bow sequences*, which have the flipped recursion from the Stern recursion.

Definition 1. The general *bow sequence*, $b_{\alpha,\beta}(n)$, for α, $\beta \in \mathbb{Z}$ is defined by:

$$b_{\alpha,\beta}(0) = 0, \quad b_{\alpha,\beta}(1) = \alpha, \quad b_{\alpha,\beta}(2) = \beta; \tag{4}$$
$$b_{\alpha,\beta}(2n) = b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1), \quad \text{for } n \geq 2; \tag{5}$$
$$b_{\alpha,\beta}(2n+1) = b_{\alpha,\beta}(n), \quad \text{for } n \geq 1. \tag{6}$$

We define $b_{\alpha,\beta}(0) = 0$ to simplify results, although the designation does not affect later terms in the sequence, as $b_{\alpha,\beta}(0)$ does not enter into the recurrence. It is also worth noting that (5) fails in the case where $n = 1$, since if $\alpha \neq 0$, then $b_{\alpha,\beta}(2) \neq b_{\alpha,\beta}(1) + b_{\alpha,\beta}(2)$. Table 2 lists the first 17 values of the general bow sequence.

Many properties of the bow sequences are related to known properties of the Stern sequence. In particular, in Section 3 we note that although pairs of consecutive terms in the Stern sequence are always relatively prime, pairs of consecutive terms in the bow sequences may share a common factor, but triples may only share factors which divide $\gcd(\alpha, \beta)$. We also show that, much like the Stern sequence, the maximum values of the bow sequences on intervals are related to the Fibonacci numbers.

Next, in Section 4, we derive properties of the summatory function. Then we derive the generating function for the general bow sequence, and give interpretations of the generating function for two basic cases in Section 5. In particular, we note that $b_{0,1}(n)$ is the number of ways of writing $n-2$ as the sum $\displaystyle\sum_i c_i 2^i$ where $c_i \in \{0, 2, 3\}$, while $b_{1,0}(n)$ is the number of ways of writing $n-3$ as a finite sum $\displaystyle\sum_{i=0}^{k} c_i 2^i$ where $c_i \in \{1, 3, 4\}$. Lastly, we look at an alternative representation of $b_{0,1}(n)$ in Section 6.

n	$b_{\alpha,\beta}(n)$
0	0
1	α
2	β
3	α
4	$\alpha+\beta$
5	β
6	$2\alpha+\beta$
7	α
8	$\alpha+2\beta$
9	$\alpha+\beta$
10	$2\alpha+2\beta$
11	β
12	$3\alpha+\beta$
13	$2\alpha+\beta$
14	$2\alpha+2\beta$
15	α
16	$2\alpha+3\beta$

Table 2: The general bow sequence

2 Preliminaries

We begin by considering the values of the general bow sequence. Note that the terms of the general bow sequence are not necessarily distinct. In particular, we can see from Table 2 that $b_{\alpha,\beta}(10) = b_{\alpha,\beta}(14)$.

By considering the recursion, we see that $b_{\alpha,\beta}(n)$ is linear in α and β. Accordingly, we can write

$$b_{\alpha,\beta}(n) = \alpha b_{1,0}(n) + \beta b_{0,1}(n). \tag{7}$$

For simplicity, we shall define

$$b_0(n) := b_{0,1}(n),$$

and

$$b_1(n) := b_{1,0}(n).$$

In this paper we consider the sequences $b_0(n)$ A106345 and $b_1(n)$ A281185, as all other cases of the general bow sequence can be determined from these two cases by applying equation (7).

We can present the values of the general bow sequence as a "triatomic" array where the first row is $b_{\alpha,\beta}(1)$, $b_{\alpha,\beta}(2)$, $b_{\alpha,\beta}(3)$, and the second row is $b_{\alpha,\beta}(3)$, $b_{\alpha,\beta}(4)$, $b_{\alpha,\beta}(5)$, $b_{\alpha,\beta}(6)$, $b_{\alpha,\beta}(7)$. In general the r-th row contains $b_{\alpha,\beta}(2^r-1)$, ..., $b_{\alpha,\beta}(2^{r+1}-1)$. The odd entries in each row are copied from the previous row in Table 3. The even entries are found by adding the two numbers in the previous row that occur to the right of the entry. For example, the second entry in the third row, $b_{\alpha,\beta}(8) = \alpha + 2\beta$, is obtained by adding the second and third entries in the second row of the array, namely, $\alpha+\beta$ and β. The last even entry in each row is obtained by summing the first two entries in that same row. The first three rows of the array are given in Table 3.

α				β				α
α		$\alpha+\beta$		β		$2\alpha+\beta$		α
α	$\alpha+2\beta$	$\alpha+\beta$	$2\alpha+2\beta$	β	$3\alpha+\beta$	$2\alpha+\beta$	$2\alpha+2\beta$	α

Table 3: First three rows of the triatomic array for the general bow sequence

The triatomic arrays for $b_0(n)$ and $b_1(n)$ are quite different from each other. As it turns out, $b_0(n)$ is much more regular than $b_1(n)$, but the two sequences share many properties. Consider Table 4 and Table 5:

0								1								0
0				1				1				1				0
0		2		1		2		1		1		1		2		0
0	3	2	3	1	3	2	2	1	2	1	3	1	2	2	3	0

Table 4: First four rows of the triatomic array for $b_0(n)$

1								0								1
1				1				0				2				1
1		1		1		2		0		3		2		2		1
1	2	1	3	1	2	2	3	0	5	3	4	2	3	2	3	1

Table 5: First four rows of the triatomic array for $b_1(n)$

3 Greatest common divisors and maximum values

3.1 Greatest common divisors of three consecutive terms

One interesting property of the bow sequences is that although two consecutive terms may share a nontrivial factor, three consecutive terms can only share factors which divide

$\gcd(\alpha, \beta)$.

Theorem 2. *For all* $\{\alpha, \beta\}$ *and* $n > 0$,

$$\gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) = \gcd(\alpha, \beta). \tag{8}$$

Proof. Observe from Table 2 that $\gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) = \gcd(\alpha, \beta)$ for $n = 1,\ 2$. Now assume that this holds true for $n < k$.

Case 1: $k = 2n+1$, with $n < k$. Then,

$$\begin{aligned}
&\gcd(b_{\alpha,\beta}(k), b_{\alpha,\beta}(k+1), b_{\alpha,\beta}(k+2)) \\
&= \gcd(b_{\alpha,\beta}(2n+1), b_{\alpha,\beta}(2n+2), b_{\alpha,\beta}(2n+3)) \\
&= \gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2), b_{\alpha,\beta}(n+2)) \\
&= \gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) \\
&= \gcd(\alpha, \beta).
\end{aligned}$$

Case 2: $k = 2n$, with $n < k$. Then,

$$\begin{aligned}
&\gcd(b_{\alpha,\beta}(k), b_{\alpha,\beta}(k+1), b_{\alpha,\beta}(k+2)) \\
&= \gcd(b_{\alpha,\beta}(2n), b_{\alpha,\beta}(2n+1), b_{\alpha,\beta}(2n+2)) \\
&= \gcd(b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2)) \\
&= \gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) \\
&= \gcd(\alpha, \beta).
\end{aligned}$$

Thus $\gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) = \gcd(\alpha, \beta)$ for all $\{\alpha, \beta\}$ and $n > 0$. ◻

Remark 3. This result bears some similarity to a result for pairs of consecutive terms of the Stern sequence. As noted by Stern [6, 13, 19], for $n \geq 0$, every pair of consecutive terms of the Stern sequence is relatively prime, or $\gcd(s(n), s(n+1)) = 1$.

3.2 Greatest common divisors of pairs

Surprisingly, Theorem 2 has implications for pairs. Although we have seen in Table 4 and Table 5 that pairs frequently share common factors, when the pairs

$$(b_0(n), b_0(n+1)),\ \text{and}\ (b_1(n), b_1(n+1))$$

are taken together, the resulting quadruple does not have a common factor.

Theorem 4. *For* $n \geq 0$, $\gcd(b_0(n), b_0(n+1), b_1(n), b_1(n+1)) = 1$.

Proof. Suppose p is prime, and let

$$p \mid \gcd(b_0(n), b_0(n+1), b_1(n), b_1(n+1)).$$

Then, p must necessarily divide $b_0(n)$ and $b_0(n+1)$, so we know by Theorem 2 that $p \nmid b_0(n+2)$. Similarly, $p \nmid b_1(n+2)$. So let

$$b_0(n+2) \equiv r \pmod{p} \qquad \text{and} \qquad b_1(n+2) \equiv s \pmod{p},$$

where $1 \leq r, s \leq p-1$.

Consider $b_{s,-r}(n) = sb_1(n) - rb_0(n)$. Thus $p \mid b_{s,-r}(n)$, and $p \mid b_{s,-r}(n+1)$. Then, since

$$b_{s,-r}(n+2) \equiv sr + (-r)s \equiv 0 \pmod{p},$$

we know that $p \mid b_{s,-r}(n+2)$. But this implies that $p \mid \gcd(r, s)$, which is a contradiction. Thus $\gcd(b_0(n), b_0(n+1), b_1(n), b_1(n+1)) = 1$. □

Remark 5. Note that Theorem 4 fails if we consider only three of the four terms. For example, consider $n = 2149948$. Then we have

$$\begin{aligned}(b_0(n), b_0(n+1), b_1(n), b_1(n+1)) &= (2070, 1815, 1430, 1254)\\ &= (2 \cdot 3^2 \cdot 5 \cdot 23, 3 \cdot 5 \cdot 11^2, 2 \cdot 5 \cdot 11 \cdot 13, 2 \cdot 3 \cdot 11 \cdot 19).\end{aligned}$$

When we consider the triples, we find that

$$\begin{aligned}\gcd(b_0(n), b_0(n+1), b_1(n)) &= 5,\\ \gcd(b_0(n), b_0(n+1), b_1(n+1)) &= 3,\\ \gcd(b_0(n), b_1(n), b_1(n+1)) &= 2,\\ \gcd(b_0(n+1), b_1(n), b_1(n+1)) &= 11.\end{aligned}$$

However, clearly $\gcd(b_0(n), b_0(n+1), b_1(n), b_1(n+1)) = 1$.

Similarly, we have the following theorem.

Theorem 6. *For $n \geq 0$, $\gcd(b_0(n), b_0(n+2), b_1(n), b_1(n+2)) = 1$.*

Proof. Suppose p is prime, and let

$$p \mid \gcd(b_0(n), b_0(n+2), b_1(n), b_1(n+2)).$$

Then, p must necessarily divide $b_0(n)$ and $b_0(n+2)$, so we know by Theorem 2 that $p \nmid b_0(n+1)$. Similarly, $p \nmid b_1(n+1)$. So let

$$b_0(n+1) \equiv r \pmod{p} \qquad \text{and} \qquad b_1(n+1) \equiv s \pmod{p},$$

where $1 \leq r, s \leq p-1$.

Consider $b_{s,-r}(n) = sb_1(n) - rb_0(n)$. Thus $p \mid b_{s,-r}(n)$, and $p \mid b_{s,-r}(n+2)$. Then, since

$$b_{s,-r}(n+1) \equiv sr + (-r)s \equiv 0 \pmod{p},$$

we know that $p \mid b_{s,-r}(n+1)$. But this implies that $p \mid \gcd(r, s)$, which is a contradiction. Thus $\gcd(b_0(n), b_0(n+2), b_1(n), b_1(n+2)) = 1$. □

3.3 Maximum values on intervals

For the next theorem, we will need the Fibonacci numbers A000045, defined as usual by

$$\begin{aligned} F_0 &= 0, \quad F_1 = 1; \\ F_n &= F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2. \end{aligned}$$

Theorem 7. *For $r \geq 1$,*

$$b_{\alpha,\beta}(2^r) = \alpha F_{r-1} + \beta F_r. \tag{9}$$

In particular, for $r \geq 1$,

$$b_0(2^r) = F_r, \quad b_1(2^r) = F_{r-1}, \quad \textit{and} \quad b_{1,1}(2^r) = F_{r+1}.$$

Proof. First, note that by applying (5) and (6) we find for $n \geq 1$,

$$b_{\alpha,\beta}(4n) = b_{\alpha,\beta}(2n) + b_{\alpha,\beta}(2n+1) = b_{\alpha,\beta}(2n) + b_{\alpha,\beta}(n).$$

Thus, for $r \geq 2$,

$$b_{\alpha,\beta}(2^r) = b_{\alpha,\beta}(2^{r-1}) + b_{\alpha,\beta}(2^{r-2}).$$

Secondly, recall by (7) that

$$b_{\alpha,\beta}(2^r) = \alpha b_1(2^r) + \beta b_0(2^r).$$

Thus we consider only these two cases. All that remains is to check the initial conditions. We find that

$$(b_0(2), b_0(4)) = (0, 1) = (F_0, F_1),$$

and

$$(b_1(2), b_1(4)) = (1, 1) = (F_1, F_2).$$

Thus the initial conditions are satisfied. $\square$

Next, we estimate the size of $b_{\alpha,\beta}(n)$. Let $I_r = (2^{r-1}, 2^r] \cap \mathbb{Z}$. To be specific,

$$I_r = \{2^{r-1} + 1, \ldots, 2^r\}.$$

Note that

$$I_r = (2I_{r-1} - 1) \cup 2I_{r-1}.$$

We have the following upper bound.

Theorem 8. *For $r \geq 1$,*

$$\max_{n\in I_r} |b_{\alpha,\beta}(n)| \leq \max\{|\alpha|, |\beta|\} F_{r+1}. \tag{10}$$

Moreover, for $\beta \geq \alpha \geq 0$,

$$\max_{n\in I_r} |b_{\alpha,\beta}(n)| = b_{\alpha,\beta}(2^r) = \alpha F_{r-1} + \beta F_r. \tag{11}$$

Proof. Since $b_{\alpha,\beta}(n) \geq 0$ for α, $\beta \geq 0$, $|b_{\alpha,\beta}(n)| \leq b_{|\alpha|,|\beta|}(n)$, and we may assume that α, $\beta \geq 0$. Observe that for α, $\beta \geq 0$ and $\gamma = \max\{\alpha, \beta\}$,

$$\begin{aligned} b_{\alpha,\beta}(n) \leq b_{\gamma,\gamma}(n) &= \gamma b_{1,1}(n) \\ &= \gamma\left(b_0(n) + b_1(n)\right). \end{aligned}$$

For $r = 1, 2$, it is easy to see that $\max_{n\in I_r} b_{1,1}(n)$ is F_{r+1}. Now assume that this holds for all $r < k$. Let $r = k$. Then

$$\begin{aligned} \max_{n\in I_k} b_{1,1}(n) &= \max_{n\in I_{k-1}} \{b_{1,1}(2n), b_{1,1}(2n-1)\} \\ &= \max_{n\in I_{k-1}} \{b_{1,1}(n) + b_{1,1}(n+1), b_{1,1}(n-1)\}. \end{aligned}$$

However, one of n or $n+1$ is odd. Thus, if $n, n+1 \in I_{k-1}$, then

$$\begin{aligned} \max_{n\in I_{k-1}} \{b_{1,1}(n) + b_{1,1}(n+1), b_{1,1}(n-1)\} &\leq \max\{F_k + F_{k-1}, F_k\} \\ &= \max\{F_{k+1}, F_k\} \\ &= F_{k+1}. \end{aligned}$$

The previous argument fails if $n+1$ is not in I_{k-1}. However, then

$$n + 1 = 2^{k-1} + 1,$$

and

$$b_{1,1}(2^{k-1} + 1) = b_{1,1}(2^{k-2}) = F_{k-1}$$

by Theorem 7.

Thus we have shown that

$$\max_{n\in I_k} b_{\gamma,\gamma}(n) \leq \gamma F_{k+1}.$$

Now we will show that $\max_{n\in I_r} b_0(n) \le F_r$. For $r = 1, 2$, it is easy to see that $\max_{n\in I_r} b_0(n) = F_r$. Now assume that this statement also holds for all $r < k$. Let $r = k$. Then

$$\begin{aligned}\max_{n\in I_k} b_0(n) &= \max_{n\in I_{k-1}} \{b_0(2n), b_0(2n-1)\}\\ &= \max_{n\in I_{k-1}} \{b_0(n) + b_0(n+1), b_0(n-1)\}.\end{aligned}$$

Similarly, one of n or $n+1$ is odd, with the same result if $n+1 = 2^{k-1}+1$. Thus, if $n, n+1 \in I_{k-1}$, then

$$\begin{aligned}\max_{n\in I_{k-1}} \{b_0(n) + b_0(n+1), b_0(n-1)\} &\le \max\{F_{k-1} + F_{k-2}, F_{k-1}\}\\ &= \max\{F_k, F_{k-1}\}\\ &= F_k.\end{aligned}$$

With $\delta > 0$ and $b_{0,\delta}(n) = \delta b_0(n)$, we have $\max_{n\in I_r} b_{0,\delta}(n) \le \delta F_r$.

Now we shall show that the maximum occurs at $n = 2^r$. Let $\beta \ge \alpha \ge 0$. Then, by setting $\gamma = \alpha$ and $\delta = \beta - \alpha$ in the previous results above, we find

$$\begin{aligned}b_{\alpha,\beta}(2^r) &= \alpha b_{1,1}(2^r) + (\beta - \alpha) b_0(2^r)\\ &\le \alpha F_{r+1} + (\beta - \alpha) F_r\\ &= \alpha F_{r-1} + \beta F_r.\end{aligned}$$

However, by Theorem 7 we know that $b_{\alpha,\beta}(2^r) = \alpha F_{r-1} + \beta F_r$, thus the maximum occurs at $n = 2^r$. □

Remark 9. As noted by Lucas [14] and Lehmer [13], the maximum value of $s(n)$ on I_r is F_{r+1}.

4 Properties of the summatory function

Next, we consider the sums of the rows of the triatomic array. We define a new function as follows:

Definition 10. $D_{\alpha,\beta}(r)$ is defined as the sum of the r^{th} row of the triatomic array of the bow sequence,

$$D_{\alpha,\beta}(r) := \sum_{n\in I_{r+1}} b_{\alpha,\beta}(n) = \sum_{n=2^r+1}^{2^{r+1}} b_{\alpha,\beta}(n). \tag{12}$$

Lemma 11. *For $r \geq 5$,*

$$D_{\alpha,\beta}(r) = 3D_{\alpha,\beta}(r-1) - \alpha F_{r-5} - \beta F_{r-4}. \tag{13}$$

Proof. Starting with the definition of $D_{\alpha,\beta}(r)$ and applying the recurrence, we find

$$\begin{aligned} D_{\alpha,\beta}(r) &= \sum_{n \in I_{r+1}} b_{\alpha,\beta}(n) \\ &= \sum_{n \in I_r} b_{\alpha,\beta}(2n) + b_{\alpha,\beta}(2n-1) \\ &= \sum_{n \in I_r} b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n-1) \\ &= \left(\sum_{n \in I_r} 3b_{\alpha,\beta}(n) \right) + b_{\alpha,\beta}(2^r+1) - b_{\alpha,\beta}(2^{r-1}+1) + b_{\alpha,\beta}(2^{r-1}) - b_{\alpha,\beta}(2^r) \\ &= 3D_{\alpha,\beta}(r-1) + b_{\alpha,\beta}(2^{r-1}) - b_{\alpha,\beta}(2^{r-2}) + b_{\alpha,\beta}(2^{r-1}) - b_{\alpha,\beta}(2^r). \end{aligned}$$

Then, by applying Theorem 7 and simplifying we get

$$\begin{aligned} D_{\alpha,\beta}(r) &= 3D_{\alpha,\beta}(r-1) + 2(\alpha F_{r-2} + \beta F_{r-1}) - \alpha F_{r-3} - \beta F_{r-2} - \alpha F_{r-1} - \beta F_r \\ &= 3D_{\alpha,\beta}(r-1) + \alpha\left(F_{r-1} - 3F_{r-3}\right) + \beta\left(F_r - 3F_{r-2}\right) \\ &= 3D_{\alpha,\beta}(r-1) + \alpha\left(F_{r-2} - 2F_{r-3}\right) + \beta\left(F_{r-1} - 2F_{r-2}\right) \\ &= 3D_{\alpha,\beta}(r-1) + \alpha\left(F_{r-4} - F_{r-3}\right) + \beta\left(F_{r-3} - F_{r-2}\right) \\ &= 3D_{\alpha,\beta}(r-1) - \alpha F_{r-5} - \beta F_{r-4}. \end{aligned}$$

□

Lemma 11 suggests a relationship of the following form.

Theorem 12. *For $r \geq 1$,*

$$D_{\alpha,\beta}(r) = \alpha\left(\frac{7}{5} \cdot 3^{r-1} + \frac{3}{5} \cdot F_r - \frac{4}{5} \cdot F_{r-1}\right) + \beta\left(\frac{2}{5} \cdot 3^r - \frac{1}{5} \cdot F_r + \frac{3}{5} \cdot F_{r-1}\right).$$

Proof. By considering Table 2, we check that this holds for small r. Suppose that the formula holds for $r < k$. Then by Lemma 11,

$$\begin{aligned}
D_{\alpha,\beta}(k) &= 3\alpha\left(\frac{7}{5}\cdot 3^{k-2} + \frac{3}{5}\cdot F_{k-1} - \frac{4}{5}\cdot F_{k-2}\right) + 3\beta\left(\frac{2}{5}\cdot 3^{k-1} - \frac{1}{5}\cdot F_{k-1} + \frac{3}{5}\cdot F_{k-2}\right) \\
&\quad + \alpha F_{k-1} - 3\alpha F_{k-3} + \beta F_k - 3\beta F_{k-2} \\
&= \alpha\left(\frac{7}{5}\cdot 3^{k-1} + \frac{9}{5}\cdot F_{k-1} - \frac{12}{5}\cdot F_{k-2} + F_{k-1} - 3F_{k-3}\right) \\
&\quad + \beta\left(\frac{2}{5}\cdot 3^{k} - \frac{3}{5}\cdot F_{k-1} + \frac{9}{5}\cdot F_{k-2} + F_k - 3F_{k-2}\right) \\
&= \alpha\left(\frac{7}{5}\cdot 3^{k-1} + \frac{3}{5}\cdot F_{k} - \frac{4}{5}\cdot F_{k-1}\right) + \beta\left(\frac{2}{5}\cdot 3^{k} - \frac{1}{5}\cdot F_{k} + \frac{3}{5}\cdot F_{k-1}\right).
\end{aligned}$$

Thus the theorem holds for $r \geq 1$. $\square$

Remark 13. Theorem 12 implies that

$$\frac{D_{\alpha,\beta}(r)}{3^r} = \frac{7\alpha + 6\beta}{15} + \mathcal{O}\left(\left(\frac{\phi}{3}\right)^r\right) = \frac{7\alpha + 6\beta}{15} + o(1).$$

Remark 14. Interestingly, as noted by Stern [19] and Lehmer [13], the sum of the r^{th} row of the diatomic array for the Stern sequence is $3^r + 1$.

Remark 15. Note that $D_{-6,7}(r) = -5F_r + 9F_{r-1}$, which means that the average value of $b_{-6,7}(n) \to 0$ as $n \to \infty$.

Now suppose we sum all the terms of the general bow sequence up to the N^{th} term. Note that summatory functions of k-regular sequences are again k-regular sequences, as they are obtained by convolution with the all 1 sequence, which is also k-regular. The result is k-regular, as noted by Allouche and Shallit [2], which explains in some sense Theorem 17. Define a new function as follows:

Definition 16. $E_{\alpha,\beta}(N)$ is defined as the finite sum of the first N terms of the bow sequence,

$$E_{\alpha,\beta}(N) := \sum_{k=1}^{N} b_{\alpha,\beta}(k). \tag{14}$$

Theorem 17. *For $N \geq 1$ we have the following:*

$$E_{\alpha,\beta}(2N) = 3E_{\alpha,\beta}(N) - \alpha - b_{\alpha,\beta}(N) + b_{\alpha,\beta}(N+1). \tag{15}$$

Proof. One can calculate quickly that this formula holds for $N < 4$. Let $N \geq 4$. We start with the definition, separate terms and apply the recursion to find

$$\begin{aligned}
E_{\alpha,\beta}(2N) &= \sum_{k=1}^{2N} b_{\alpha,\beta}(k) \\
&= b_{\alpha,\beta}(1) + b_{\alpha,\beta}(2) + b_{\alpha,\beta}(3) + \sum_{k=4}^{2N} b_{\alpha,\beta}(k) \\
&= b_{\alpha,\beta}(1) + b_{\alpha,\beta}(2) + b_{\alpha,\beta}(3) + \sum_{k=2}^{N} (b_{\alpha,\beta}(2k) + b_{\alpha,\beta}(2k+1)) - b_{\alpha,\beta}(2N+1) \\
&= b_{\alpha,\beta}(1) + b_{\alpha,\beta}(2) + b_{\alpha,\beta}(3) + \sum_{k=2}^{N} (2b_{\alpha,\beta}(k) + b_{\alpha,\beta}(k+1)) - b_{\alpha,\beta}(2N+1).
\end{aligned}$$

Using Table 2 we see that

$$\begin{aligned}
E_{\alpha,\beta}(2N) &= 2\alpha + \beta + \sum_{k=2}^{N} (2b_{\alpha,\beta}(k) + b_{\alpha,\beta}(k+1)) - b_{\alpha,\beta}(2N+1) \\
&= \sum_{k=1}^{N} 2b_{\alpha,\beta}(k) + \sum_{k=1}^{N} b_{\alpha,\beta}(k+1) - b_{\alpha,\beta}(2N+1).
\end{aligned}$$

We reindex the second sum and apply (14) to obtain

$$\begin{aligned}
E_{\alpha,\beta}(2N) &= 2E_{\alpha,\beta}(N) + \sum_{k=2}^{N+1} b_{\alpha,\beta}(k) - b_{\alpha,\beta}(2N+1) \\
&= 2E_{\alpha,\beta}(N) + E_{\alpha,\beta}(N) - \alpha + b_{\alpha,\beta}(N+1) - b_{\alpha,\beta}(2N+1).
\end{aligned}$$

Then by applying the recursion we get the desired result

$$E_{\alpha,\beta}(2N) = 3E_{\alpha,\beta}(N) - \alpha - b_{\alpha,\beta}(N) + b_{\alpha,\beta}(N+1).$$

□

Corollary 18. *For $r \geq 3$ we have*

$$E_{\alpha,\beta}(2^{r+1}) = 3E_{\alpha,\beta}(2^r) - \alpha(F_{r-3} + 1) - \beta F_{r-2}. \tag{16}$$

Proof. By Theorem 17 we have

$$E_{\alpha,\beta}(2^{r+1}) = 3E_{\alpha,\beta}(2^r) - \alpha - b_{\alpha,\beta}(2^r) + b_{\alpha,\beta}(2^r + 1).$$

By applying Theorem 7 and (6), and simplifying we obtain

$$\begin{aligned} E_{\alpha,\beta}(2^{r+1}) &= 3E_{\alpha,\beta}(2^r) - \alpha - (\alpha F_{r-1} + \beta F_r) + (\alpha F_{r-2} + \beta F_{r-1}) \\ &= 3E_{\alpha,\beta}(2^r) - \alpha - \alpha F_{r-3} - \beta F_{r-2} \\ &= 3E_{\alpha,\beta}(2^r) - \alpha(F_{r-3} + 1) - \beta F_{r-2}. \end{aligned}$$

□

Corollary 18 suggests the following relationship.

Theorem 19. *For $r \geq 1$ we have*

$$E_{\alpha,\beta}(2^r) = \alpha\left(\frac{7}{10} \cdot 3^{r-1} - \frac{1}{5} \cdot F_r + \frac{3}{5} \cdot F_{r-1} + \frac{1}{2}\right) + \beta\left(\frac{1}{5} \cdot 3^r + \frac{2}{5} \cdot F_r - \frac{1}{5} \cdot F_{r-1}\right). \quad (17)$$

Proof. By considering Table 2, we check that this holds for small $r \geq 1$. Suppose that the formula holds for $r < k$. Then, by Corollary 18,

$$\begin{aligned} E_{\alpha,\beta}(2^r) = 3\alpha&\left(\frac{7}{10} \cdot 3^{r-2} - \frac{1}{5} \cdot F_{r-1} + \frac{3}{5} \cdot F_{r-2} + \frac{1}{2}\right) \\ &+ 3\beta\left(\frac{1}{5} \cdot 3^{r-1} + \frac{2}{5} \cdot F_{r-1} - \frac{1}{5} \cdot F_{r-2}\right) - \alpha(F_{r-4} + 1) - \beta F_{r-3} \\ =\ &\alpha\left(\frac{7}{10} \cdot 3^{r-1} - \frac{3}{5} \cdot F_{r-1} + \frac{9}{5} \cdot F_{r-2} + \frac{3}{2} - F_{r-4} - 1\right) \\ &+ \beta\left(\frac{1}{5} \cdot 3^r + \frac{6}{5} \cdot F_{r-1} - \frac{3}{5} \cdot F_{r-2} - F_{r-3}\right). \end{aligned}$$

Then, by simplifying the Fibonacci terms, we find

$$\begin{aligned} E_{\alpha,\beta}(2^r) &= \alpha\left(\frac{7}{10} \cdot 3^{r-1} - \frac{3}{5} \cdot F_{r-1} + \frac{4}{5} \cdot F_{r-2} + F_{r-3} + \frac{1}{2}\right) \\ &\quad + \beta\left(\frac{1}{5} \cdot 3^r + \frac{1}{5} \cdot F_{r-1} + \frac{2}{5} \cdot F_{r-2}\right) \\ &= \alpha\left(\frac{7}{10} \cdot 3^{r-1} - \frac{1}{5} \cdot F_r + \frac{3}{5} \cdot F_{r-1} + \frac{1}{2}\right) + \beta\left(\frac{1}{5} \cdot 3^r + \frac{2}{5} \cdot F_r - \frac{1}{5} \cdot F_{r-1}\right). \end{aligned}$$

Thus the theorem holds for $r \geq 1$. □

Remark 20. Theorem 19 implies that

$$\frac{E_{\alpha,\beta}(2^r)}{3^r} = \frac{7\alpha + 6\beta}{30} + \mathcal{O}\left(\left(\frac{\phi}{3}\right)^r\right) = \frac{7\alpha + 6\beta}{30} + o(1).$$

Remark 21. By comparison, as noted by Stern [19] and Lehmer [13], the sum of the first 2^r terms of the Stern sequence is

$$\frac{1}{2}\left(3^r + 1\right).$$

5 Generating functions

We use the following notation for generating functions:

$$G^{a_1,a_2,a_3,\ldots,a_m}(x) := x\prod_{j=0}^{\infty}(1 + x^{a_1\cdot 2^j} + x^{a_2\cdot 2^j} + \cdots + x^{a_m\cdot 2^j}) \tag{18}$$

$$:= \sum_{n=1}^{\infty} c^{a_1,a_2,a_3,\ldots,a_m}(n)x^n. \tag{19}$$

Combinatorially, this means that $c^{a_1,a_2,a_3,\ldots,a_m}(n)$ is the number of ways of writing $n-1$ as the sum $\sum_{i\geq 0}\gamma_i 2^i$ where $\gamma_i \in \{0, a_1, a_2, a_3, \ldots, a_m\}$.

Example 22. Consider

$$\frac{1}{x}G^1(x) = \prod_{n=0}^{\infty}(1 + x^{2^j}) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Thus we recover the fact that every integer $n > 0$ has a unique representation

$$n = \sum_{i=0}^{\infty}\gamma_i 2^i,$$

where $\gamma_i \in \{0, 1\}$.

We now consider the generating function for the general bow sequence. First, let

$$B_{\alpha,\beta}(x) = \sum_{n\geq 0} b_{\alpha,\beta}(n)x^n = \alpha B_{1,0}(x) + \beta B_{0,1}(x). \tag{20}$$

It follows from (10) that $B_{1,0}(x)$ and $B_{0,1}(x)$ have radius of convergence 1.

By repeatedly applying (5) and (6) and tweaking the limits, we have

$$\begin{aligned}
B_{\alpha,\beta}(x) &= \alpha x + \beta x^2 + \sum_{n\geq 3} b_{\alpha,\beta}(n)x^n \\
&= \alpha x + \beta x^2 + \sum_{n\geq 1} b_{\alpha,\beta}(2n+1)x^{2n+1} + \sum_{n\geq 2} b_{\alpha,\beta}(2n)x^{2n} \\
&= \alpha x + \beta x^2 + \sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n+1} + \sum_{n\geq 2}(b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1))x^{2n} \\
&= \alpha x + \beta x^2 + \sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n+1} + \sum_{n\geq 2} b_{\alpha,\beta}(n)x^{2n} + \sum_{n\geq 2} b_{\alpha,\beta}(n+1)x^{2n} \\
&= \alpha x + \beta x^2 + \sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n+1} + \sum_{n\geq 2} b_{\alpha,\beta}(n)x^{2n} + \sum_{n\geq 3} b_{\alpha,\beta}(n)x^{2n-2}
\end{aligned}$$

$$\begin{aligned}
&= \alpha x + \beta x^2 + x \sum_{n\geq 1} b_{\alpha,\beta}(n) x^{2n} + \sum_{n\geq 2} b_{\alpha,\beta}(n) x^{2n} + \frac{1}{x^2} \sum_{n\geq 3} b_{\alpha,\beta}(n) x^{2n} \\
&= \alpha x + \beta x^2 + x \sum_{n\geq 1} b_{\alpha,\beta}(n) x^{2n} + \sum_{n\geq 1} b_{\alpha,\beta}(n) x^{2n} - \alpha x^2 \\
&\quad + \frac{1}{x^2} \sum_{n\geq 1} b_{\alpha,\beta}(n) x^{2n} - \alpha - \beta x^2 \\
&= -\alpha + \alpha x - \alpha x^2 + \left(\frac{1}{x^2} + 1 + x \right) B_{\alpha,\beta}(x^2) \\
&= -\alpha(1 - x + x^2) + \frac{1}{x^2}(1 + x^2 + x^3) B_{\alpha,\beta}(x^2).
\end{aligned}$$

Since $b_{\alpha,\beta}(0) = 0$ and $b_{\alpha,\beta}(1) = \alpha$, we can write $B_{\alpha,\beta}(x) = \alpha x + x^2 C_{\alpha,\beta}(x)$, with $C_{\alpha,\beta}(0) = \beta$, so $B_{\alpha,\beta}(x^2) = \alpha x^2 + x^4 C_{\alpha,\beta}(x^2)$. Thus we have

$$\begin{aligned}
B_{\alpha,\beta}(x) &= \alpha x + x^2 C_{\alpha,\beta}(x) \\
&= \frac{1}{x^2}(1 + x^2 + x^3)(\alpha x^2 + x^4 C_{\alpha,\beta}(x^2)) - \alpha(1 - x + x^2).
\end{aligned}$$

Solving for $C_{\alpha,\beta}(x)$, we find

$$\begin{aligned}
C_{\alpha,\beta}(x) &= \frac{1}{x^2}\Big(\alpha(1 - x + x^2 + x^3) + x^2 C_{\alpha,\beta}(x^2)(1 + x^2 + x^3) - \alpha(1 - x + x^2) \Big) \\
&= \alpha x + C_{\alpha,\beta}(x^2)(1 + x^2 + x^3).
\end{aligned}$$

We can feed this identity back into the equation to get

$$\begin{aligned}
C_{\alpha,\beta}(x) &= \alpha x + (1 + x^2 + x^3)(\alpha x^2 + C_{\alpha,\beta}(x^4)(1 + x^4 + x^6)) \\
&= \alpha x + (1 + x^2 + x^3)\alpha x^2 + (1 + x^2 + x^3)(1 + x^4 + x^6) C_{\alpha,\beta}(x^4).
\end{aligned}$$

After N steps,

$$C_{\alpha,\beta}(x) = \alpha x + \alpha \sum_{k=0}^{N} x^{2\cdot 2^k} \prod_{j=0}^{k} (1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}) + \prod_{k=0}^{N+1} (1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}) C_{\alpha,\beta}(x^{2^{N+2}}).$$

Since, $C_{\alpha,\beta}(x) = \beta + x \cdot P(x)$ for some $P(x)$, $C_{\alpha,\beta}(x^{2^{N+2}}) - \beta = x^{2^{N+2}} P(x^{2^{N+2}})$. Thus for $|x| < 1$, $C_{\alpha,\beta}(x^{2^{N+2}}) - \beta \to 0$ as $N \to \infty$; hence we have

$$C_{\alpha,\beta}(x) = \alpha x + \alpha \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k} (1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}) + \beta \prod_{k=0}^{\infty} (1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}).$$

Thus we have the following theorem.

Theorem 23. *The generating function for the general bow sequence is:*

$$B_{\alpha,\beta}(x) = \alpha x + \alpha x^3 + \alpha x^2 \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k}(1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}) \tag{21}$$
$$+ \beta x^2 \prod_{k=0}^{\infty}(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}).$$

Since $B_{\alpha,\beta}(x) = \alpha B_{1,0}(x) + \beta B_{0,1}(x)$, we can state two combinatorial implications. First, as also noted by Anders, Lansing, Reznick, and the author [3] we have

$$B_{0,1}(x) := \sum_{n=0}^{\infty} b_0(n)x^n = x^2 \prod_{k=0}^{\infty}(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}) = xG^{2,3}(x) \tag{22}$$

for the case when $\alpha = 0$, $\beta = 1$.

We see that $b_0(n) = c^{2,3}(n-1)$, where $c^{2,3}(m)$ is the number of ways of writing $m-1$ as the sum $\sum_i c_i 2^i$ where $c_i \in \{0, 2, 3\}$. We interpret this statement as:

Corollary 24. *Combinatorially, for $n \geq 2$, $b_0(n)$ is the number of ways of writing $n-2$ as the sum $\sum_i c_i 2^i$ where $c_i \in \{0, 2, 3\}$.*

We also have

$$B_{1,0}(x) := \sum_{n=0}^{\infty} b_1(n)x^n = x + x^3 + x^2 \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k}(1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}) \tag{23}$$

for the case when $\alpha = 1$, $\beta = 0$. We interpret this statement as:

Corollary 25. *Combinatorially, for $n \geq 4$, $b_1(n)$ is the number of ways of writing $n-2-2^{k+1}$ as the sum $\sum_{i=0}^{k} c_i 2^i$ where $c_i \in \{0, 2, 3\}$ and $k \in \mathbb{N}$.*

Remark 26. By rearranging, we find that $b_1(n)$ is also the number of ways of writing $n-2$ as the sum $\sum_{i=0}^{k} c_i 2^i$ where $k \in \mathbb{N}$, $c_i \in \{0, 2, 3\}$ for $i \leq k-1$, and $c_k \in \{2, 4, 5\}$.

Alternatively, since $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$, we get the following formula

$$B_{1,0}(x) = x + x^3 + x^3 \sum_{k=0}^{\infty} \prod_{j=0}^{k}(x^{1\cdot 2^j} + x^{3\cdot 2^j} + x^{4\cdot 2^j}), \tag{24}$$

which means that $b_1(n)$ is the number of ways of writing $n-3$ as a finite sum $\sum_{i=0}^{k} c_i 2^i$ where $c_i \in \{1,3,4\}$.

Remark 27. Similarly, Reznick [17] showed for the Stern sequence that

$$S(x) = xT(x) = x\prod_{j=0}^{\infty}(1 + x^{2^j} + x^{2\cdot 2^j}). \tag{25}$$

By equation (18) we see that $S(x) = G^{1,2}(x)$.

6 Another representation of $b_0(n)$

In this section, we discuss another representation of $b_0(n)$, found independently in unpublished work by Paul Barry [5], and referenced in Sloane's *On-Line Encyclopedia of Integer Sequences* [18]. The sequence was defined by Barry, but was not shown to satisfy the bow recurrence. We show that $b_0(n)$ can be determined by computing this finite sum, which is independent of the recurrence. Before we give this new equation we must first state a few formulas which will be necessary in the proof of our theorem. The first is the well-known dePolignac's formula.

Theorem 28 (dePolignac's Formula)**.** *[11, 16] For $n \geq 1$, the exponent of the highest power of a prime p dividing $n!$ is*

$$s_p(n) := \sum_{k=1}^{\infty}\left\lfloor \frac{n}{p^k}\right\rfloor. \tag{26}$$

We will need the following corollary for the proof of the next theorem.

Corollary 29. *For $m \geq 1$,*

$$s_2(2m+1) = s_2(2m) = s_2(m) + m. \tag{27}$$

Proof. We write $(2m+1)! = (2m+1)(2m)!$, thus $s_2(2m+1) = s_2(2m)$.

Then, pulling the first term out of the sum, we find

$$\begin{aligned} s_2(2m) &= \left\lfloor \frac{2m}{2}\right\rfloor + \sum_{k=2}^{\infty}\left\lfloor \frac{2m}{2^k}\right\rfloor \\ &= m + s_2(m). \end{aligned}$$

Thus $s_2(2m+1) = s_2(2m) = s_2(m) + m$. $\square$

Remark 30. The second equality in Corollary 29 can also be shown directly by noting that

$$(2m)! = 2^m m! \prod_{j=1}^{m} (2j-1).$$

Definition 31. Define $\epsilon(m)$ to be the smallest non-negative residue of m modulo 2, thus $m \in \{0, 1\}$.

Let

$$c(n) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \epsilon\left(\binom{k}{n-2k}\right). \tag{28}$$

The terms of this sequence are 1, 0, 1, 1, 1, 0, 2, 1, 2, ... We can see that this looks the same as $b_0(n+2)$. In fact, we shall prove that $c(n-2) = b_0(n)$.

Let $c(n-2) = a(n)$. We have the following lemma.

Lemma 32.

$$\binom{2a+1}{2b+1} \equiv \binom{2a+1}{2b} \equiv \binom{2a}{2b} \equiv \binom{a}{b} \pmod{2}.$$

Proof. To show that the first three have the same parity, we note that

$$\frac{(2a+1)!}{(2b+1)!(2a-2b)!}, \ \frac{(2a+1)!}{(2b)!(2a+1-2b)!}, \text{ and } \frac{(2a)!}{(2b)!(2a-2b)!}$$

share all the same even factors in both their numerator and denominator. Thus the highest power of 2 dividing $\binom{2a+1}{2b+1}$, $\binom{2a+1}{2b}$, and $\binom{2a}{2b}$ is the same, and they are all congruent modulo 2.

Then, to show that $\binom{a}{b} \equiv \binom{2a}{2b} \pmod{2}$, we notice that if $s_2(a) = r$, then by Corollary 29 we know that $s_2(2a) = r+a$. Similarly, if $s_2(b) = t$, then $s_2(2b) = t+b$. Lastly, if $s_2(a-b) = s$, then $s_2(2a-2b) = s + (a-b)$.

The highest power of 2 dividing $\binom{a}{b}$ is $r - s - t$, and the highest power of 2 dividing $\binom{2a}{2b}$ is $r + a - (t+b) - s - (a-b) = r - s - t$. Thus the highest power of 2 dividing $\binom{a}{b}$ and $\binom{2a}{2b}$ is the same, and they are congruent modulo 2. □

Remark 33. Note that Lemma 32 also follows from Lucas's Theorem, stated below. In particular, we can see that $\binom{2a}{2b} \equiv \binom{a}{b} \pmod{2}$ since multiplication by 2 simply shifts all the indices in the base 2 expansions of a and b up by 1.

Theorem 34 (Lucas's Theorem). *[9] For non-negative integers m and n and a prime p, the congruence relation*

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod p$$

holds, where $m = \sum_{i=0}^{k} m_i p^i$ and $n = \sum_{i=0}^{k} n_i p^i$ with $m_i, n_i \in \{0, \ldots, p-1\}$ are the base p expansions of m and n, using the convention that $\binom{m}{n} = 0$ if $m < n$.

6.1 A New Formula for $b_0(n)$

We use Theorem 28, Corollary 29, and Lemma 32 to prove that $b_0(n)$ satisfies the following formula.

Theorem 35. *For $n \geq 2$, $b_0(n)$ can be described by the following formula*

$$b_0(n) = a(n) := \sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{k}{n-2-2k}\right). \tag{29}$$

Proof. Clearly $b_0(2) = 1 = a(2)$. Defining $a(0) = a(1) = 0$, we shall show that the bow recursion holds for $a(n)$.

First we shall show that $a(2n+1) = a(n)$, or

$$\sum_{k=0}^{\lfloor (2n+1)/2-1 \rfloor} \epsilon\left(\binom{k}{2n-1-2k}\right) = \sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{k}{n-2-2k}\right). \tag{30}$$

Consider the terms in the left hand side of (30)

$$\sum_{k=0}^{\lfloor (2n+1)/2-1 \rfloor} \epsilon\left(\binom{k}{2n-1-2k}\right) = \sum_{k=0}^{n-1} \epsilon\left(\binom{k}{2n-1-2k}\right).$$

In general, when k is even, the term $\binom{k}{2a+1} \equiv 0 \mod 2$. Thus we need only consider the terms where k is odd. So let $k = 2a+1$ and now consider the sum

$$\sum_{a=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{2a+1}{2n-1-2(2a+1)}\right).$$

We know that

$$\sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{2k+1}{2n-1-2(2k+1)}\right) = \sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{k}{n-2-2k}\right) \tag{31}$$

because the terms are equal, by Lemma 32 with $a = k$, and $b = n - 2 - 2k$. Thus, the recurrence $a(2n+1) = a(n)$ holds.

Second, we must show that $a(2n) = a(n) + a(n+1)$ holds, or

$$\sum_{k=0}^{n-1} \epsilon\left(\binom{k}{2n-2-2k}\right) = \sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{k}{n-2-2k}\right) + \sum_{k=0}^{\lfloor (n+1)/2-1 \rfloor} \epsilon\left(\binom{k}{n-1-2k}\right). \quad (32)$$

We shall first consider the right hand side.

By applying Lemma 32 with $a = k$ and $b = n - 2 - 2k$, we find

$$\sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{k}{n-2-2k}\right) = \sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{2k+1}{2n-4-4k}\right). \quad (33)$$

Similarly applying Lemma 32 with $a = k$ and $b = n - 1 - 2k$, we obtain

$$\sum_{k=0}^{\lfloor (n+1)/2-1 \rfloor} \epsilon\left(\binom{k}{n-1-2k}\right) = \sum_{k=0}^{\lfloor (n+1)/2-1 \rfloor} \epsilon\left(\binom{2k}{2n-2-4k}\right). \quad (34)$$

Summing (33) and (34) together we observe that these are the odd and even terms of the sum

$$\sum_{k=0}^{n-1} \epsilon\left(\binom{k}{2n-2-2k}\right). \quad (35)$$

Therefore (32) holds and we have shown that $a(2n) = a(n) + a(n+1)$, and hence $b_0(n) = a(n)$. ☐

7 Acknowledgments

Many of the results in this paper were part of the author's doctoral dissertation [7] at the University of Illinois at Urbana-Champaign. The author wishes to thank Professor Bruce Reznick for his guidance, support, and enthusiasm. The author would also like to thank the referee for their careful reading and thoughtful additions.

References

[1] J.-P. Allouche and J. Shallit, The ring of k-regular sequences, *Theoret. Comput. Sci.* **98** (1992), 163–197.

[2] J.-P. Allouche and J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, 2003.

[3] K. Anders, M. Dennison, J. Lansing, and B. Reznick, Congruence properties of binary partition functions, *Ann. Comb.* **17** (2013), 15–26.

[4] P. Bachmann, *Niedere Zahlentheorie, vol. 1*, 1902. Reprinted by Chelsea, 1968.

[5] P. Barry, unpublished, cited in [18], sequence A106345, 2005.

[6] N. Calkin and H. Wilf, Recounting the rationals, *Amer. Math. Monthly* **107** (2000), 360–363.

[7] M. Dennison, A sequence related to the Stern sequence, Ph.D. dissertation, University of Illinois at Urbana-Champaign, 2010.

[8] G. de Rham, Un peu de mathématiques à propos d'une courbe plane, *Elem. Math.* **2** (1947), 73–76, 89–97.

[9] N. J. Fine, Binomial coefficients modulo a prime, *Amer. Math. Monthly* **54** (1947), 589–592.

[10] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, 1994.

[11] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 2nd ed., Clarendon Press, 1945.

[12] A. Hinz, S. Klavžar, U. Milutinović, D. Parisse, and C. Petr, Metric properties of the Tower of Hanoi graphs and Stern's diatomic sequence, *Euro. J. Comb.* **26** (2005), 693–708.

[13] D. H. Lehmer, On Stern's diatomic series, *Amer. Math. Monthly* **36** (1929), 59–67.

[14] E. Lucas, Sur les suites de Farey, *Bull. Soc. Math. France* **6** (1878), 118–119.

[15] H. Minkowski, Zur Geometrie der Zahlen, Ver. III, Int. Math-Kong., Heidelberg, 1904, pp. 164–173. Also in *Gesammelte Abhandlungen, Vol. 2*, Chelsea, 1967, pp. 45–52.

[16] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An Introduction to the Theory of Numbers*, 5th ed., Wiley, 1991.

[17] B. Reznick, Regularity properties of the Stern enumeration of the rationals, *J. Integer Sequences* **11** (2008), Article 08.4.1.

[18] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. Published electronically at `https://oeis.org`.

[19] M. A. Stern, Ueber eine Zahlentheoretische Funktion, *J. Reine Angew. Math.* **55** (1858), 193–220.

2010 *Mathematics Subject Classification*: Primary 11B37.
Keywords: Fibonacci number, generating function, recursion, Stern sequence.

(Concerned with sequences A000045, A002487, A106345, and A281185.)

Received December 19 2017; revised versions received December 19 2018; March 24 2019. Published in *Journal of Integer Sequences*, March 28 2019.

Return to Journal of Integer Sequences home page.