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Finite Test Sets for Morphisms That Are Squarefree on Some of Thue's Squarefree Ternary Words

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Abstract

Let S be one of $\{aba, cbc\}$ and $\{aba, aca\}$, and let w be an infinite squarefree word over $\Sigma = \{a, b, c\}$ with no factor in S. Suppose that $f : \Sigma \to T^*$ is a non-erasing morphism. We prove that the word $f(w)$ is squarefree if and only if f is squarefree on factors of w of length 8 or less.

1 Introduction

The papers of Thue on squarefree words [12, 13] are foundational to the area of combinatorics on words. A word w is *squarefree* if we cannot write $w = xyyz$, where y is a non-empty word. The longest squarefree words over the 2-letter alphabet $\{a, b\}$ are aba and bab, each of length 3, but Thue showed that arbitrarily long squarefree words exist over the 3-letter alphabet $\{a, b, c\}$. Infinite squarefree words over finite alphabets are routinely encountered in combinatorics on words, and are frequently used as building blocks in constructions. (See, for example, [9, 10].)

Let w be an infinite squarefree word over $\Sigma = \{a, b, c\}$. Thue showed that w must contain every squarefree word of length 2 over Σ . However, he showed that the same is not true for squarefree words of length 3 over Σ. For each of $S_1 = \{aba, cbc\}, S_2 = \{aba, aca\}$, and

$S_3 = \{aba, bab\}$, Thue constructed an infinite squarefree word over Σ with no factor in S_i. Constructions giving squarefree words equivalent to Thue's word with no factors in S_1 were independently discovered by Braunholtz [4] and Istrail [8]; Berstel [1] shows this equivalence. Their word is called **vtm** (for 'variation of Thue-Morse') by Blanchet-Sadri et al. [3], and has been used as the basis for various constructions [3, 6, 7]. These constructions require showing that $f(\mathbf{vtm})$ is squarefree for particular morphisms f. In this paper, we give a testable characterization of morphisms f such that $f(\mathbf{vtm})$ is squarefree; we do the same in the case where **vtm** is replaced by an infinite squarefree word over Σ with no factors in S_2. We leave as an open problem whether there is a characterization when we replace **vtm** by a word over Σ with no factor in S_3.

Theorem 1. *Let w be an infinite squarefree word over Σ such that either w has no factor in S_1, or w has no factor in S_2. Suppose that $f : \Sigma \to T^*$ is a non-erasing morphism. The word $f(w)$ is squarefree if and only if f is squarefree on factors of w of length 8 or less.*

Our theorem says that to establish squarefreeness of $f(w)$, the morphism f need only be checked for squarefreeness on a finite test set. Crochemore [5] proved a variety of similar theorems; in particular, a morphism f defined on Σ^* preserves squarefreeness exactly when f preserves squarefreeness on words of Σ^* of length at most 5. Note that while Crochemore's theorem requires testing the squarefreeness of $f(v)$ for every squarefree word $v \in \Sigma^*$ up to a certain length, we only test words v that are factors of w. Thus, while aba is squarefree, we do not require $f(aba)$ to be squarefree, for example. Finite test sets for morphisms preserving overlap-freeness have also been well-studied [11].

2 Preliminaries

Let $S = S_1$ or $S = S_2$. For the remainder of this paper, let w be an infinite squarefree word over Σ with no factor in S. Write $w = a_0a_1a_2a_3\cdots$ with $a_i \in \Sigma$. For the remainder of this section suppose that $f : \Sigma \to T^*$ is a non-erasing morphism that is squarefree on factors of w of length 8 or less.

Lemma 2. *Suppose $f(\xi)$ is a factor of $f(x)$, where $x \in \Sigma$ and ξ is a factor of w. Then $|\xi| \leq 3$.*

Proof. If x is a letter of ξ, but $f(\xi)$ is a factor of $f(x)$, we have

$$\begin{aligned} |f(x)| &\leq |f(\xi)| \\ &\leq |f(x)|. \end{aligned}$$

Since f is non-erasing, this forces $x = \xi$, giving $|\xi| = 1$.

If x is not a letter of ξ, then ξ is a squarefree word over a two-letter alphabet, so that $|\xi| \leq 3$. $\square$

Lemma 3. *Suppose that $f(x)$ is a prefix or a suffix of $f(y)$ where $x, y \in \Sigma$. Then $x = y$.*

Proof. We give the proof where $f(x)$ is a prefix of $f(y)$. (The other case is similar.) Suppose $x \neq y$. Then xy must be a factor of w, and xy is squarefree. However, $f(xy)$ begins with the square $f(x)f(x)$, contradicting the squarefreeness of f on factors of w of length at most 8. $\square$

Lemma 4. *There is no solution* $(\alpha, \beta, \gamma, \xi, p, s, t, x, y, z)$ *to*

$$\begin{aligned}
&\alpha\xi xyz\beta \text{ is a factor of } w;\\
&\alpha, \beta, \gamma, x, y, z \in \Sigma, \xi \in \Sigma^*;\\
&t \text{ is a suffix of } f(\gamma);\\
&s \text{ is a suffix of } f(\alpha);\\
&p \text{ is a non-empty prefix of } f(\beta);\\
&t = sf(\xi xyz)p.
\end{aligned}$$

Proof. Suppose, for the sake of getting a contradiction, that $(\alpha, \beta, \gamma, \xi, p, s, t, x, y, z)$ is a solution.

Here p is a prefix of $f(\beta)$, but also a suffix of t, that is a suffix of $f(\gamma)$. Thus we see that $f(\gamma\beta)$ contains square pp, so that $\gamma\beta$ is not a factor of w. This forces $\gamma = \beta$. On the other hand, since $z\beta$ is a factor of w, we conclude that $z \neq \gamma$. Again, $f(z)p$ is a prefix of $f(z\beta)$, but also a suffix of $f(\gamma)$, so that $f(\gamma z\beta)$ contains a square. Since $yz\beta$ is a factor of w, it follows that $y \neq \gamma$. Similarly, $f(\gamma yz\beta)$ contains a square, but $xyz\beta$ is a factor of w, so that that $x \neq \gamma$. Finally, we see that $f(\gamma xyz\beta)$ contains a square. Let δ be the last letter of $\alpha\xi$. We conclude that $\delta \neq \gamma$. However, now δxyz is a squarefree word of length 4 over the two-letter alphabet $\Sigma - \{\gamma\}$. This is impossible. $\square$

The symmetrical lemma is proved analogously:

Lemma 5. *There is no solution* $(\alpha, \beta, \gamma, \xi, p, s, t, x, y, z)$ *to*

$$\begin{aligned}
&\alpha\xi xyz\beta \text{ is a factor of } w;\\
&\alpha, \beta, \gamma, x, y, z \in \Sigma, \xi \in \Sigma^*;\\
&t \text{ is a suffix of } f(\gamma);\\
&s \text{ is a non-empty suffix of } f(\alpha);\\
&p \text{ is a non-empty of } f(\beta);\\
&t = sf(xyz\xi)p.
\end{aligned}$$

Suppose that $f(w)$ contains a non-empty square xx, with $|x|$ as short as possible. Write $f(w) = uxxv$, such that

$$\begin{aligned}
u &= A_0A_1 \cdots A_i'\\
ux &= A_0A_1 \cdots A_j'\\
uxx &= A_0A_1 \cdots A_k',
\end{aligned}$$

where $i \le j \le k$ are non-negative integers, and for each non-negative integer ℓ, $A_\ell = f(a_\ell)$, and A'_ℓ is a prefix of A_ℓ, but $A'_\ell \ne A_\ell$. This notation is not intended to exclude the possibilities that $i = 0$, $i = j$ and/or $j = k$.

Remark 6. Since f is squarefree on factors of w of length at most 8, but $f(a_i \cdots a_k)$ contains the square xx, we must have $k - i \ge 8$.

Remark 7. We cannot have $i = j$; otherwise suffix x of ux is a factor of $A_j = A_i$, and $A_{i+1} \cdots A_{k-1}$ is a factor of suffix x of uxx. Then, $f(a_{i+1}a_{i+2} \cdots a_{k-1})$ is a factor of $f(a_i)$, forcing $(k-1) - (i+1) + 1 \le 3$ by Lemma 2, so that $k - i \le 4$, a contradiction. Reasoning, in the same way, we show that $i < j < k$.

For $\ell \in \{i, j, k\}$, let A''_ℓ be the suffix of A_ℓ such that $A_\ell = A'_\ell A''_\ell$. By our choice of A'_ℓ, $A''_\ell \ne \epsilon$. Then

$$x = A''_i A_{i+1} \cdots A_{j-1} A'_j = A''_j A_{j+1} \cdots A_{k-1} A'_k. \tag{1}$$

Remark 8. Since $k - i \ge 8$, we must have $k - j - 1 \ge 3$ and/or $j - i - 1 \ge 3$.

Lemma 9. *We must have* $|A''_i| + |A'_k| \le |x|$ *and* $|A''_j| + |A'_j| \le |x|$.

Proof. We give the proof that $|A''_j| + |A'_j| \le |x|$. (The proof of the other assertion is similar.) Suppose for the sake of getting a contradiction that $|A''_j| + |A'_j| > |x|$. Then $|A''_j| > |x| - |A'_j| = |A''_i A_{i+1} \cdots A_{j-1}|$. It follows that $A''_j = A''_i A_{i+1} \cdots A_{j-1} A'''_j$ for some non-empty prefix A'''_j of A_j. Similarly, one shows that $A'_j = A''''_j A_{j+1} \cdots A_{k-1} A'_k$ for some non-empty suffix A''''_j of A_j. Now $|a_{i+1} \cdots a_{j-1}| = (j-1) - (i+1) + 1 = j - i - 1$, and $|a_{j+1} \cdots a_{k-1}| = (k-1) - (j+1) + 1 = k - j - 1$. However, either $j - i - 1 \ge 3$, or $k - j - 1 \ge 3$. If $j - i - 1 \ge 3$, then $A''_j = A''_i A_{i+1} \cdots A_{j-1} A'''_j$ for some non-empty prefix A'''_j of A_j contradicts Lemma 4, letting $s = A''_i$, $\alpha = a_i$, $\xi = a_{i+1} \cdots a_{j-4}$, $xyz = a_{j-3}a_{j-2}a_{j-1}$, $p = A'''_j$, $\beta = a_j$.

In the case where $k - j - 1 \ge 3$, we get the analogous contradiction using Lemma 5. □

Lemma 10. *We have* $j - i = k - j$, $A''_i = A''_j$, $A'_j = A'_k$, *and* $A_{i+\ell} = A_{j+\ell}$, $1 \le \ell \le j - i - 1$.

Proof. To begin with we show that $A'_j = A'_k$. Since both words are suffixes of x, it suffices to show that $|A'_j| = |A'_k|$. Suppose not. Suppose that $|A'_k| < |A'_j|$.

By the previous lemma, $|A'_j| \le |A_{j+1} \cdots A_{k-2} A_{k-1} A'_k|$. Let m be greatest such that $|A_m \cdots A_{k-1} A'_k| \ge |A'_j|$. Thus $j + 1 \le m \le k - 1$, and

$$|A_{m+1} \cdots A_{k-1} A'_k| < |A'_j| \le |A_m A_{m+1} \cdots A_{k-1} A'_k|.$$

If $A'_j = A_m A_{m+1} \cdots A_{k-1} A'_k$, then A_m is a non-empty prefix of A'_j, forcing $a_m = a_j$ so that $A_m = A'_j = A_j$, by Lemma 3. This is contrary to our choice of A'_j.

Similarly, suppose $|A'_k| < |A'_j|$. Suppose that $|A'_k| < |A'_j|$.

Again by the previous lemma, $|A'_k| \le |A_{i+1} \cdots A_{j-2} A_{j-1} A'_j|$. Let m be greatest such that $|A_m \cdots A_{j-1} A'_j| \ge |A'_k|$. Thus $i + 1 \le m \le j - 1$, and

$$|A_{m+1} \cdots A_{j-1} A'_j| < |A'_k| \le |A_m A_{m+1} \cdots A_{j-1} A'_j|.$$

If $A'_k = A_m A_{m+1} \cdots A_{j-1} A'_j$, then A_m is a non-empty prefix of A'_k, forcing $a_m = a_k$ so that $A_m = A'_k = A_k$, by Lemma 3, contrary to our choice of A'_k.

We thus conclude that $A'_j = A'_k$, as desired.

Next, we show that $j - 1 = k - j$ and $A_{i+\ell} = A_{j+\ell}$, $1 \le \ell \le j - i - 1$. Suppose that $j - i \le k - j$. (The other case is similar.) Suppose now that we have shown that for some ℓ, $0 \le \ell < j - i - 1$ that

$$A_{j-\ell} \cdots A_{j-1} A'_j = A_{k-\ell} \cdots A_{k-1} A'_k. \tag{2}$$

This is true when $\ell = 0$; i.e., $A'_j = A'_k$.

From (1), one of $A_{j-\ell-1} A_{j-\ell} \cdots A_{j-1} A'_j$ and $A_{k-\ell-1} A_{k-\ell} \cdots A_{k-1} A'_k$ is a suffix of the other. Together with (2), this implies that one of $A_{j-\ell-1}$ and $A_{k-\ell-1}$ is a suffix of the other. By Lemma 3, this implies that $a_{j-\ell-1} = a_{k-\ell-1}$, and by combining this with (2),

$$A_{j-\ell-1} \cdots A_{j-1} A'_j = A_{k-\ell-1} \cdots A_{k-1} A'_k. \tag{3}$$

By induction we conclude that

$$A_{j-\ell} \cdots A_{j-1} A'_j = A_{k-\ell} \cdots A_{k-1} A'_k, 0 \le \ell \le j - i - 1,$$

which implies $A_{j-\ell} = A_{k-\ell}$, $1 \le \ell \le j - i - 1$. In particular, we note that

$$A_{i+1} \cdots A_{j-1} = A_{k-j+i+1} \cdots A_{k-1}. \tag{4}$$

If we now have $k - j > j - i$, then $k - j + i > j$, and (1) and (4) imply that $A''_i = A''_j A_{j+1} \cdots A_{k-j+i}$. Then A_{k-j+i} is a suffix of A''_i and Lemma 3 forces $A_{k-j+i} = A_i$. Then (1) and (4) force $A_i = A'_i$, contrary to our choice of A'_i. We conclude that $k - j = j - i$. From (1) and (4) we conclude that $A''_i = A''_j$, as desired. $\square$

Corollary 11. *The word w contains a factor $\alpha z \beta z \gamma$, $\alpha, \beta, \gamma \in \Sigma$, $\alpha, \gamma \neq \beta$, $|z| \ge 3$, such that $\alpha\beta\gamma$ is not a factor of w.*

Proof. By Lemma 10, w contains a factor $\alpha z \beta z \gamma$, where $\alpha = a_i$, $\beta = a_j$, $\gamma = a_k$, $z = a_{i+1} \cdots a_{j-1} = a_{j+1} \cdots a_{k-1}$. This gives $|z| = j - i - 1 = k - j - 1 \ge 3$.

Since w is squarefree, we cannot have $a_i = a_j$; otherwise w contains the square $(a_0 z)^2$; similarly, $a_j \neq a_k$. To see that $\alpha\beta\gamma$ is not a factor of w, we note that f is squarefree on factors of w of length at most 8, but $f(\alpha\beta\gamma) = A_i A_j A_k$ contains the square $(A''_i A'_j)(A''_j A'_k)$; this is a square since $A''_i = A''_j$ and $A'_j = A'_k$. Since $|A_i A_j A_k| = 3 \le 8$, we conclude that $A_i A_j A_k$ is not a factor of w. $\square$

3 Results

Lemma 12. *If $S = \{aba, cbc\}$, the only squarefree words of length 3 that are not factors of w are aba and cbc. In addition, w contains no factor of the form $azbza$ or $czbzc$.*

Proof. Thue [13] showed that a squarefree word over Σ not containing aba or cbc as a factor contains every other length 3 squarefree word as a factor.

Suppose w contains a factor $azbza$. Since aba is not a factor of w, $z \neq \epsilon$. Since az and bz are factors of w, and hence squarefree, the first letter of z cannot be a or b, and must be c. Similarly, the last letter of z must be c. But w contains zbz, and thus cbc. This is a contradiction. Therefore w contains no factor $azbza$.

Replacing a by c and vice versa in the preceding argument shows that w contains no factor $czbzc$. ☐

Combining the last two lemmas gives this corollary:

Corollary 13. *If $S = \{aba, cbc\}$ and $f : \Sigma^* \to T^*$ is squarefree on factors of w of length at most 8, then $f(w)$ is squarefree.*

Lemma 14. *If $S = \{aba, aca\}$, the only squarefree words of length 3 that are not factors of w are aba and aca. In addition, w contains no factor of the form $azbza$ or $azcza$, $|z| \geq 3$.*

Proof. Thue [13] showed that a squarefree word over Σ not containing aba or aca as a factor contains every other length 3 squarefree word as a factor.

Suppose w contains a factor $azbza$, $|z| \geq 3$. Since aba is not a factor of w, $z \neq \epsilon$. Since az and bz are factors of w, and hence squarefree, the first letter of z cannot be a or b, and must be c. Similarly, the last letter of z must be c. However, since az is a factor of w, but aca is not, the second letter of z cannot be a and must be b. Write $z = cbz'c$. Then $azbza = acbz'cbcbz'ca$ contains the square $cbcb$, that is impossible. We conclude that w contains no factor $azbza$, $|z| \geq 3$.

Replacing c by b and vice versa in the preceding argument shows that w contains no factor $azcza$, $|z| \geq 3$. ☐

Corollary 15. *If $S = \{aba, aca\}$ and $f : \Sigma^* \to T^*$ is squarefree on factors of w of length at most 8, then $f(w)$ is squarefree.*

Lemma 16. *The squarefree word $azbza$ where $z = cabcbacabcacbacabcbac$ has no factors aba or bab. It follows that any analogous theorem for S_3, with an analogous proof, would require us to replace 8 by a value of at least $|azbza| = 45$.*

Proof. This is established by a finite check. ☐

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