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On the Problem of Pillai with Tribonacci Numbers and Powers of 3

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Abstract

Let $(T_n)_{n\geq 0}$ be the sequence of tribonacci numbers defined by $T_0 = 0$, $T_1 = T_2 = 1$, and $T_{n+3} = T_{n+2}+T_{n+1}+T_n$ for all $n \geq 0$. In this note, we find all integers c admitting at least two representations as a difference between a tribonacci number and a power of 3.

1 Introduction

We consider the sequence $(T_n)_{n\geq 0}$ of tribonacci numbers defined by

$$T_0 = 0,\ T_1 = 1,\ T_2 = 1,\ \text{and } T_{n+3} = T_{n+2} + T_{n+1} + T_n \text{ for all } n \geq 0.$$

The tribonacci sequence is sequence A000073 on the Online Encyclopedia of Integer Sequences (OEIS) [18]. The first few terms of the tribonacci sequence are

$$(T_n)_{n\geq 0} = 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, \ldots.$$

In this paper, we study the Diophantine equation

$$T_n - 3^m = c, \tag{1}$$

for a fixed integer c and variable integers n and m. In particular, we are interested in finding those integers c admitting at least two representations as a difference between a tribonacci number and a power of 3. This equation is a variation of the Pillai equation

$$a^x - b^y = c, \tag{2}$$

where x, y are non-negative integers and $a,\ b,\ c$ are fixed positive integers.

In the 1930's, Pillai [19, 20] conjectured that for any given integer $c \geq 1$, the number of positive integer solutions (a, b, x, y), with $x \geq 2$ and $y \geq 2$ to the equation (2) is finite. This conjecture is still open for all $c \neq 1$. The case $c = 1$ is the conjecture of Catalan which was proved by Mihăilescu [17]. The work of Pillai work was an extension of the work of Herschfeld [14, 15], who had already studied a particular case of the problem with $(a, b) = (2, 3)$. Since then, different variations of the Pillai equation have been studied. Several recent results for the different variations of the Pillai problem involving Fibonacci numbers, tribonacci numbers, Pell numbers, k-generalized Fibonacci numbers, and other linearly recurrent sequences, with powers of 2, have been completely studied. For example, see [3, 4, 5, 6, 7, 8, 10, 11].

We discard the situation when $n = 1$ and just count the solutions for $n = 2$ since $T_1 = T_2 = 1$. The reason for the above convention is to avoid trivial parametric families such as $1 - 3^m = T_1 - 3^m = T_2 - 3^m$. Thus, we always assume that $n \geq 2$. The main aim of this paper is to prove the following result.

Theorem 1. *The only integers c having at least two representations of the form $T_n - 3^m$ with $n \geq 2$ and $m \geq 0$, are $c \in \{-2, 0, 1, 4\}$. Furthermore, all the representations of the above integers as $T_n - 3^m$ with integers $n \geq 2$ and $m \geq 0$ are given by*

$$\begin{aligned} -2 &= T_5 - 3^2 = T_2 - 3^1, \\ 0 &= T_9 - 3^4 = T_2 - 3^0, \\ 1 &= T_4 - 3^1 = T_3 - 3^0, \\ 4 &= F_6 - 3^2 = T_5 - 3^1. \end{aligned} \tag{3}$$

2 Preliminary results

2.1 The tribonacci sequence

The characteristic polynomial of the tribonacci sequence $(T_n)_{n\geq 0}$ is given by

$$\Psi(x) := x^3 - x^2 - x - 1.$$

$\Psi(x)$ is irreducible in $\mathbb{Q}[x]$, and has a positive real zero

$$\alpha = \frac{1}{3}\left(1 + (19 + 3\sqrt{33})^{1/3} + (19 - 3\sqrt{33})^{1/3}\right),$$

lying strictly outside the unit circle and two complex conjugate zeros β and γ lying strictly inside the unit circle. Furthermore, $|\beta| = |\gamma| = \alpha^{-1/2}$. According to Dresden and Zu [9], a Binet-like formula for the k-generalized Fibonacci sequences is established. For the tribonacci sequence, it states that

$$T_n = C_\alpha \alpha^{n-1} + C_\beta \beta^{n-1} + C_\gamma \gamma^{n-1} \quad \text{for all } n \geq 1, \tag{4}$$

where $C_X = (X-1)/(4X-6)$. Dresden and Zu [9], also showed that the contribution of the complex conjugate zeros β and γ to the right-hand side of (4) is very small. More precisely,

$$\left|T_n - C_\alpha \alpha^{n-1}\right| < \frac{1}{2} \quad \text{for all } n \geq 1. \tag{5}$$

The minimal polynomial of C_α over the integers is given by

$$44X^3 - 44X^2 + 12X - 1,$$

has zeros $C_\alpha,\ C_\beta,\ C_\gamma$ with $|C_\alpha|,\ |C_\beta|,\ |C_\gamma| < 1$. Numerically,

$$\begin{gathered} 1.83 < \alpha < 1.84, \\ 0.73 < |\beta| = |\gamma| = \alpha^{-1/2} < 0.74, \\ 0.61 < |C_\alpha| < 0.62, \\ 0.19 < |C_\beta| = |C_\gamma| < 0.20. \end{gathered}$$

It is also a well known fact (see [3, 8]) that

$$\alpha^{n-2} \leq T_n \leq \alpha^{n-1} \quad \text{holds for all } n \geq 1. \tag{6}$$

Let $\mathbb{K} := \mathbb{Q}(\alpha, \beta)$ be the splitting field of the polynomial Ψ over $\mathbb{Q}$. Then, $[\mathbb{K}, \mathbb{Q}] = 6$. Furthermore, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. The Galois group of $\mathbb{K}$ over $\mathbb{Q}$ is given by

$$\mathcal{G} := \mathrm{Gal}(\mathbb{K}/\mathbb{Q}) \cong \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\} \cong S_3.$$

Thus, we identify the automorphisms of $\mathcal{G}$ with the permutations of the zeros of the polynomial Ψ. For example, the permutation $(\alpha\gamma)$ corresponds to the automorphism $\sigma : \alpha \to \gamma,\ \gamma \to \alpha,\ \beta \to \beta$.

2.2 Linear forms in logarithms

In order to prove Theorem 1, we need to use several times a Baker–type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such bounds in the literature like that of Baker and Wüstholz [2]. We use the one of Matveev [16]. Matveev [16] proved the following theorem, which is one of our main tools in this paper.

Let γ be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d}(x - \gamma^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of γ. Then the *logarithmic height* of γ is given by

$$h(\gamma) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\gamma^{(i)}|, 1\}\right)\right).$$

In particular, if $\gamma = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\gamma) = \log \max\{|p|, q\}$. Some of the properties of the logarithmic height function $h(\cdot)$, that will be used in the next sections of this paper without reference are as follows:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s|h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned} \tag{7}$$

Theorem 2 (Matveev)**.** *Let $\gamma_1, \ldots, \gamma_t$ be positive real numbers in a number field $\mathbb{K} \subseteq \mathbb{R}$ of degree D, and let $b_1, \ldots, b_t$ be nonzero integers. Assume that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1, \tag{8}$$

is nonzero. Then,

$$\log |\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\}$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \quad \textit{for all} \quad i = 1, \ldots, t.$$

2.3 Baker-Davenport reduction procedure

During the course of our calculations, we get some upper bounds on our variables which are too large. Thus, we need to reduce them. To do so, we use some results from the theory of continued fractions. Specifically, for a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő [12], which itself is a generalization of a result of Baker and Davenport [1].

For a real number X, we write $\|X\| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3 (Dujella, Pethő)**.** *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and let A, B, and μ be some real numbers with $A > 0$ and $B > 1$. Let further $\varepsilon := \|\mu q\| - M\|\tau q\|$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Finally, the following lemma is also useful. It is Lemma 7 in [13].

Lemma 4 (Gúzman, Luca)**.** *Let $m \geqslant 1$, $Y > (4m^2)^m$, and $Y > x/(\log x)^m$. Then*

$$x < 2^m Y (\log Y)^m.$$

3 Proof of Theorem 1

Let n, m, n_1, and m_1 be non-negative integers such that $(n, m) \neq (n_1, m_1)$ and

$$T_n - 3^m = T_{n_1} - 3^{m_1}.$$

Without loss of generality, we assume that $m \geq m_1$. If $m = m_1$, then $T_n = T_{n_1}$, so $(n, m) = (n_1, m_1)$, which gives a contradiction to our assumption. Thus, $m > m_1$. Since

$$T_n - T_{n_1} = 3^m - 3^{m_1}, \tag{9}$$

and the right-hand side is positive, we get that the left-hand side is also positive and so $n > n_1$. Thus, $n \geq 3$ and $n_1 \geq 2$.

Using the equation (9) and the inequality (6), we get

$$\alpha^{n-4} \leq T_{n-2} \leq T_n - T_{n_1} = 3^m - 3^{m_1} < 3^m, \tag{10}$$
$$\alpha^{n-1} \geq T_n \geq T_n - T_{n_1} = 3^m - 3^{m_1} \geq 3^{m-1}, \tag{11}$$

from which we get that

$$1 + \left(\frac{\log 3}{\log \alpha}\right)(m-1) < n < \left(\frac{\log 3}{\log \alpha}\right) m + 4. \tag{12}$$

If $n \leq 300$, then $m \leq 200$. We ran a Mathematica program for $2 \leq n_1 < n \leq 300$ and $0 \leq m_1 < m \leq 200$ and found only the solutions from the list (3). From now, we assume that $n > 300$. Note that the inequality (12) implies that $m < 0.6n + 0.4$. Therefore, to solve the Diophatine equation (1), it suffices to find an upper bound for n.

3.1 Bounding n

From (4) and (5), we get

$$\begin{aligned}\left|C_\alpha\alpha^{n-1}-3^m\right| &= \left|(C_\alpha\alpha^{n-1}-T_n)+(T_{n_1}-3^{m_1})\right|\\ &= \left|(C_\alpha\alpha^{n-1}-T_n)+(T_{n_1}-C_\alpha\alpha^{n_1-1})+(C_\alpha\alpha^{n_1-1}-3^{m_1})\right|\\ &< 1+\frac{7}{10}\alpha^{n_1-1}+3^{m_1}\\ &< \alpha^{n_1}+3^{m_1}\\ &< 2\max\{\alpha^{n_1},3^{m_1}\}.\end{aligned}$$

In the above we have used the fact that $|C_\alpha|<0.62<0.7$. Multiplying through by 3^{-m}, using the relation (10) and using the fact that $\alpha<3$, we get

$$\left|C_\alpha\alpha^{n-1}3^{-m}-1\right| \ < \ 2\max\left\{\frac{\alpha^{n_1}}{3^m},3^{m_1-m}\right\} \ < \ \max\{\alpha^{n_1-n+6},3^{m_1-m+1}\}. \tag{13}$$

For the left-hand side, we apply the result of Matveev, Theorem 2 with the following data:

$$t:=3,\quad \gamma_1:=C_\alpha,\quad \gamma_2:=\alpha,\quad \gamma_3:=3,\quad b_1:=1,\quad b_2:=n-1,\quad \text{and}\quad b_3:=-m.$$

Through out we work with the field $\mathbb{K}:=\mathbb{Q}(\alpha)$ with $D=3$. Since $\max\{1,n-1,m\}\le n$, we take $B:=n$. The minimal polynomial of C_α over the integers is given by

$$44x^3-44x^2+12x-1.$$

Since $|C_\alpha|,\ |C_\beta|,\ |C_\gamma|<1$, we get that $h(C_\alpha)=\frac{1}{3}\log 44$. So we can take $A_1:=3h(\gamma_1)=\log 44$. We can also take $A_2:=3h(\gamma_2)=\log\alpha$ and $A_3:=3h(\gamma_3)=3\log 3$. We put

$$\Lambda:=C_\alpha\alpha^{n-1}3^{-m}-1.$$

First we check that $\Lambda\neq 0$, if it were, then $C_\alpha\alpha^{n-1}=3^m\in\mathbb{Z}$. Conjugating this relation by the automorphism $(\alpha\beta)$, we obtain that $C_\beta\beta^{n-1}=3^m$, which is a contradiction because $|C_\beta\beta^{n-1}|<1$ while $3^m\ge 3$ for all $m\ge 1$. Thus, $\Lambda\neq 0$. Hence, by Theorem 2, the left-hand side of (13) is bounded as follows:

$$\log|\Lambda|>-1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2(1+\log 3)(1+\log n)(\log 44)(\log\alpha)(3\log 3).$$

By comparing with (13), we get

$$\min\{(n-n_1-5)\log\alpha,(m-m_1-1)\log 3\}<2.06\times 10^{13}(1+\log n),$$

which gives

$$\min\{(n-n_1)\log\alpha,(m-m_1)\log 3\}<2.12\times 10^{13}(1+\log n).$$

Now we split the argument into two cases.

Case 1. $\min\{(n-n_1)\log\alpha,(m-m_1)\log 3\}=(n-n_1)\log\alpha$.

In this case, we rewrite (9) as

$$\begin{aligned}\left|C_\alpha\alpha^{n-1}-C_\alpha\alpha^{n_1-1}-3^m\right| &= \left|C_\alpha\alpha^{n-1}-T_n)+(T_{n_1}-C_\alpha\alpha^{n_1-1})-3^{m_1}\right|\\ &< 1+3^{m_1}\le 3^{m_1+1},\end{aligned}$$

which implies

$$\left|C_\alpha(\alpha^{n-n_1}-1)\alpha^{n_1-1}3^{-m}-1\right|<3^{m_1-m+1}. \tag{14}$$

We put

$$\Lambda_1 := C_\alpha(\alpha^{n-n_1}-1)\alpha^{n_1-1}3^{-m}-1.$$

As before, we take $\mathbb{K}:=\mathbb{Q}(\alpha)$, so we have $D=3$. We have $\Lambda_1\neq 0$, for if $\Lambda_1=0$, then

$$C_\alpha(\alpha^{n-n_1}-1)\alpha^{n_1-1}=3^m.$$

By conjugating the above relation by the Galois automorphism $(\alpha\beta)$, we get that

$$C_\beta(\beta^{n-n_1}-1)\beta^{n_1-1}=3^m.$$

The absolute value of the left-hand side is at most $|C_\beta(\beta^{n-n_1}-1)\beta^{n_1-1}|\le|C_\beta\beta^{n-1}|+|C_\beta\beta^{n_1-1}|<2$, while the absolute value of the right-hand side is at least $3^m\ge 3$ for all $m\ge 1$, which is a contradiction.

We apply Theorem 2 on the left-hand side of (14) with the following data:

$$t:=3,\quad \gamma_1:=C_\alpha(\alpha^{n-n_1}-1),\quad \gamma_2:=\alpha,\quad \gamma_3:=3,\quad b_1:=1,\quad b_2:=n_1-1,\quad \text{and}\quad b_3:=-m.$$

Since

$$\begin{aligned}h(\gamma_1) &\le h(C_\alpha)+h(\alpha^{n-n_1}-1)\\ &< \frac{1}{3}\log 44+\frac{1}{3}(n-n_1)\log\alpha+\log 2\\ &< \frac{1}{3}(\log 11+\log 32)+\frac{1}{3}\times 2.12\times 10^{13}(1+\log n)\\ &< \frac{1}{3}\times 2.50\times 10^{13}(1+\log n).\end{aligned} \tag{15}$$

So, we can take $A_1:=2.50\times 10^{13}(1+\log n)$. Furthermore, as before, we take $A_2:=\log\alpha$ and $A_3:=3\log 3$. Finally, since $\max\{1,n_1-1,m\}\le n$, we can take $B:=n$. Then we get

$$\log|\Lambda_1|>-1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2(1+\log 3)(1+\log n)(2.50\times 10^{13}(1+\log n))(\log\alpha)(3\log 3).$$

Then,

$$\log|\Lambda_1| > -1.36 \times 10^{25}(1+\log n)^2.$$

By comparing the above relation with (14), we get that

$$(m-m_1)\log 3 < 1.40 \times 10^{26}(1+\log n)^2. \tag{16}$$

Case 2. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 3\} = (m-m_1)\log 3$.

In this case, we rewrite (9) as

$$\begin{aligned}\left|C_\alpha\alpha^n - (3^{m-m_1}-1)\cdot 3^{m_1}\right| &= \left|(C_\alpha\alpha^{n-1} - T_n) + (T_{n_1} - C_\alpha\alpha^{n_1-1}) + C_\alpha\alpha^{n_1-1}\right| \\ &< 1 + \frac{7}{10}\alpha^{n-1} \quad < \quad \alpha^{n-1} \quad \text{(beacause } n \geq 3\text{)},\end{aligned}$$

which implies that

$$\begin{aligned}|C_\alpha(3^{m-m_1}-1)^{-1}\alpha^{n-1}3^{-m_1} - 1| &< \frac{\alpha^{n_1}}{3^m - 3^{m_1}} \ \leq \ \frac{3\alpha^{n_1}}{3^m} \\ &< 3\alpha^{n_1-n+4} \ < \ \alpha^{n_1-n+6}.\end{aligned} \tag{17}$$

We put

$$\Lambda_2 := C_\alpha(3^{m-m_1}-1)^{-1}\alpha^{n-1}3^{-m_1} - 1.$$

Clearly, $\Lambda_2 \neq 0$, for if $\Lambda_2 = 0$, then $C_\alpha = (\alpha^{-1})^{n-1}(3^m - 3^{m_1})$ implying that C_α is an algebraic integer, a contradiction. We again apply Theorem 2 with the following data:

$$t := 3, \quad \gamma_1 := C_\alpha(3^{m-m_1}-1)^{-1}, \quad \gamma_2 := \alpha, \quad \gamma_3 := \alpha, \quad b_1 := 1, \quad b_2 := n, \quad \text{and} \quad b_3 := -m_1.$$

We note that

$$\begin{aligned}h(\gamma_1) = h(C_\alpha(3^{m-m_1}-1)^{-1}) \ &\leq \ h(C_\alpha) + h(3^{m-m_1}-1) \\ &= \frac{1}{3}\log 44 + h(3^{m-m_1}-1) \quad < \quad \log(3^{m-m_1+2}) \\ &= (m-m_1+2)\log 3 \ < \ 2.50 \times 10^{13}(1+\log n).\end{aligned}$$

So, we can take $A_1 := 7.5 \times 10^{13}(1+\log n)$. Further, as in the previous applications, we take $A_2 := \log\alpha$ and $A_3 := 3\log 3$. Finally, since $\max\{1, n-1, m_1\} \leq n$, we can take $B := n$. Then, we get

$$\log|\Lambda_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1+\log 3)(1+\log n)(7.5 \times 10^{13}(1+\log n))(\log\alpha)(3\log 3).$$

Thus,

$$\log|A_2| > -4.08 \times 10^{26}(1+\log n)^2.$$

Now, by comparing with (17), we get that

$$(n - n_1)\log\alpha < 4.10 \times 10^{26}(1 + \log n)^2. \tag{18}$$

Therefore, in both Case 1 and Case 2, we have

$$\begin{aligned} \min\{(n-n_1)\log\alpha, (m-m_1)\log 3\} &< 2.12 \times 10^{13}(1+\log n), \\ \max\{(n-n_1)\log\alpha, (m-m_1)\log 3\} &< 4.10 \times 10^{26}(1+\log n)^2. \end{aligned} \tag{19}$$

Finally, we rewrite the equation (9) as

$$\left|C_\alpha\alpha^{n-1} - C_\alpha\alpha^{n_1-1} - 3^m + 3^{m_1}\right| = \left|(C_\alpha\alpha^{n-1} - T_n) + (T_{n_1} - C_\alpha\alpha^{n_1-1})\right| \; < \; 1.$$

Dividing through by $3^m - 3^{m_1}$, we get

$$\begin{aligned} \left|\frac{C_\alpha(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\alpha^{n_1-1}3^{-m_1} - 1\right| &< \frac{1}{3^m - 3^{m_1}} \le \frac{3}{3^m} \\ &\le 3\alpha^{-(n-4)} \; \le \; \alpha^{6-n}, \end{aligned} \tag{20}$$

since $3 < \alpha \le \alpha^{n_1}$. We again apply Theorem 2 on the left-hand side of (20) with the following data:

$$t := 3, \quad \gamma_1 := \frac{C_\alpha(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}, \quad \gamma_2 := \alpha, \quad \gamma_3 := 3, \quad b_1 := 1, \quad b_2 := n_1 - 1, \quad \text{and} \quad b_3 := -m_1.$$

By using the algebraic properties of the logarithmic height function, we get

$$\begin{aligned} 3h(\gamma_1) = 3h\left(\frac{C_\alpha(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\right) \; &\le \; h\left(C_\alpha(\alpha^{n-n_1}-1)\right) + h(3^{m-m_1}-1) \\ &< \log 352 + (n-n_1)\log\alpha + 3(m-m_1)\log 3 \\ &< 6.80 \times 10^{26}(1+\log n)^2, \end{aligned}$$

where in the above inequalities, we used the argument from (15) as well as the bounds (19). Thus, we can take $A_1 := 6.80 \times 10^{26}(1 + \log n)$, and again as before $A_2 := \log\alpha$ and $A_3 := 3\log 3$. If we put

$$\Lambda_3 := \frac{C_\alpha(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\alpha^{n_1-1}3^{-m_1} - 1,$$

we need to show that $\Lambda_3 \neq 0$. If not, $\Lambda_3 = 0$ leads to

$$C_\alpha(\alpha^{n-1} - \alpha^{n_1-1}) = 3^m - 3^{m_1}.$$

A contradiction is reached upon a conjugation by the automorphism $(\alpha\beta)$ in $\mathbb{K}$ and by taking absolute values on both sides. Thus, $\Lambda_3 \neq 0$. Applying Theorem 2 gives

$$\log|\Lambda_3| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1+\log 3)(1+\log n)(6.80 \times 10^{26}(1+\log n)^2)(\log\alpha)(3\log 3),$$

a comparison with (20) gives

$$(n-6) < 3.70 \times 10^{39}(1+\log n)^3,$$

or

$$n < 3.8 \times 10^{39}(1+\log n)^3. \tag{21}$$

Now, by applying Lemma 4 on (21) with the data $m := 3$, $Y := 3.8 \times 10^{39}$, and $x := n$, leads to $n < 3 \times 10^{46}$.

3.2 Reducing the bound for n

We need to reduce the above bound for n and to do so we make use of Lemma 3 several times. To begin, we return to (13) and put

$$\Gamma := (n-1)\log\alpha - m\log 3 + \log C_\alpha.$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\} \geq 20$. We go back to the inequalities for Λ, Λ_1, and Λ_2, Since we assume that $\min\{n-n_1, m-m_1\} \geq 20$ we get $|e^\Gamma - 1| = |\Lambda| < \frac{1}{4}$. Hence, $|\Lambda| < \frac{1}{2}$ and since the inequality $|y| < 2|e^y - 1|$ holds for all $y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we get

$$0 < |\Gamma| < 2\max\{\alpha^{n_1-n+6}, 3^{m_1-m+1}\} \leq \max\{\alpha^{n_1-n+8}, 3^{m_1-m+2}\}.$$

By substituting for Γ in the above inequality and dividing through by $\log 3$, we get the inequality

$$0 < \left|(n-1)\left(\frac{\log\alpha}{\log 3}\right) - m + \frac{\log C_\alpha}{\log 3}\right| < \max\left\{\frac{\alpha^8}{(\log 3)\alpha^{n-n_1}}, \frac{9}{(\log 3)3^{m-m_1}}\right\}.$$

We apply Lemma 3 with the following data

$$\tau := \frac{\log\alpha}{\log 3}, \quad \mu := \frac{\log C_\alpha}{\log 3}, \quad \text{and} \quad (A, B) := \left(\frac{\alpha^8}{\log 3}, \alpha\right) \text{ or } \left(\frac{9}{\log 3}, 3\right).$$

Let $\tau = [a_0; a_1, a_2, \ldots] = [0; 1, 1, 4, 13, 1, 6, 1, 4, 1, 10, 7, 1, 24, 3, 3, 2, 12, 4, 4, \ldots]$ be the continued fraction expansion of τ. We choose $M := 3 \times 10^{46}$ which is the upper bound on n. With the help of Mathematica, we find out that the convergent

$$\frac{p}{q} = \frac{p_{88}}{q_{88}} = \frac{383979914200993729068715782793592146551951600940}{692255294546383107303758900444711151890883197059}$$

is such that $q = q_{88} > 6M$. Furthermore, it yields $\varepsilon > 0.0428119$, and therefore either

$$n - n_1 \leq \frac{\log\left((\alpha^8/\log 3)q/\varepsilon\right)}{\log\alpha} < 193, \quad \text{or} \quad m - m_1 \leq \frac{\log\left((9/\log 3)q/\varepsilon\right)}{\log 3} < 105.$$

Thus, we have that either $n - n_1 \le 193$ or $m - m_1 \le 105$.

Now we distinguish between the cases $n-n_1 \le 193$ and $m-m_1 \le 105$. First, we assume that $n - n_1 \le 193$. In this case we consider the inequality for Λ_1, (14) and also assume that $m - m_1 \ge 20$. We put

$$\Gamma_1 := (n_1 - 1)\log\alpha - m\log 3 + \log\left(C_\alpha(\alpha^{n-n_1} - 1)\right).$$

Then, inequality (14) implies that

$$|\Gamma_1| < \frac{6}{3^{m-m_1}}.$$

If we substitute for Γ_1 in the above inequality and divide through by $\log 3$, we then get

$$0 < \left|(n_1 - 1)\left(\frac{\log\alpha}{\log 3}\right) - m + \frac{\log(C_\alpha(\alpha^{n-n_1} - 1))}{\log 3}\right| < \frac{6}{(\log 3)3^{m-m_1}}.$$

Again we apply Lemma 3 with the same τ as in the case of Γ. We use the 88-th convergent $p/q = p_{88}/q_{88}$ of τ as before. But in this case we choose $(A, B) := \left(\frac{6}{\log 3}, 3\right)$ and use

$$\mu_\ell := \frac{\log(C_\alpha(\alpha^\ell - 1))}{\log 3},$$

instead of μ for each possible value of $\ell := n - n_1 \in [1, 2, \ldots, 193]$. For all values of ℓ, we get $\varepsilon > 0.0000420218$. Hence, by Lemma 3, we get

$$m - m_1 \;<\; \frac{\log\left((6/\log 3)q/\varepsilon\right)}{\log 3} \;<\; 110.$$

Thus, $n - n_1 \le 193$ implies that $m - m_1 \le 110$.

Now let us turn to the case $m-m_1 \le 105$ and we consider the inequlity for Λ_2, (17). We put

$$\Gamma_2 := (n - 1)\log\alpha - m_1\log 3 + \log\left(\frac{C_\alpha}{3^{m-m_1} - 1}\right),$$

and we also assume that $n - n_1 \ge 20$. We then have

$$|\Gamma_2| < \frac{\alpha^8}{\alpha^{n-n_1}}.$$

If we substitute for Γ_2 in the above inequality and divide through by $\log 3$, we then get

$$0 < \left|(n - 1)\left(\frac{\log\alpha}{\log 3}\right) - m_1 + \frac{\log(C_\alpha/(3^{m-m_1} - 1))}{\log 3}\right| < \frac{\alpha^8}{(\log 3)\alpha^{n-n_1}}.$$

We apply again Lemma 3 with the same τ, q, M, $(A,B) := \left(\frac{\alpha^8}{\log 3}, \alpha\right)$, and

$$\mu_\ell := \frac{\log(C_\alpha/(3^\ell - 1))}{\log 3} \quad \text{for } \ell = 1, 2, \ldots, 105.$$

We get $\varepsilon > 0.00218297$, therefore

$$n - n_1 < \frac{\log\left((\alpha^8/\log 3)q/\varepsilon\right)}{\log \alpha} < 198.$$

To conclude, we first get that either $n - n_1 \leq 193$ or $m - m_1 \leq 105$. If $n - n_1 \leq 193$, then $m - m_1 \leq 110$, and if $m - m_1 \leq 105$, then $n - n_1 \leq 198$. Thus, we conclude that we always have $n - n_1 \leq 198$ and $m - m_1 \leq 110$.

Finally, we go to the inequality of Λ_3, (20). We put

$$\Gamma_3 := (n_1 - 1)\log \alpha - m_1 \log 3 + \log\left(\frac{C_\alpha(\alpha^{n-n_1} - 1)}{3^{m-m_1} - 1}\right).$$

Since $n > 300$, the inequality (20) implies that

$$|\Gamma_3| < \frac{2}{\alpha^{n-6}} = \frac{\alpha^8}{\alpha^n}.$$

Substituting for Γ_3 in the above inequality and dividing through by $\log 3$, we get

$$0 < \left|(n_1 - 1)\left(\frac{\log \alpha}{\log 3}\right) - m_1 + \frac{\log(C_\alpha(\alpha^k - 1)/(3^\ell - 1))}{\log 3}\right| < \frac{\alpha^8}{(\log 3)\alpha^n},$$

where $(k, \ell) := (n - n_1, m - m_1)$. We again apply Lemma 3 with the same τ, q, M, $(A,B) := \left(\frac{\alpha^8}{\log 3}, \alpha\right)$, and

$$\mu_{k,l} := \frac{\log(C_\alpha(\alpha^k - 1)/(3^\ell - 1))}{\log 3} \quad \text{for} \quad 1 \leq k \leq 198 \quad \text{and} \quad 1 \leq \ell \leq 110.$$

For the cases, we get $\varepsilon > 0.0000115272$, so we obtain

$$n \leq \frac{\log\left((\alpha^8/\log 3)q/\varepsilon\right)}{\log \alpha} < 207.$$

Hence, $n \leq 207$. However, this contradicts our working assumption that $n > 300$. This completes the proof of Theorem 1. $\square$

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