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New Estimates for the nth Prime Number

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Abstract

In this paper we establish new upper and lower bounds for the nth prime number p_n, which improve several existing bounds of similar shape. As the main tool, we use some explicit estimates recently obtained for the prime counting function. A further main tool is the use of estimates concerning the reciprocal of $\log p_n$. As an application, we derive new estimates for $\vartheta(p_n)$, where $\vartheta(x)$ is Chebyshev's ϑ-function.

1 Introduction

Let p_n denote the nth prime number and let $\pi(x)$ be the number of primes not exceeding x. In 1896, Hadamard [10] and de la Vallée-Poussin [19] independently proved the asymptotic formula $\pi(x) \sim x/\log x$ as $x \to \infty$, which is known as the *prime number theorem*. (Here $\log x$ is the natural logarithm of x.) As a consequence of the prime number theorem, one gets the asymptotic expression

$$p_n \sim n \log n \qquad (n \to \infty). \tag{1.1}$$

Here p_n is the nth prime. Cipolla [5] found a more precise result. He showed that for every positive integer m there exist unique monic polynomials $T_1, \ldots, T_m$ with rational coefficients and $\deg(T_k) = k$ with

$$p_n = n\left(\log n + \log\log n - 1 + \sum_{k=1}^{m} \frac{(-1)^{k+1}T_k(\log\log n)}{k\log^k n}\right) + O\left(\frac{n(\log\log n)^{m+1}}{\log^{m+1} n}\right). \tag{1.2}$$

The polynomials T_k can be computed explicitly. In particular, $T_1(x) = x - 2$ and $T_2(x) = x^2 - 6x + 11$ (see Cipolla [5] or Salvy [18] for further details). Since the computation of the nth prime number is difficult for large n, we are interested in explicit estimates for p_n. The asymptotic formula (1.2) yields

$$p_n > n \log n, \tag{1.3}$$
$$p_n < n(\log n + \log\log n), \tag{1.4}$$
$$p_n > n(\log n + \log\log n - 1) \tag{1.5}$$

for all sufficiently large values of n. The first result concerning a lower bound for the nth prime number is due to Rosser [15, Theorem 1]. He showed that the inequality (1.3) holds for every positive integer n. In the literature, this result is often called *Rosser's theorem*. Moreover, he proved [15, Theorem 2] that

$$p_n < n(\log n + 2\log\log n) \tag{1.6}$$

for every $n \geq 4$. The next results concerning the upper and lower bounds that correspond to the first three terms of (1.2) are due to Rosser and Schoenfeld [16, Theorem 3]. They refined Rosser's theorem and the inequality (1.6) by showing that

$$p_n > n(\log n + \log\log n - 1.5)$$

for every $n \geq 2$ and that the inequality

$$p_n < n(\log n + \log\log n - 0.5) \tag{1.7}$$

holds for every $n \geq 20$. The inequality (1.7) implies that (1.4) is fulfilled for every $n \geq 6$. Based on their estimates for the Chebyshev functions $\psi(x)$ and $\vartheta(x)$, Rosser and Schoenfeld [17] announced to have new estimates for the nth prime number p_n but they have never published the details. In the direction of (1.5), Robin [14, Lemme 3, Théorème 8] showed that

$$p_n \geq n(\log n + \log\log n - 1.0072629) \tag{1.8}$$

for every $n \geq 2$, and that the inequality (1.5) holds for every integer n such that $2 \leq n \leq \pi(10^{11})$. Massias and Robin [11, Théorème A] gave a series of improvements of (1.7) and (1.8). For instance, they have found that $p_n \geq n(\log n + \log\log n - 1.002872)$ for every $n \geq 2$. Dusart [6, p. 54] showed that the inequality

$$p_n \leq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 1.8}{\log n}\right) \tag{1.9}$$

holds for every $n \geq 27\,076$. Further, he [7, Theorem 3] made a breakthrough concerning the inequality (1.5) by showing that this inequality holds for every $n \geq 2$. The current best estimates for the nth prime, which correspond to the first terms in (1.2), are also given by

Dusart [8, Propositions 5.15 and 5.16]. He used explicit estimates for Chebyshev's ϑ-function to show that the inequality

$$p_n \leq n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} \right), \tag{1.10}$$

which corresponds to the first four terms of (1.2), holds for every $n \geq 688\,383$ and that

$$p_n \geq n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2.1}{\log n} \right) \tag{1.11}$$

for every $n \geq 3$. The goal of this paper is to improve the inequalities (1.10) and (1.11) with regard to Cipolla's asymptotic expansion (1.2). For this purpose, we use estimates for the quantity $1/\log p_n$ and some estimates [3] for the prime counting function $\pi(x)$ to obtain the following refinement of (1.10).

Theorem 1. *For every integer $n \geq 46\,254\,381$, we have*

$$p_n < n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 10.667}{2\log^2 n} \right). \tag{1.12}$$

Under the assumption that the Riemann hypothesis is true, Dusart [9, Theorem 3.4] found that

$$p_n < n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n}{2\log^2 n} \right). \tag{1.13}$$

for every integer $n \geq 3468$. Using Theorem 1 and a computer for smaller values of n, we get

Corollary 2. *The inequality (1.13) holds unconditionally for every $n \geq 3468$.*

In the other direction, we find the following result which yields a lower bound for the nth prime number in a bounded range.

Theorem 3. *For every integer n satisfying $2 \leq n \leq \pi(10^{19}) = 234\,057\,667\,276\,344\,607$, we have*

$$p_n > n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11.25}{2\log^2 n} \right).$$

Finally, we use Theorem 3 to give the following improvement of (1.11).

Theorem 4. *For every integer $n \geq 2$, we have*

$$p_n > n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11.321}{2\log^2 n} \right). \tag{1.14}$$

We get the following corollary which was already known under the assumption that the Riemann hypothesis is true (see Dusart [9, Theorem 3.4]).

Corollary 5. *For every $n \geq 2$, we have*

$$p_n > n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2}{2\log^2 n} \right).$$

In Section 6 we apply the Theorems 1 and 4 to find some refined estimates for $\vartheta(p_n)$, where $\vartheta(x) = \sum_{p \leq x} \log p$ is Chebyshev's ϑ-function.

Notation 6. Throughout this paper, let n denote a positive integer. For better readability, in the majority of the proofs we use the notation

$$w = \log\log n, \quad y = \log n, \quad z = \log p_n.$$

2 Effective estimates for the reciprocal of $\log p_n$

Let m be a positive integer. Using Panaitopol's asymptotic formula for the prime counting function $\pi(x)$ — see [12] — we see that

$$p_n = n \left(\log p_n - 1 - \frac{1}{\log p_n} - \frac{3}{\log^2 p_n} - \cdots - \frac{k_m}{\log^m p_n} \right) + O\left(\frac{n}{\log^{m+1} n} \right), \tag{2.1}$$

where the positive integers $k_1, \ldots, k_m$ are given by the recurrence formula

$$k_m + 1!k_{m-1} + 2!k_{m-2} + \cdots + (m-1)!k_1 = m \cdot m!.$$

So, in order to prove Theorems 1 and 4, we first use some results of [3] concerning effective estimates for $\pi(x)$ which imply estimates for the nth prime number p_n in the direction of (2.1). Then we apply the estimates for the quantity $1/\log p_n$ obtained in this section. Cipolla [5, p. 139] showed that

$$\frac{1}{\log p_n} = \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + o\left(\frac{1}{\log^2 n} \right).$$

Concerning this asymptotic formula, we give the following inequality involving $1/\log p_n$, where the polynomials $P_1, \ldots, P_4 \in \mathbb{Z}[x]$ are given by

$$\begin{aligned}
P_1(x) &= 3x^2 - 6x + 5, \\
P_2(x) &= 5x^3 - 24x^2 + 39x - 14, \\
P_3(x) &= 7x^4 - 48x^3 + 120x^2 - 124x + 51, \\
P_4(x) &= 9x^5 - 80x^4 + 280x^3 - 480x^2 + 405x - 124.
\end{aligned}$$

Proposition 7. *For every integer $n \geq 688\,383$, we have*

$$\frac{1}{\log p_n} \geq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{1}{\log p_n} \sum_{k=1}^{4} \frac{(-1)^{k+1} P_k(\log\log n)}{k(k+1)\log^{k+2} n}.$$

Proof. We just give a sketch of the proof. For details, see [2, Proposition 2.2]. We write $w = \log\log n$, $y = \log n$, and $z = \log p_n$. By (1.10), the inequality $\log(1+x) \leq \sum_{k=1}^{7} (-1)^{k+1} x^k/k$, which holds for every $x > -1$, and the fact that $(w-1)/y + (w-2)/y^2 > -1$, we see that

$$-y^2 + (y-w)z \leq -w^2 + (y-w) \sum_{k=1}^{7} \frac{(-1)^{k+1}}{k} \left(\frac{w-1}{y} + \frac{w-2}{y^2} \right)^k.$$

Finally, we extend the right-hand side of the last inequality to complete the proof. □

Corollary 8. *For every integer $n \geq 456\,914$, we have*

$$\frac{1}{\log p_n} \geq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_1(\log\log n)}{2\log^3 n \log p_n} - \frac{P_2(\log\log n)}{6\log^4 n \log p_n}.$$

Proof. See [2, Korollar 2.6]. □

Corollary 9. *For every integer $n \geq 71$, we have*

$$\frac{1}{\log p_n} \geq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n}.$$

Proof. Since the inequality

$$\frac{P_1(\log\log n)}{2\log n} - \frac{P_2(\log\log n)}{6\log^2 n} \geq 0 \tag{2.2}$$

holds for every $n \geq 3$, Corollary 8 implies the validity of the required inequality for every $n \geq 456\,914$. We finish by checking the remaining cases with a computer. □

Using a similar method as in the proof of Proposition 7, we find the following inequality involving the reciprocal of $\log p_n$. Here, we have

$$\begin{aligned} P_5(x) &= 3x^2 - 6x + 5.2, \\ P_6(x) &= x^3 - 6x^2 + 11.4x - 4.2, \\ P_7(x) &= 2x^3 - 7.2x^2 + 8.4x - 4.41, \\ P_8(x) &= x^3 - 4.2x^2 + 4.41x. \end{aligned}$$

Proposition 10. *For every integer $n \geq 2$, we have*

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_5(\log\log n)}{2\log^3 n \log p_n} - \sum_{k=4}^{6} \frac{P_{k+2}(\log\log n)}{2\log^k n \log p_n}.$$

Proof. First, we consider the case where $n \geq 33$. We write again $w = \log\log n$, $y = \log n$, and $z = \log p_n$. Notice that $\log(1+t) \geq t - t^2/2$ for every $t \geq 0$. If we combine the last fact with (1.11) and $(w-1)/y + (w-2.1)/y^2 \geq 0$, we obtain the inequality

$$-y^2 + (y-w)z \geq -w^2 + (y-w)\sum_{k=1}^{2} \frac{(-1)^{k+1}}{k}\left(\frac{w-1}{y} + \frac{w-2.1}{y^2}\right)^k$$

which implies the required inequality. A computer check completes the proof. □

Proposition 10 implies the following both corollaries.

Corollary 11. *For every integer $n \geq 2$, we have*

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_5(\log\log n)}{2\log^3 n \log p_n} - \sum_{k=4}^{5} \frac{P_{k+2}(\log\log n)}{2\log^k n \log p_n}.$$

Proof. See [2, Korollar 2.20]. □

Corollary 12. *For every integer $n \geq 2$, we have*

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_5(\log\log n)}{2\log^3 n \log p_n} - \frac{P_6(\log\log n)}{2\log^4 n \log p_n}.$$

Proof. See [2, Korollar 2.21]. □

3 Proof of Theorem 1

First, we introduce the following notation. Let the polynomials $P_1, \ldots, P_4 \in \mathbb{Z}[x]$ are given as in the beginning of Section 2. Let A_0 be a real number with $0.75 \leq A_0 < 1$ and let $F_0 : \mathbb{N} \to \mathbb{R}$ be defined by

$$F_0(n) = \log n - A_0 \log p_n.$$

From (1.1), it follows that $F_0(n)$ is nonnegative for all sufficiently large values of n. Let N_0 be a positive integer so that $F_0(n) \geq 0$ for every $n \geq N_0$. Furthermore, let A_1 be a real number with $0 < A_1 \leq 458.7275$, and for $w = \log\log n$ let $F_1 : \mathbb{N}_{\geq 2} \to \mathbb{R}$ be given by

$$\begin{aligned} F_1(n) = {} & \frac{A_1}{\log^5 p_n} + \frac{(w^2 - 3.85w + 14.15)(w^2 - w + 1)}{\log^4 n \log p_n} + \frac{2.85 P_1(w)}{2\log^3 n \log^2 p_n} + \frac{2.85 P_1(w)}{2\log^4 n \log p_n} \\ & + \left(\frac{13.15(w^2 - w + 1)}{\log^2 n \log^2 p_n} - \frac{70.7w}{\log^2 n \log^2 p_n}\right)\left(\frac{1}{\log n} + \frac{1}{\log p_n}\right) - \frac{P_2(w)}{6\log^4 n \log p_n}. \end{aligned}$$

Then $F_1(n)$ is nonnegative for all sufficiently large values of n, and we can define N_1 to be a positive integer so that $F_1(n) \geq 0$ for every $n \geq N_1$. Further we set $A_2 = (458.7275 - A_1)A_0^5$ and $A_3 = 3428.7225 A_0^6$. To prove Theorem 1, we first use a recently obtained estimate [3] for the prime counting function $\pi(x)$ and some results from the previous section to construct a positive integer n_0 and an arithmetic function $b_0 : \mathbb{N}_{\geq 2} \to \mathbb{R}$, both depending on some parameters, with $b_0(n) \to 10.7$ as $n \to \infty$ so that

$$p_n < n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + b_0(n)}{2\log^2 n} \right)$$

for every $n \geq n_0$. In order to do this, let $a_0 : \mathbb{N}_{\geq 2} \to \mathbb{R}$ be an arithmetic function satisfying

$$a_0(n) \geq -(\log\log n)^2 + 6\log\log n, \tag{3.1}$$

and let N_2 be a positive integer depending on the arithmetic function a_0 so that the inequalities

$$-1 < \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2}{\log^2 n} - \frac{(\log\log n)^2 - 6\log\log n + a_0(n)}{2\log^3 n} \leq 1, \tag{3.2}$$

$$\frac{\log\log n - 2}{\log^2 n} - \frac{(\log\log n)^2 - 6\log\log n + a_0(n)}{2\log^3 n} \geq 0, \quad \text{and} \tag{3.3}$$

$$p_n < n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + a_0(n)}{2\log^2 n} \right) \tag{3.4}$$

hold simultaneously for every $n \geq N_2$. Now we set

$$\begin{aligned} G_0(x) = {} & \frac{2x^3 - 21x^2 + 82.2x - 98.9}{6e^{3x}} - \frac{x^4 - 14x^3 + 53.4x^2 - 100.6x + 17}{4e^{4x}} \\ & + \frac{2x^5 - 10x^4 + 35x^3 - 110x^2 + 150x - 42}{10e^{5x}} - \frac{3x^4 - 44x^3 + 156x^2 - 96x + 64}{24e^{6x}}, \end{aligned}$$

and for $w = \log\log n$ we define

$$\begin{aligned} b_0(n) = {} & 10.7 + \frac{2A_2}{\log^3 n} + \frac{2A_3}{\log^4 n} + \frac{a_0(n)}{\log n}\left(1 - \frac{w-1}{\log n} - \frac{w-2}{\log^2 n} + \frac{2w^2 - 12w + a_0(n)}{4\log^3 n}\right) \\ & - 2\,G_0(w)\log^2 n + \frac{A_0((5.7A_0 + 8.7)w^2 - (32A_0 + 38)w + 147.1A_0 + 10.7)}{\log^2 n} \\ & + \frac{2 \cdot 70.7A_0^3(w^2 - w + 1)}{\log^4 n} + \frac{2 \cdot 70.7A_0^4(w^2 - w + 1)}{\log^4 n}. \end{aligned} \tag{3.5}$$

Then we obtain the following

Proposition 13. *For every integer $n \geq \max\{N_0, N_1, N_2, 841\,424\,976\}$, we have*

$$p_n < n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + b_0(n)}{2\log^2 n} \right).$$

In order to prove this proposition, we need the following lemma. Its proof is left to the reader.

Lemma 14. *For every $x \geq 2.103$, we have*

$$0 \leq \frac{(x^2 - 3.85x + 14.15)P_1(x)}{2} - \frac{2.85P_2(x)}{3} + \frac{P_3(x)}{12} - \frac{(x^2 - 3.85x + 14.15)P_2(x)}{6e^x} - \frac{P_4(x)}{20e^x}. \tag{3.6}$$

Now we give a proof of Proposition 13.

Proof of Proposition 13. Let $n \geq \max\{N_0, N_1, N_2, 841\,424\,976\}$. Using [3, Theorem 3] with $x = p_n$, we see that

$$p_n < n\left(\log p_n - 1 - \frac{1}{\log p_n} - \frac{2.85}{\log^2 p_n} - \frac{13.15}{\log^3 p_n} - \frac{70.7}{\log^4 p_n} - \frac{458.7275}{\log^5 p_n} - \frac{3428.7225}{\log^6 p_n}\right). \tag{3.7}$$

For convenience, we write $w = \log\log n$, $y = \log n$, and $z = \log p_n$. By Corollary 8, we have

$$\frac{1}{z^2} \geq \frac{1}{yz} - \frac{w}{y^2 z} + \frac{w^2 - w + 1}{y^2 z^2} + \frac{P_1(w)}{2y^3 z^2} - \frac{P_2(w)}{6y^4 z^2}. \tag{3.8}$$

Again using Corollary 8, we get

$$\frac{1}{yz} \geq \Phi_1(n) = \frac{1}{y^2} - \frac{w}{y^3} + \frac{w^2 - w + 1}{y^3 z} + \frac{P_1(w)}{2y^4 z} - \frac{P_2(w)}{6y^5 z}. \tag{3.9}$$

Applying (3.9) to (3.8), we see that

$$\frac{1}{z^2} \geq \Phi_2(n), \tag{3.10}$$

where

$$\Phi_2(n) = \frac{1}{y^2} - \frac{w}{y^3} - \frac{w}{y^2 z} + \frac{w^2 - w + 1}{y^3 z} + \frac{w^2 - w + 1}{y^2 z^2} + \left(\frac{P_1(w)}{2y^3 z} - \frac{P_2(w)}{6y^4 z}\right)\left(\frac{1}{y} + \frac{1}{z}\right).$$

Now (2.2) implies that

$$\frac{1}{z^2} \geq \Phi_3(n) = \frac{1}{y^2} - \frac{w}{y^3} - \frac{w}{y^2 z} + \frac{w^2 - w + 1}{y^3 z} + \frac{w^2 - w + 1}{y^2 z^2}. \tag{3.11}$$

We assumed $n \geq N_0$. Hence $F_0(n) \geq 0$, which is equivalent to

$$\frac{A_0}{y} \leq \frac{1}{z}. \tag{3.12}$$

From (3.12) and the fact that $2.85x^2 - 16x + 73.55 \geq 0$ for every $x \geq 0$, it follows

$$\frac{2.85w^2 - 16w + 73.55}{z^2} \geq \frac{A_0(5.7w^2 - 32w + 147.1)}{2yz}. \tag{3.13}$$

Let $f(x) = (5.7A_0 + 8.7)x^2 - (32A_0 + 38)x + 147.1A_0 + 10.7$. Since $0.75 \leq A_0 < 1$, we get $f(x) \geq 12.975x^2 - 70x + 121.025 \geq 0$ for every $x \geq 0$. Using (3.12) and (3.13), we get

$$\frac{2.85w^2 - 16w + 73.55}{z^2} + \frac{8.7w^2 - 38w + 10.7}{2yz} \geq \frac{A_0 f(w)}{2y^2}. \tag{3.14}$$

We recall that $A_2 = (458.7275 - A_1)A_0^5$ and $A_3 = 3428.7225A_0^6$. Hence (3.12) implies that

$$\frac{A_2}{y^5} + \frac{A_3}{y^6} + \frac{70.7A_0^3}{y^6} + \frac{70.7A_0^4}{y^6} \leq \frac{458.7275 - A_1}{z^5} + \frac{3428.7225}{z^6} + \frac{70.7}{y^3z^3} + \frac{70.7}{y^2z^4}. \tag{3.15}$$

Now we apply (3.14) and (3.15) to (3.5) and see that

$$\begin{aligned}
&\frac{10.7 - b_0(n)}{2y^2} + \frac{2.85(w^2 - w + 1)}{y^2z^2} - \frac{13.15w}{y^2z^2} + \frac{70.7}{y^2z^2} + \frac{8.7w^2 - 38w + 10.7}{2y^3z} \\
&\qquad + \frac{458.7275 - A_1}{z^5} + \frac{3428.7225}{z^6} + \frac{70.7(w^2 - w + 1)}{y^2z^3}\left(\frac{1}{y} + \frac{1}{z}\right) \\
&\geq G_0(w) - \frac{a_0(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2 - 12w + a_0(n)}{4y^3}\right).
\end{aligned} \tag{3.16}$$

The inequality (2.2) tells us that

$$\frac{13.15}{z}\left(\frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z}\right)\left(\frac{1}{y} + \frac{1}{z}\right) \geq 0. \tag{3.17}$$

Adding the left-hand side of (3.17) and the right-hand side of (3.6) with $x = w$ to the left-hand side of (3.16), we get

$$\begin{aligned}
&\frac{5.35}{y^2} - \frac{b_0(n)}{2y^2} + \frac{2.85(w^2 - w + 1)}{y^2z^2} - \frac{13.15w}{y^2z^2} + \frac{70.7}{y^2z^2} + \frac{8.7w^2 - 38w + 10.7}{2y^3z} + \frac{458.7275 - A_1}{z^5} \\
&\qquad + \frac{3428.7225}{z^6} + \frac{70.7(w^2 - w + 1)}{y^2z^3}\left(\frac{1}{y} + \frac{1}{z}\right) + \frac{13.15}{z}\left(\frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z}\right)\left(\frac{1}{y} + \frac{1}{z}\right) \\
&\qquad - \frac{2.85P_2(w)}{6y^5z} - \frac{2.85P_2(w)}{6y^4z^2} + \frac{(w^2 - 3.85w + 14.15)P_1(w)}{2y^5z} + \frac{P_3(w)}{12y^5z} - \frac{P_4(w)}{20y^6z} \\
&\qquad - \frac{(w^2 - 3.85w + 14.15)P_2(w)}{6y^6z} \\
&\geq G_0(w) - \frac{a_0(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2 - 12w + a_0(n)}{4y^3}\right).
\end{aligned}$$

Since $n \geq N_1$, we have $F_1(n) \geq 0$. Now we add $F_1(n)$ to the left-hand side of the last inequality, use the identity $8.7w^2 - 38w + 10.7 = P_1(w) + 2 \cdot 2.85(w^2 - w + 1) - 2 \cdot 13.15w$, and collect all terms containing the number 70.7 and the term $w^2 - 3.85w + 14.15$, respectively, to get

$$\begin{aligned}
&\frac{5.35}{y^2} - \frac{b_0(n)}{2y^2} + \frac{2.85(w^2-w+1)}{y^2z^2} - \frac{13.15w}{y^2z^2} + \frac{70.7}{z^2} \cdot \Phi_3(n) + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} \\
&\qquad + \frac{2.85(w^2-w+1)}{y^3z} - \frac{13.15w}{y^3z} + \left(2.85 + \frac{13.15}{z}\right)\left(\frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z}\right)\left(\frac{1}{y} + \frac{1}{z}\right) \\
&\qquad + \frac{w^2 - 3.85w + 14.15}{y} \cdot \Phi_1(n) + \frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z} + \frac{P_3(w)}{12y^5z} - \frac{P_4(w)}{20y^6z} \\
&\qquad + \frac{13.15(w^2-w+1)}{y^2z^2}\left(\frac{1}{y} + \frac{1}{z}\right) - \frac{2.85w}{y^3} \\
&\quad \geq \widetilde{G_0}(w) - \frac{a_0(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2 - 12w + a_0(n)}{4y^3}\right),
\end{aligned}$$

where $\Phi_1(n)$ and $\Phi_3(n)$ are given as in (3.9) and (3.11), respectively, and

$$\widetilde{G_0}(x) = G_0(x) + \frac{x^2 - 3.85x + 14.15}{e^{3x}} - \frac{x^3 - 3.85x^2 + 14.15x}{e^{4x}} - \frac{2.85x}{e^{3x}}.$$

Now we use (3.9) and (3.11) and collect all terms containing the numbers 2.85 and 13.15 to see that

$$\begin{aligned}
&\frac{2.5}{y^2} - \frac{b_0(n)}{2y^2} + \left(2.85 + \frac{13.15}{z}\right)\Phi_2(n) + \frac{70.7}{z^4} + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} + \frac{w^2-w+1}{y^2z} \\
&\qquad + \frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z} + \frac{P_3(w)}{12y^5z} - \frac{P_4(w)}{20y^6z} \\
&\quad \geq \widetilde{G_0}(w) - \frac{a_0(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2 - 12w + a_0(n)}{4y^3}\right).
\end{aligned}$$

Applying (3.10) and Proposition 7, we get

$$\begin{aligned}
&\frac{2.5}{y^2} - \frac{b_0(n)}{2y^2} + \frac{2.85}{z^2} + \frac{13.15}{z^3} + \frac{70.7}{z^4} + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} - \frac{1}{y} + \frac{w}{y^2} + \frac{1}{z} \\
&\quad \geq \widetilde{G_0}(w) - \frac{a_0(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2 - 12w + a_0(n)}{4y^3}\right).
\end{aligned}$$

A straightforward calculation shows that the last inequality is equivalent to

$$\begin{aligned}-\frac{1}{y}-\frac{w^2-4w-(4-b_0(n))}{2y^2}+\frac{1}{z}+\frac{2.85}{z^2}+\frac{13.15}{z^3}+\frac{70.7}{z^4}+\frac{458.7275}{z^5}+\frac{3428.7225}{z^6}\\ \geq -\frac{w^2-6w+a_0(n)}{2y^3}-\frac{1}{2}\left(\frac{w-1}{y}+\frac{w-2}{y^2}-\frac{w^2-6w+a_0(n)}{2y^3}\right)^2\\ +\frac{1}{3}\left(\frac{w-1}{y}+\frac{w-2}{y^2}\right)^3-\frac{1}{4}\left(\frac{w-1}{y}\right)^4+\frac{1}{5}\left(\frac{w-1}{y}\right)^5.\end{aligned}$$

We add $(w-1)/y+(w-2)/y^2$ to both sides of this inequality. Since $\log(1+x)\leq\sum_{k=1}^{5}(-1)^{k+1}x/k$ for every $x>-1$, $g(x)=x^3/3$ is increasing, and $h(x)=-x^4/4+x^5/5$ is decreasing on the interval $[0,1]$, we can use (3.1)–(3.3) to get

$$\begin{aligned}y+w-1+\frac{w-2}{y}-\frac{w^2-6w+b_0(n)}{2y^2}+\frac{1}{z}+\frac{2.85}{z^2}+\frac{13.15}{z^3}+\frac{70.7}{z^4}+\frac{458.7275}{z^5}+\frac{3428.7225}{z^6}\\ \geq y+w-1+\log\left(1+\frac{w-1}{y}+\frac{w-2}{y^2}-\frac{w^2-6w+a_0(n)}{2y^3}\right).\end{aligned}$$

Finally, we use (3.4) and (3.7) to arrive at the desired result. □

Next we use Proposition 13 and the following both lemmata to prove Theorem 1. In the first lemma we determine a suitable value of N_0 for $A_0=0.87$.

Lemma 15. *For every integer $n\geq 1\,338\,564\,587$, we have*

$$\log n\geq 0.87\log p_n.$$

Proof. We set

$$f(x)=e^x-0.87\left(e^x+x+\log\left(1+\frac{x-1}{e^x}+\frac{x-2}{e^{2x}}\right)\right).$$

Since $f'(x)\geq 0$ for every $x\geq 2.5$ and $f(3.046)\geq 0.00137$, we see that $f(x)\geq 0$ for every $x\geq 3.046$. Substituting $x=\log\log n$ in $f(x)$ and using (1.10), we see that the desired inequality holds for every $n\geq\exp(\exp(3.046))$. We check the remaining cases with a computer. □

Now we use Lemma 15 to find a suitable value of N_1 for $A_1=155.32$.

Lemma 16. *Let $A_1=155.32$. Then $F_1(n)\geq 0$ for every $n\geq 100\,720\,878$.*

Proof. First, let $n\geq\exp(\exp(3.05))$. We have

$$F_1(n)=\frac{155.32}{z^5}+\frac{f(w)}{6y^4z}+\frac{34.85w^2-184.8w+40.55}{2y^3z^2}+\frac{13.15w^2-83.85w+13.15}{y^2z^3}.$$

where $f(x) = 6x^4 - 34.1x^3 + 163.65x^2 - 198.3x + 141.65$. Since $f(x) \geq 0$ for every $x \geq 3.05$, it suffices to show that

$$\frac{155.32}{z^5} + \frac{6w^4 - 34.1w^3 + 268.2w^2 - 752.7w + 263.3}{6y^3z^2} + \frac{13.15w^2 - 83.85w + 13.15}{y^2z^3} \geq 0. \tag{3.18}$$

In order to do this, we set

$$\begin{aligned} g(x) = (6x^4 - 34.1x^3 + 268.2x^2 - 752.7x + 263.3)(e^x + x) \\ + 6e^x(13.15x^2 - 83.85x + 13.15 + 155.32 \cdot 0.87^2). \end{aligned}$$

It is easy to see that $h_1(x) = 6x^4 - 10.1x^3 + 244.8x^2 - 561.6x - 208.229752 \geq 0$ for every $x \geq 2.6$ and that $h_2(x) = 30x^4 - 136.4x^3 + 804.6x^2 - 1505.4x + 263.3 \geq 0$ for every $x \geq 2.2$. Hence $g'(x) = h_1(x)e^x + h_2(x) \geq 0$ for every $x \geq 2.6$. We also have $g(3.05) \geq 0.9$. Therefore, $g(x) \geq 0$ for every $x \geq 3.05$. Since $6x^4 - 34.1x^3 + 268.2x^2 - 752.7x + 263.3 \geq 0$ for every $x \geq 3.05$, we can use (1.3) to get $g(w)/(6y^3z^3) \geq 0$. Now we apply Lemma 15 to obtain (3.18). We finish by direct computation. $\square$

Finally, we give a proof of Theorem 1.

Proof of Theorem 1. For convenience, we write $w = \log\log n$ and $y = \log n$. Setting $A_0 = 0.87$ and $A_1 = 155.32$, we use Lemma 15 and Lemma 16 to get $N_0 = 1\,338\,564\,587$ and $N_1 = 100\,720\,878$, respectively. The proof of this theorem goes in two steps.

Step 1. We set $a_0(n) = -w^2 + 6w$. Then $N_2 = 688\,383$ is a suitable choice for N_2. By (3.5), we get

$$b_0(n) \geq 10.7 + g(n), \tag{3.19}$$

where

$$\begin{aligned} g(n) = &-\frac{2w^3 - 18w^2 + 64.2w - 98.9}{3y} + \frac{w^4 - 12w^3 + 63.16w^2 - 203.17w + 258.29}{2y^2} \\ &- \frac{2w^5 - 10w^4 + 30w^3 - 70w^2 + 90w - 1554.24}{5y^3} \\ &- \frac{8w^3 - 2137.44w^2 + 2185.45w - 37836.25}{12y^4}. \end{aligned}$$

We define

$$\begin{aligned} g_1(x,t) = 3.54e^{4x} &+ 20(18x^2 + 98.9)e^{3x} - 20(2t^3 + 64.2t)e^{3t} \\ &+ 30(x^4 + 63.16x^2 + 258.29)e^{2x} - 30(12t^3 + 203.17t)e^{2t} \\ &+ 12(10x^4 + 70x^2 + 1554.24)e^x - 12(2t^5 + 30t^3 + 90t)e^t \\ &+ 5(2137.44x^2 + 37836.25) - 5(8t^3 + 2185.45t). \end{aligned}$$

If $t_0 \le x \le t_1$, then $g_1(x,x) \ge g_1(t_0,t_1)$. We check with a computer that $g_1(i \cdot 10^{-5}, (i+1) \cdot 10^{-5}) \ge 0$ for every integer i with $0 \le i \le 699\,999$. Therefore,

$$g(n) + 0.059 = \frac{g_1(w,w)}{60y^4} \ge 0 \qquad (0 \le w \le 7). \tag{3.20}$$

Next we prove that $g_1(x,x) \ge 0$ for every $x \ge 7$. For this purpose, let $W_1(x) = 3.54e^x - 20(2x^3 - 18x^2 + 64.2x - 98.9)$. It is easy to show that $W_1(x) \ge 792$ for every $x \ge 7$. Hence we get

$$\begin{aligned} g_1(x,x) \ge{} & (792e^x + 30(x^4 - 12x^3 + 63.16x^2 - 203.17x + 258.29))e^{2x} \\ & - 12(2x^5 - 10x^4 + 30x^3 - 70x^2 + 90x - 1554.24)e^x \\ & - 5(8x^3 - 2137.44x^2 + 2185.45x - 37836.25). \end{aligned}$$

Since $792e^t + 30(t^4 - 12t^3 + 63.16t^2 - 203.17t + 258.29) \ge 875\,011$ for every $t \ge 7$, we obtain $g(n) + 0.059 = g_1(w,w)/(60y^4) \ge 0$ for $w \ge 7$. Combined with (3.20), it gives that $g(n) \ge -0.059$ for every $n \ge 3$. Applying this to (3.19), we get $b_0(n) \ge 10.641$ for every $n \ge 3$. Hence, by Proposition 13, we get

$$p_n < n\left(y + w - 1 + \frac{w-2}{y} - \frac{w^2 - 6w + 10.641}{2y^2}\right)$$

for every $n \ge 1\,338\,564\,587$. For every integer n such that $39\,529\,802 \le n \le 1\,338\,564\,586$ we check the last inequality with a computer.

Step 2. We set $a_0(n) = 10.641$. Using the first step, we can choose $N_2 = 39\,529\,802$. By (3.5), we have

$$b_0(n) \ge 10.7 + h(n), \tag{3.21}$$

where $h(n)$ is given by

$$\begin{aligned} h(n) = {} & -\frac{2w^3 - 21w^2 + 82.2w - 130.823}{3y} + \frac{w^4 - 14w^3 + 77.16w^2 - 236.45w + 279.57}{2y^2} \\ & - \frac{2w^5 - 10w^4 + 35w^3 - 110w^2 + 203.205w - 1660.65}{5y^3} \\ & + \frac{3w^4 - 44w^3 + 2309.28w^2 - 2568.52w + 38175.947}{12y^4}. \end{aligned}$$

We set

$$\begin{aligned} h_1(x,t) = {} & 1.98e^{4x} + 20(21x^2 + 130.823)e^{3x} - 20(2t^3 + 82.2t)e^{3t} \\ & + 30(x^4 + 77.16x^2 + 279.57)e^{2x} - 30(14t^3 + 236.45t)e^{2t} \\ & + 12(10x^4 + 110x^2 + 1660.65)e^x - 12(2t^5 + 35t^3 + 203.205t)e^t \\ & + 5(3x^4 + 2309.28x^2 + 38175.947) - 5(44t^3 + 2568.52t). \end{aligned}$$

Clearly, $h_1(x,x) \geq h_1(t_0,t_1)$ for every x such that $t_0 \leq x \leq t_1$. We use a computer to verify that $h_1(i \cdot 10^{-6}, (i+1)\cdot 10^{-6}) \geq 0$ for every integer i with $0 \leq i \leq 7\,999\,999$. Therefore,

$$h(n) + 0.033 = \frac{h_1(w,w)}{60y^4} \geq 0 \qquad (0 \leq w \leq 8). \tag{3.22}$$

We next show that $h_1(x,x) \geq 0$ for every $x \geq 8$. Since $1.98e^t - 20(2t^3 - 21t^2 + 82.2t - 130.823) \geq 1766$ for every $t \geq 8$, we have

$$\begin{aligned} h_1(x,x) \geq{}& 1766e^{3x} + 30(x^4 - 14x^3 + 77.16x^2 - 236.45x + 279.57)e^{2x} \\ & - 12(2x^5 - 10x^4 + 35x^3 - 110x^2 + 203.205x - 1660.65)e^x \\ & + 5(3x^4 - 44x^3 + 2309.28x^2 - 2568.52x + 38175.947). \end{aligned}$$

Note that $1766e^t + 30(t^4 - 14t^3 + 77.16t^2 - 236.45t + 279.57) \geq 5\,271\,998$ for every $t \geq 8$. Hence $h(n) + 0.033 = h_1(w,w)/(60y^4) \geq 0$ for $w \geq 8$. Combined with (3.22) and (3.21), this gives $b_0(n) \geq 10.667$ for every $n \geq 3$. Applying this to Proposition 13, we complete the proof of the required inequality for every $n \geq 1\,338\,564\,587$. We verify the remaining cases with a computer. □

Denoting the right-hand side of (1.10) by $D_{\text{up}}(n)$ and the right-hand side (1.12) by $A_{up}(n)$, we use A006988 to compare the error term of the approximation from Theorem 1 with Dusart's approximation from (1.10) for the 10^nth prime number:

n	p_n	$\lceil D_{\text{up}}(n) - p_n \rceil$	$\lceil A_{up}(n) - p_n \rceil$
10^{10}	252 097 800 623	20 510 784	4 613 984
10^{11}	2 760 727 302 517	172 884 400	38 768 198
10^{12}	29 996 224 275 833	1 469 932 710	311 593 524
10^{13}	323 780 508 946 331	12 732 767 836	2 542 231 421
10^{14}	3 475 385 758 524 527	112 026 014 682	21 049 069 521
10^{15}	37 124 508 045 065 437	998 861 791 991	176 995 293 694
10^{16}	394 906 913 903 735 329	9 004 342 407 404	1 507 803 850 451
10^{17}	4 185 296 581 467 695 669	81 924 060 077 026	12 998 658 322 559
10^{18}	44 211 790 234 832 169 331	751 154 982 343 786	113 204 602 033 556
10^{19}	465 675 465 116 607 065 549	6 932 757 377 044 654	994 838 584 902 026
10^{20}	4 892 055 594 575 155 744 537	64 346 895 915 006 577	8 812 315 669 274 243

4 Proof of Theorem 3

In order to do prove Theorem 3, we introduce the *logarithmic integral* $\mathrm{li}(x)$ which is defined for every real $x \geq 0$ as

$$\mathrm{li}(x) = \int_0^x \frac{\mathrm{d}t}{\log t} = \lim_{\varepsilon \to 0+} \left\{ \int_0^{1-\varepsilon} \frac{\mathrm{d}t}{\log t} + \int_{1+\varepsilon}^x \frac{\mathrm{d}t}{\log t} \right\}.$$

Proof of Theorem 3. Let $x_0 = 3\,273\,361\,096$. First, we verify the required inequality for every integer n with $x_0 \le n \le \pi(10^{19})$. For $x > 1$, the logarithmic integral $\mathrm{li}(x)$ is increasing with $\mathrm{li}((1, \infty)) = \mathbb{R}$. Thus, we can define the inverse function $\mathrm{li}^{-1} : \mathbb{R} \to (1, \infty)$ by

$$\mathrm{li}(\mathrm{li}^{-1}(x)) = x. \tag{4.1}$$

Further, let

$$f(x) = x - \mathrm{li}\left(x\left(\log x + \log\log x - 1 + \frac{\log\log x - 2}{\log x} - \frac{(\log\log x)^2 - 6\log\log x + 11.25}{2\log^2 x}\right)\right).$$

We show that $f(x) > 0$ for every $x \ge x_0$. We have $f(x_0) > 0.000001$. So it suffices to show that $f'(x) \ge 0$ for every $x \ge x_0$. Setting

$$g_1(a, b) = \log\left(1 + \frac{\log a - 1}{a} + \frac{\log a - 2}{a^2} - \frac{\log^2 b - 6\log b + 11.25}{2b^3}\right)$$

and $g(z) = g_1(z, z)$, we see that $(z + \log z + g(z))f'(e^z) = h(z)$, where

$$h(z) = g(z) - \frac{\log z - 1}{z} + \frac{\log^2 z - 4\log z + 5.25}{2z^2} - \frac{\log^2 z - 7\log z + 14.25}{z^3}.$$

Since $z + \log z + g(z) > 0$ for every $z \ge 2.1$, it suffices to verify that $h(z) \ge 0$ for every $z \ge \log x_0$. We have $h(\log x_0) \ge 0.000026$ and

$$\begin{aligned}(-4)z^7 e^{g(z)} h'(z) = z^4 &+ (4\log^3 z - 46\log^2 z + 197\log z - 323.5)z^3 \\ &+ (-6\log^3 z + 60\log^2 z - 175.5\log z + 90)z^2 \\ &+ (-2\log^4 z + 10\log^3 z + 19\log^2 z - 183.5\log z + 234.876)z \\ &+ 6\log^4 z - 82\log^3 z + 443\log^2 z - 1114.5\log z + 1119.375.\end{aligned} \tag{4.2}$$

In order to show that $h'(z) > 0$ for every $z \in J = [\log x_0, 29.8]$, it suffices to show that the right-hand side of (4.2) is negative. Since $z + 4\log^3 z - 46\log^2 z + 197\log z - 325.5 < 1.43$ for every $z \in J$, we get

$$\begin{aligned}(-4)z^7 e^{g(z)} h'(z) < 1.43z^3 &+ (-6\log^3 z + 60\log^2 z - 175.5\log z + 90)z^2 \\ &+ (-2\log^4 z + 10\log^3 z + 19\log^2 z - 183.5\log z + 234.876)z \\ &+ 6\log^4 z - 82\log^3 z + 443\log^2 z - 1114.5\log z + 1119.375.\end{aligned}$$

Notice that $1.43z - 6\log^3 z + 60\log^2 z - 175.5\log z + 90 \le -0.444$ for every $z \in J$. Hence

$$\begin{aligned}(-4)z^7 e^{g(z)} h'(z) < -0.444z^2 &+ (-2\log^4 z + 10\log^3 z + 19\log^2 z - 183.5\log z + 234.876)z \\ &+ 6\log^4 z - 82\log^3 z + 443\log^2 z - 1114.5\log z + 1119.375.\end{aligned}$$

We have $-0.444z - 2\log^4 z + 10\log^3 z + 19\log^2 z - 183.5\log z + 234.876 \leq -47.701$ for every $z \in J$. Hence $(-4)z^7 e^{g(z)} h'(z) < 0$ for every $z \in J$ which yields that $h'(z) > 0$ for every $z \in J$. Combined with $h(\log x_0) > 0$, it turns out that $h(z) > 0$ for every $z \in [\log x_0, 29.8]$. Similar, we get $h'(z) < 0$ for every $z \geq 29.88$. Together with $\lim_{z\to\infty} h(z) = 0$, we see that $h(z) \geq 0$ for every $z \geq 29.88$. It remains to consider the case where $z \in (29.8, 29.88)$. If $a \leq z \leq b$, then

$$h(z) \geq h_1(a,b) = g_1(a,b) - \frac{\log b - 1}{b} + \frac{\log^2 a - 4\log a + 5.25}{2a^2} - \frac{\log^2 b - 7\log b + 14.25}{b^3}.$$

Now we check with a computer that $h_1(29.8, 29.88) > 0$. Hence $f(x) > 0$ for every $x \geq x_0$. Since $\mathrm{li}(x)$ is increasing for $x > 1$, we can use (4.1) to get

$$x\left(\log x + \log\log x - 1 + \frac{\log\log x - 2}{\log x} - \frac{(\log\log x)^2 - 6\log\log x + 11.25}{2\log^2 x}\right) < \mathrm{li}^{-1}(x)$$

for every $x \geq x_0$. Applying [13, Lemma 7] to the last inequality, we see that the desired inequality holds for every integer n satisfying $3\,273\,361\,096 \leq n \leq \pi(10^{19})$. For every integer n such that $2 \leq n < 3\,273\,361\,096$ we check the desired inequality with a computer. □

5 Proof of Theorem 4

Compared with the proof of Theorem 3, the proof of Theorem 4 is rather technical and we need to introduce some notation. First, let

$$\begin{aligned}
P_9(x) &= P_5(x) + 2\cdot 3.15(x^2 - x + 1),\\
P_{10}(x) &= (x^2 - x + 1)P_9(x) + (x^2 - x + 1)P_5(x) - 3.15P_6(x) - P_7(x) + 12.85P_5(x),\\
P_{11}(x) &= 3.15P_7(x) + 12.85P_6(x),\\
P_{12}(x) &= 2(x^2 - x + 1)P_6(x) - P_5(x)P_9(x),
\end{aligned}$$

where the polynomials P_5, P_6, P_7, and P_8 were defined as in Section 2. Let $B_1, \ldots, B_{10}$ be real positive constants satisfying

$$B_6 + B_7 + B_8 + B_9 + B_{10} \leq 3.15. \tag{5.1}$$

Writing $w = \log\log n$, $y = \log n$, and $z = \log p_n$, we define $H_i : \mathbb{N}_{\geq 2} \to \mathbb{R}$, where $1 \leq i \leq 10$, by

- $H_1(n) = \dfrac{B_1 w}{y^3 z} - \dfrac{P_{10}(w)}{2y^5 z} + \dfrac{P_{11}(w)}{2y^5 z^2} + \dfrac{P_{12}(w)}{4y^6 z} + \dfrac{12.85P_6(w)}{2y^4 z^3}$,
- $H_2(n) = \dfrac{B_2 w}{y^3 z} + \dfrac{12.85w}{y^2 z^2} - \dfrac{71.3}{z^4}$,

- $H_3(n) = \dfrac{B_3 w}{y^3 z} - \dfrac{3.15 P_5(w)}{2y^3z^2} - \dfrac{12.85(w^2-w+1)}{y^3z^2}$,
- $H_4(n) = \dfrac{B_4 w}{y^3 z} + \dfrac{3.15 P_6(w) - 12.85 P_5(w)}{2y^4z^2}$,
- $H_5(n) = \dfrac{B_5 w}{y^3 z} + \dfrac{P_6(w) - 3.15 P_5(w)}{2y^4z} - \dfrac{12.85(w^2-w+1)}{y^4z} - \dfrac{(w^2-w+1)^2}{y^4z}$,
- $H_6(n) = \dfrac{B_6 w}{y^2 z} + \dfrac{(12.85 - B_1 - B_2 - B_3 - B_4 - B_5)w}{y^3z} - \dfrac{3.15(w^2-w+1)}{y^2z^2}$,
- $H_7(n) = \dfrac{B_7 w}{y^2 z} - \dfrac{12.85 P_5(w)}{2y^3z^3}$,
- $H_8(n) = \dfrac{B_8 w}{y^2 z} - \dfrac{12.85(w^2-w+1)}{y^2z^3}$,
- $H_9(n) = \dfrac{B_9 w}{y^2 z} - \dfrac{463.2275}{z^5}$,
- $H_{10}(n) = \dfrac{B_{10} w}{y^2 z} - \dfrac{4585}{z^6}$.

Then $H_i(n)$, $1 \leq i \leq 10$, is nonnegative for all sufficiently large values of n. Let K_1 be a positive integer so that $H_i(n) \geq 0$, $1 \leq i \leq 10$, for every $n \geq K_1$. Let $a_1 : \mathbb{N}_{\geq 2} \to \mathbb{R}$ be an arithmetic function and let K_2 be a positive integer, which depends on a_1, so that the inequalities

$$a_1(n) > -(\log\log n)^2 + 6\log\log n, \tag{5.2}$$

$$0 \leq \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2}{\log^2 n} - \frac{(\log\log n)^2 - 6\log\log n + a_1(n)}{2\log^3 n} \leq 1, \quad \text{and} \tag{5.3}$$

$$p_n > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + a_1(n)}{2\log^2 n}\right) \tag{5.4}$$

hold simultaneously for every $n \geq K_2$. Furthermore, we define the function $G_1 : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} G_1(x) = \frac{3.15x}{e^{3x}} - \frac{12.85}{e^{3x}} + \frac{12.85x}{e^{4x}} - \frac{x^2-x+1}{e^{3x}} + \frac{(x^2-x+1)x}{e^{4x}} - \frac{P_9(x)}{2e^{4x}} + \frac{P_9(x)x}{2e^{5x}} \\ + \frac{(x-1)^2}{2e^{2x}} - \frac{x^2-6x}{2e^{3x}} - \sum_{k=2}^{4} \frac{(-1)^k}{k}\left(\frac{x-1}{e^x} + \frac{x-2}{e^{2x}}\right)^k + \frac{(x-2)^4}{4e^{8x}}. \end{aligned}$$

In order to prove Theorem 4, we set

$$\begin{aligned} b_1(n) = 11.3 - 2\,G_1(\log\log n)\log^2 n + \frac{a_1(n)}{\log n} \\ - \frac{2A_0(3.15 - (B_6 + B_7 + B_8 + B_9 + B_{10}))\log\log n}{\log n} \end{aligned} \tag{5.5}$$

and prove the following proposition.

Proposition 17. *For every integer $n \geq \max\{N_0, K_1, K_2, 3520\}$, we have*

$$p_n > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + b_1(n)}{2\log^2 n}\right).$$

The following lemma is helpful for the proof of Proposition 17. The proof is left to the reader.

Lemma 18. *Let $w = \log\log n$. For every integer $n \geq 6$, we have*

$$\frac{12.85P_6(w)}{2\log^6 n\log p_n} + \frac{3.15P_7(w)}{2\log^6 n\log p_n} + \frac{P_8(w)}{2\log^6 n\log p_n} \geq 0,$$

and for every integer $n \geq 17$, we have

$$\frac{P_6(w)P_9(w)}{4\log^7 n\log p_n} + \frac{12.85P_7(w)}{2\log^7 n\log p_n} + \frac{3.15P_8(w)}{2\log^7 n\log p_n} + \frac{3.15P_8(w)}{2\log^6 n\log^2 p_n} \geq \frac{(w-2)^4}{4\log^8 n}.$$

Now we give a proof of Proposition 17.

Proof of Proposition 17. Let $n \geq \max\{N_0, K_1, K_2, 3520\}$. By [3, Theorem 2], we have

$$p_n > n\left(\log p_n - 1 - \frac{1}{\log p_n} - \frac{3.15}{\log^2 p_n} - \frac{12.85}{\log^3 p_n} - \frac{71.3}{\log^4 p_n} - \frac{463.2275}{\log^5 p_n} - \frac{4585}{\log^6 p_n}\right). \tag{5.6}$$

For convenience, we write $w = \log\log n$, $y = \log n$, and $z = \log p_n$. From Corollary 12, it follows that

$$-\frac{1}{z} \geq \Psi_1(n) = -\frac{1}{y} + \frac{w}{y^2} - \frac{w^2 - w + 1}{y^2 z} - \frac{P_5(w)}{2y^3 z} + \frac{P_6(w)}{2y^4 z}. \tag{5.7}$$

Similarly to the proof of (3.10), we use Proposition 10 to get

$$-\frac{1}{z^2} \geq \Psi_2(n), \tag{5.8}$$

where

$$\Psi_2(n) = -\frac{1}{y^2} + \frac{w}{y^3} + \frac{w}{y^2 z} - \left(\frac{1}{y} + \frac{1}{z}\right)\left(\frac{w^2 - w + 1}{y^2 z} + \frac{P_5(w)}{2y^3 z} - \frac{1}{2z}\sum_{k=4}^{6}\frac{P_{k+5}(w)}{y^k}\right).$$

Using $P_8(\log\log x) \geq 0$ for every $x \geq 3$, $P_7(\log\log x) \geq 0$ for every $x \geq 3520$, and Corollary 11, we get

$$\begin{aligned}-\frac{1}{z^3} \geq \Psi_3(n) = &-\frac{1}{y^3} + \frac{w}{y^4} + \frac{w}{y^3 z} + \frac{w}{y^2 z^2} - \frac{w^2 - w + 1}{y^4 z} - \frac{w^2 - w + 1}{y^3 z^2} - \frac{w^2 - w + 1}{y^2 z^3} \\ &- \frac{P_5(w)}{2y^5 z} - \frac{P_5(w)}{2y^4 z^2} - \frac{P_5(w)}{2y^3 z^3} + \frac{P_6(w)}{2y^6 z} + \frac{P_6(w)}{2y^5 z^2} + \frac{P_6(w)}{2y^4 z^3} + \frac{P_7(w)}{2y^7 z}.\end{aligned} \tag{5.9}$$

By (5.1), $3.15-(B_6+B_7+B_8+B_9+B_{10})\geq 0$. Since $n\geq N_0$ is assumed, we have $F_0(n)\geq 0$. Hence, by (3.12) and (5.5), we see that

$$\frac{d(n)}{2y^2}\leq G_1(w)-\frac{a_1(n)}{2y^3}+\frac{(3.15-(B_6+B_7+B_8+B_9+B_{10}))w}{y^2z}, \tag{5.10}$$

where $d(n)=11.3-b_1(n)$. We have $n\geq K_1$. This means that $\sum_{i=1}^{10}H_i(n)\geq 0$. So we can add $\sum_{i=1}^{10}H_i(n)$ to the right-hand side of (5.10) and use Lemma 18 to get

$$\begin{aligned}\frac{d(n)}{2y^2}\leq G_1(w)&-\frac{a_1(n)}{2y^3}+12.85\left(\Psi_3(n)+\frac{1}{y^3}-\frac{w}{y^4}+\frac{P_5(w)}{2y^5z}-\frac{P_6(w)}{2y^5z^2}\right)\\ &+3.15\left(\Psi_2(n)+\frac{1}{y^2}-\frac{w}{y^3}+\frac{w^2-w+1}{y^3z}-\frac{P_6(w)}{2y^5z}-\frac{P_7(w)}{2y^5z^2}\right)\\ &-\frac{71.3}{z^4}-\frac{463.2275}{z^5}-\frac{4585}{z^6}-\frac{(w^2-w+1)^2}{y^4z}+\frac{P_6(w)}{2y^4z}+\frac{P_6(w)P_9(w)}{4y^7z}\\ &+\frac{P_8(w)}{2y^6z}-\frac{P_{10}(w)}{2y^5z}+\frac{P_{11}(w)}{2y^5z^2}+\frac{P_{12}(w)}{4y^6z}-\frac{(w-2)^4}{4y^8},\end{aligned}$$

where $\Psi_2(n)$ and $\Psi_3(n)$ are given as in (5.8) and (5.9), respectively. Applying the defining formulas of P_{10},P_{11},P_{12}, and G_1 to the last inequality, we find

$$\begin{aligned}\frac{d(n)}{2y^2}\leq &-\frac{a_1(n)}{2y^3}+\frac{w^2-w+1}{y^2}\cdot\Psi_1(n)+\frac{P_9(w)}{2y^3}\cdot\Psi_1(n)+12.85\Psi_3(n)+\frac{(w-1)^2}{2y^2}\\ &-\frac{w^2-6w}{2y^3}-\frac{71.3}{z^4}-\frac{463.2275}{z^5}-\frac{4585}{z^6}+3.15\left(\Psi_2(n)+\frac{1}{y^2}+\frac{w^2-w+1}{y^3z}\right)\\ &-\sum_{k=2}^{4}\frac{(-1)^k}{k}\left(\frac{w-1}{y}+\frac{w-2}{y^2}\right)^k+\sum_{k=4}^{6}\frac{P_{k+5}(w)}{2y^kz},\end{aligned}$$

where $\Psi_1(n)$ is given as in (5.7). Note that w^2-w+1 and $P_9(w)$ are nonnegative. Therefore, we can apply (5.7) and (5.9) to the last inequality and get

$$\begin{aligned}\frac{d(n)}{2y^2}\leq &-\frac{a_1(n)}{2y^3}-\frac{w^2-w+1}{y^2z}-\frac{P_9(w)}{2y^3z}-\frac{12.85}{z^3}-\frac{71.3}{z^4}-\frac{463.2275}{z^5}-\frac{4585}{z^6}\\ &+\frac{(w-1)^2}{2y^2}-\frac{w^2-6w}{2y^3}-\sum_{k=2}^{4}\frac{(-1)^k}{k}\left(\frac{w-1}{y}+\frac{w-2}{y^2}\right)^k\\ &+3.15\left(\Psi_2(n)+\frac{1}{y^2}+\frac{w^2-w+1}{y^3z}\right)+\sum_{k=4}^{6}\frac{P_{k+5}(w)}{2y^kz}.\end{aligned}$$

Since $P_9(x)=P_5(x)+2\cdot 3.15(x^2-x+1)$ and $d(n)=11.3-b_1(n)$, the last inequality is equivalent to

$$\frac{5-b_1(n)}{2y^2} \leq -\frac{a_1(n)}{2y^3} - \frac{w^2-w+1}{y^2z} + \frac{(w-1)^2}{2y^2} - \frac{w^2-6w}{2y^3} - \sum_{k=2}^{4} \frac{(-1)^k}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^k$$
$$+ 3.15\Psi_2(n) - \frac{12.85}{z^3} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6} - \frac{P_5(w)}{2y^3z} + \sum_{k=4}^{6} \frac{P_{k+5}(w)}{2y^kz}.$$

Using (5.8) and Proposition 10, we get the inequality

$$\frac{5-b_1(n)}{2y^2} \leq -\frac{1}{z} - \frac{3.15}{z^2} - \frac{12.85}{z^3} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6} + \frac{1}{y} - \frac{w}{y^2} + \frac{(w-1)^2}{2y^2}$$
$$- \frac{w^2-6w}{2y^3} - \sum_{k=2}^{4} \frac{(-1)^k}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^k - \frac{a_1(n)}{2y^3}$$

which is equivalent to

$$\frac{w-2}{y} \leq \frac{w-1}{y} + \frac{w-2}{y^2} - \frac{w^2-6w+a_1(n)}{2y^3} - \sum_{k=2}^{4} \frac{(-1)^k}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^k$$
$$+ \frac{w^2-6w+b_1(n)}{2y^2} - \frac{1}{z} - \frac{3.15}{z^2} - \frac{12.85}{z^3} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6}. \tag{5.11}$$

Since $\log(1+t) \geq \sum_{k=1}^{4}(-1)^{k+1}t^k/k$ for every $t > -1$ and both $g_1(x) = -x^2/2 + x^3/3$ and $g_2(x) = -x^4/4$ are decreasing on the interval $[0,1]$, we can use (5.2) and (5.3) to see that the inequality (5.11) implies

$$\frac{w-2}{y} - \frac{w^2-6w+b_1(n)}{2y^2} \leq \log\left(1 + \frac{w-1}{y} + \frac{w-2}{y^2} - \frac{w^2-6w+a_1(n)}{2y^3}\right) - \frac{1}{z} - \frac{3.15}{z^2}$$
$$- \frac{12.85}{z^3} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6}.$$

Now we add $y+w-1$ to both sides of the last inequality und use (5.5) to get

$$y+w-1+\frac{w-2}{y} - \frac{w^2-6w+b_1(n)}{y} \leq z-1-\frac{1}{z} - \frac{3.15}{z^2} - \frac{12.85}{z^3} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6}.$$

Finally, we multiply the last inequality by n and apply (5.6) to complete the proof. □

Now, we give a proof of Theorem 4.

Proof of Theorem 4. Clearly, Theorem 3 implies the validity of the inequality (1.14) for every integer n satisfying $2 \leq n \leq \pi(10^{19})$. Next, we prove the inequality (1.14) for every $n \geq M_0$, where $M_0 = \pi(10^{19})+1 = 234\,057\,667\,276\,344\,608$. In order to do this, let $A_0 = 0.914$. Then, similar to the proof of Lemma 15, we get $\log n \geq 0.914 \log p_n$ for every integer $n \geq M_0$. So can chose $N_0 = M_0$. In the following table we give explicit values for B_i:

i	1	2	3	4	5	6	7	8	9	10
B_i	0.132	3.021	1.11	0.023	1.993	0.055	0.0006	0.0199	0.055	0.0125

Then $H_i(n) \geq 0$ for every $n \geq M_0$ and each integer i satisfying $1 \leq i \leq 10$. So we can set $K_1 = M_0$. The proof that $H_i(n) \geq 0$ for every $n \geq M_0$ and each integer i with $1 \leq i \leq 10$ can be found in Section 7. Furthermore, the above table indicates

$$3.15 - (B_6 + B_7 + B_8 + B_9 + B_{10}) = 3.007. \tag{5.12}$$

Step 1. We set $a_1(n) = 0.2y - w^2 + 6w$. Then, by (1.11) and (5.2)–(5.5), we can choose $K_2 = 33$. Using (5.5) and (5.12), we obtain

$$b_1(n) = 11.5 - \frac{2w^3 - 18w^2 + 65.390388w - 97.1}{3y} + \rho(n),$$

where

$$\begin{aligned}\rho(n) = {}& \frac{w^4 - 12w^3 + 46.6w^2 - 112w + 40}{2y^2} + \frac{2w^4 - 21.3w^3 + 40.3w^2 - 41.5w + 12}{y^3} \\ &+ \frac{9w^4 - 56w^3 + 129w^2 - 132w + 52}{3y^4} + \frac{2w^4 - 14w^3 + 36w^2 - 40w + 16}{y^5}.\end{aligned} \tag{5.13}$$

In this step, we show that $b_1(n) \leq 11.5$ for every $n \geq M_0$. For this purpose, we set

$$\begin{aligned}\alpha(x,t) = {}& 2(2x^3 + 65.390388x)e^{4x} - 2(18t^2 + 97.1)e^{4t} \\ &+ 3(12x^3 + 112x)e^{3x} - 3(t^4 + 46.6t^2 + 40)e^{3t} \\ &+ 6(21.3x^3 + 41.5x)e^{2x} - 6(2t^4 + 40.3t^2 + 12)e^{2t} \\ &+ 2(56x^3 + 132x)e^{x} - 2(9t^4 + 129t^2 + 52)e^{t} \\ &+ 6(14x^3 + 40x) - 6(2t^4 + 36t^2 + 16).\end{aligned}$$

Note that this function satisfies the identity

$$\alpha(w,w) = 6(11.5 - b_1(n))y^5. \tag{5.14}$$

If $t_0 \leq x \leq t_1$, then $\alpha(x,x) \geq \alpha(t_0,t_1)$. We check with a computer that $\alpha(3.6 + i \cdot 10^{-3}, 3.6 + (i+1) \cdot 10^{-3}) \geq 0$ for every integer i satisfying $0 \leq i \leq 5399$. Hence by (5.14),

$$b_1(n) \leq 11.5 \qquad (3.6 \leq w \leq 9). \tag{5.15}$$

Next, we show that $\alpha(x,x) \geq 0$ for every $x \geq 9$. Since $2(2x^3 - 18x^2 + 65.390388x - 97.1) \geq 982$ for every $x \geq 9$, we have

$$\begin{aligned}\alpha(x,x) \geq {}& 982e^{4x} - 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40)e^{3x} \\ &- 6(2x^4 - 21.3x^3 + 40.3x^2 - 41.5x + 12)e^{2x} \\ &- 2(9x^4 - 56x^3 + 129x^2 - 132x + 52)e^{x} \\ &- 6(2x^4 - 14x^3 + 36x^2 - 40x + 16).\end{aligned}$$

Note that $982e^x - 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40) \geq 7\,955\,369$ for every $x \geq 9$. Therefore, $\alpha(x,x) \geq 0$ for every $x \geq 9$. Combined with (5.14) and (5.15), it gives $b_1(n) \leq 11.5$ for every $n \geq M_0 > \exp(\exp(3.6))$. Applying this to Proposition 17, we get

$$p_n > n\left(y + w - 1 + \frac{w-2}{y} - \frac{w^2 - 6w + 11.5}{2y^2}\right)$$

for every $n \geq M_0$.

Step 2. We set $a_1(n) = 11.5$. Then $K_2 = 47$ is a suitable choice for K_2. Combined with (5.5) and (5.12), it gives

$$b_1(n) = 11.3 - \frac{2w^3 - 21w^2 + 83.390388w - 131.6}{3y} + \rho(n),$$

where $\rho(n)$ is defined as in (5.13). We set

$$\begin{aligned}\beta(x,t) = 0.15e^{5x} &+ 2(2x^3 + 83.390388x)e^{4x} - 2(21t^2 + 131.6)e^{4t}\\ &+ 3(12x^3 + 112x)e^{3x} - 3(t^4 + 46.6t^2 + 40)e^{3t}\\ &+ 6(21.3x^3 + 41.5x)e^{2x} - 6(2t^4 + 40.3t^2 + 12)e^{2t}\\ &+ 2(56x^3 + 132x)e^{x} - 2(9t^4 + 129t^2 + 52)e^{t}\\ &+ 6(14x^3 + 40x) - 6(2t^4 + 36t^2 + 16).\end{aligned}$$

Then $\beta(w,w) = 6(11.325 - b_1(n))y^5$. Similarly to the first step, we get

$$b_1(n) \leq 11.325 \qquad (3.686 \leq w \leq 7).$$

Therefore, it suffices to verify that $\beta(x,x) \geq 0$ for every $x \geq 7$. Notice that $0.15e^x + 2(2x^3 - 21x^2 + 83.390388x - 131.6) \geq 382$ for every $x \geq 7$. Thus we get

$$\begin{aligned}\beta(x,x) \geq 382e^{4x} &- 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40)e^{3x}\\ &- 6(2x^4 - 21.3x^3 + 40.3x^2 - 41.5x + 12)e^{2x}\\ &- 2(9x^4 - 56x^3 + 129x^2 - 132x + 52)e^{x}\\ &- 6(2x^4 - 14x^3 + 36x^2 - 40x + 16).\end{aligned}$$

Since $382e^x - 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40) \geq 419\,440$ for every $x \geq 7$, we conclude that $\beta(x,x) \geq 0$ for every $x \geq 7$. Hence $b_1(n) \leq 11.325$ for every $n \geq M_0 > \exp(\exp(3.686))$. So, by Proposition 17,

$$p_n > n\left(y + w - 1 + \frac{w-2}{y} - \frac{w^2 - 6w + 11.325}{2y^2}\right)$$

for every $n \geq M_0$.

Step 3. Here we set $a_1(n) = 11.325$. Then we can choose $K_2 = 47$. By (5.5) and (5.12),

$$b_1(n) = 11.3 - \frac{2w^3 - 21w^2 + 83.390388w - 131.075}{3y} + \rho(n),$$

where $\rho(n)$ is defined as in (5.13). To show that $b_1(n) \leq 11.321$ for every $n \geq M_0$, we set

$$\begin{aligned}\gamma(x,t) = 0.126e^{5x} &+ 2(2x^3 + 83.390388x)e^{4x} - 2(21t^2 + 131.075)e^{4t} \\ &+ 3(12x^3 + 112x)e^{3x} - 3(t^4 + 46.6t^2 + 40)e^{3t} + 6(21.3x^3 + 41.5x)e^{2x} \\ &- 6(2t^4 + 40.3t^2 + 12)e^{2t} + 2(56x^3 + 132x)e^{x} - 2(9t^4 + 129t^2 + 52)e^{t} \\ &+ 6(14x^3 + 40x) - 6(2t^4 + 36t^2 + 16).\end{aligned}$$

Notice that $\gamma(w,w) = 6(11.321 - b_1(n))y^5$. Analogously to the first step, we obtain $b_1(n) \leq 11.321$ for w satisfying $3.68 \leq w \leq 7$. Next we find $b_1(n) \leq 11.321$ for $w \geq 7$. Note that $0.126e^x + 2(2x^3 - 21x^2 + 83.390388x - 131.075) \geq 357.491$ for every $x \geq 7$. Therefore,

$$\begin{aligned}\gamma(x,x) \geq 357e^{4x} &- 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40)e^{3x} \\ &- 6(2x^4 - 21.3x^3 + 40.3x^2 - 41.5x + 12)e^{2x} \\ &- 2(9x^4 - 56x^3 + 129x^2 - 132x + 52)e^{x} \\ &- 6(2x^4 - 14x^3 + 36x^2 - 40x + 16).\end{aligned}$$

Since $357e^x - 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40) \geq 392\,024$ for every $x \geq 7$, we get $\gamma(x,x) \geq 0$ for every $x \geq 7$. So $b_1(n) \leq 11.321$ for every $n \geq M_0 > \exp(\exp(3.68))$. Now Proposition 17 implies the required inequality for every $n \geq M_0$ which completes the proof. $\square$

Denoting the right-hand side of (1.11) by $D_{\text{low}}(n)$ and the right-hand side of (1.14) by $A_{low}(n)$, we use A006988 to compare the error term of the approximation from Theorem 4 with the approximation from (1.11) for the 10^nth prime number:

n	p_n	$\lceil p_n - D_{\text{low}}(n) \rceil$	$\lceil p_n - A_{low}(n) \rceil$
10^{10}	252 097 800 623	22 918 665	1 553 620
10^{11}	2 760 727 302 517	221 928 766	12 203 725
10^{12}	29 996 224 275 833	2 149 187 973	116 712 205
10^{13}	323 780 508 946 331	20 674 500 003	1 107 237 510
10^{14}	3 475 385 758 524 527	198 184 329 536	10 418 290 134
10^{15}	37 124 508 045 065 437	1 896 434 754 032	97 120 372 631
10^{16}	394 906 913 903 735 329	18 139 062 711 550	901 415 873 097
10^{17}	4 185 296 581 467 695 669	173 543 282 219 005	8 342 526 771 836
10^{18}	44 211 790 234 832 169 331	1 661 592 139 340 947	77 153 499 580 018
10^{19}	465 675 465 116 607 065 549	15 924 846 933 652 812	713 638 559 773 813
10^{20}	4 892 055 594 575 155 744 537	152 800 345 036 619 338	6 606 690 561 425 196

Remark 19. Compared to Theorem 4, the asymptotic expansion (1.2) implies a better lower bound for the nth prime number, which corresponds to the first five terms, namely that

$$p_n > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11}{2\log^2 n}\right) \tag{5.16}$$

for all sufficiently large values of n. Let r_3 denote the smallest positive integer such that the inequality (5.16) holds for every $n \geq r_3$. Under the assumption that the Riemann hypothesis is true, Arias de Reyna and Toulisse [1, Theorem 6.4] proved that $3.9\cdot 10^{30} < r_3 \leq 3.958\cdot 10^{30}$.

6 New estimates for $\vartheta(p_n)$

Chebyshev's ϑ-function is defined by

$$\vartheta(x) = \sum_{p \leq x} \log p,$$

where p runs over primes not exceeding x. Notice that the prime number theorem is equivalent to

$$\vartheta(x) \sim x \qquad (x \to \infty). \tag{6.1}$$

By proving the existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\mathrm{Re}(s) = 1$, de la Vallée-Poussin [20] found an estimate for the error term in (6.1) by proving $\vartheta(x) = x + O(xe^{-c\sqrt{\log x}})$, where c is a positive absolute constant. Applying (1.2) to the last asymptotic formula, we see that

$$\vartheta(p_n) = n\left(\log n + \log_2 n - 1 + \frac{\log_2 n - 2}{\log n} - \frac{(\log_2 n)^2 - 6\log_2 n + 11}{2\log^2 n} + O\left(\frac{(\log_2 n)^3}{\log^3 n}\right)\right),$$

where $\log_2 n = \log\log n$. In this direction, many estimates for $\vartheta(p_n)$ were obtained (see for example Massias and Robin [11, Théorème B]). The current best ones are due to Dusart [8, Propositions 5.11 and 5.12]. He found that

$$\vartheta(p_n) \geq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.04}{\log n}\right)$$

for every $n \geq \pi(10^{15}) + 1 = 29\,844\,570\,422\,670$, and that the inequality

$$\vartheta(p_n) \leq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{0.782}{\log^2 n}\right)$$

holds for every $n \geq 781$. Using Theorems 1 and 4, we find the following estimates for $\vartheta(p_n)$, which improve the estimates given by Dusart.

Proposition 20. *For every integer $n \geq 2$, we have*

$$\vartheta(p_n) > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11.621}{2\log^2 n}\right),$$

and for every integer $n \geq 2581$, we have

$$\vartheta(p_n) < n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 10.367}{2\log^2 n}\right).$$

Proof. From [3, Theorem 1], it follows that

$$p_n - \frac{0.15p_n}{\log^3 p_n} < \vartheta(p_n) < p_n + \frac{0.15p_n}{\log^3 p_n}, \tag{6.2}$$

where the left-hand side inequality is valid for every integer $n \geq 841\,508\,302$ and the right-hand side inequality holds for every positive integer n. By Rosser and Schoenfeld [16, Corollary 1], we have $n > p_n/\log p_n$ for every $n \geq 7$. Applying the last inequality to the left-hand side inequality of (6.2), we get $\vartheta(p_n) > p_n - 0.15n/\log^2 n$ for every $n \geq 841\,508\,302$. Now we apply Theorem 4 to get the desired lower bound for $\vartheta(p_n)$ for every $n \geq 841\,508\,302$. By Büthe [4, Theorem 2], we have

$$\vartheta(x) \geq x - \frac{\sqrt{x}}{8\pi}\log^2 x \qquad (599 < x \leq 1.89 \times 10^{21}). \tag{6.3}$$

Now we apply Theorem 3 to (6.3) and get the required lower bound for $\vartheta(p_n)$ for every integer n with $200\,000 \leq n \leq 841\,508\,301$. We check the remaining cases for n with a computer.

Similarly to the first part of the proof, we apply the inequality $n > p_n/\log p_n$ to the right-hand side inequality of (6.2) to get $\vartheta(p_n) < p_n + 0.15n/\log^2 n$ for every $n \geq 7$. Now we use Theorem 1 to get the required upper bound for $\vartheta(p_n)$ for every $n \geq 46\,254\,381$. For smaller values of n, we use a computer. □

7 Appendix

Let $M_0 = \pi(10^{19}) + 1 = 234\,057\,667\,276\,344\,608$. In the proof of Theorem 4, we note a table in which we give explicit values of B_i. In this appendix, we show that the H_i defined at the start of paragraph 5 are non-negative for every integer $n \geq M_0$ for the given values of B_i. We start with the claim concerning H_1.

Proposition 21. *If $B_1 = 0.132$, then $H_1(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$. We have $P_{11}(x) \geq 0$ for every $x \geq 0.6$ and $P_6(x) \geq 0$ for every $x \geq 0.6$. Using Lemma 15, we get

$$H_1(n) \geq \frac{f_1(\log\log n)}{4\log^6 n \log p_n} \tag{7.1}$$

for every integer $n \geq M_0$, where $f_1(x) = 0.528xe^{3x} - 2P_{10}(x)e^x + 1.74P_{11}(x) + P_{12}(x) + 19.45233P_6(x)$. We show that $f(x) \geq 0$ for every $x \geq x_0$. For this, we set $g(x) = (57.024 + 42.768x)e^{2x} + (-24.6x^4 - 322.1x^3 - 1137.1x^2 - 1265.98x - 512.24)$. It is easy to show that $g(x) \geq 3 \cdot 10^5$ for every $x \geq x_0$. So, $f_1^{(4)}(x) = g(x)e^x + 240x - 1005.6 \geq 0$ for every $x \geq x_0$. Now it is easy to see that $f(x) \geq 0$ for every $x \geq x_0$. Applying this to (7.1), we get $H_1(n) \geq 0$ for every integer $n \geq M_0$. $\square$

Let $B_2 = 3.021$. Before we check that $H_2(n) \geq 0$ for every integer $n \geq M_0$, we introduce the following

Definition 22. For $x \geq 1$, let

$$\Phi(x) = e^x + x + \log\left(1 + \frac{x-1}{e^x} + \frac{x-2.1}{e^{2x}}\right).$$

We note the following three properties of the function $\Phi(x)$.

Lemma 23. *For every $x \geq 0.179$, we have $\Phi'(x) \geq e^x + 3/4$.*

Proof. We have $\Phi'(x) \geq e^x + 3/4$ if and only if $g(x) = e^{2x} - 3xe^x + 7e^x - 7x + 18.7 \geq 0$. Since $g''(x) = 4e^{2x} - (3x-1)e^x \geq 0$ for every $x \geq 0$ and $g'(0.179) \geq 0$, we obtain $g'(x) \geq 0$ for every $x \geq 0.179$. If we combine this with $g(0.179) \geq 26.6$, we get $g(x) \geq 0$ for every $x \geq 0.179$. $\square$

Lemma 24. *For every $x \geq 1.246$, we have $\Phi(x) \geq e^x + x$.*

Proof. The desired inequality holds if and only if $(x-1)e^x + x - 2.1 \geq 0$. Since the last inequality holds for every $x \geq 1.246$, we arrived at the end of the proof. $\square$

Lemma 25. *For every integer $n \geq 3$, we have $\Phi(\log\log n) \leq \log p_n$.*

Proof. The claim follows directly from (1.11). $\square$

Next, we use these properties to see that $H_2(n) \geq 0$ for every integer $n \geq M_0$.

Proposition 26. *Let $B_2 = 3.021$. Then $H_2(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$. We set $f_2(x) = 3.021x\Phi^3(x) + 12.85xe^x\Phi^2(x) - 71.3e^{3x}$ and use Lemmata 23 and 24 to obtain

$$\begin{aligned} f_2'(x) \geq 3.021(e^x + x)^3 &+ 21.913xe^x(e^x + x)^2 + 12.85e^x(e^x + x)^2 \\ &+ 25.7xe^{2x}(e^x + x) - 213.9e^{3x} \end{aligned} \tag{7.2}$$

for every $x \geq 1.25$. We denote the right-hand side of the last inequality by $g_2(x)$. A straightforward calculation gives $g_2^{(3)}(x) \geq (1285.551x - 4061.232)e^{3x} \geq 0$ for every $x \geq x_0$. Now it is easy to see that $g_2(x) \geq 0$ for every $x \geq x_0$. Applying this to (7.2), we see that $f_2'(x) \geq 0$ for every $x \geq x_0$. Since $f_2(x_0) \geq 268.5$, we obtain $f_2(\log\log n) \geq 0$ for every integer $n \geq M_0$. Finally, we apply Lemma 25. $\square$

Proposition 27. *If $B_3 = 1.11$, then $H_3(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$ and let $f_3(x) = 2.22x\Phi(x) - 35.15x^2 + 44.6x - 42.08$. Using Lemmata 23 and 24, we get $f_3'(x) \geq (2.22e^x - 65.86)x \geq 0$ holds for every $x \geq x_0$. Combined with $f_3(x_0) \geq 2.42$ and Lemma 25, we get that $H_3(n) \geq 0$ for every integer $n \geq M_0$. $\square$

Proposition 28. *Let $B_4 = 0.023$. Then $H_4(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$. We set $f_4(x) = 0.046xe^x\Phi(x) + 3.15x^3 - 57.45x^2 + 113.01x - 80.05$ and have $f_4(M_0) \geq 10.103$. By Lemmata 23 and 24, we get $f_4'(x) \geq (0.046(e^x(e^x + x) + e^{2x}) + 9.45x - 114.9)x \geq 0$ for every $x \geq x_0$. Hence $f_4(\log\log n) \geq 0$ for every integer $n \geq M_0$ and we can apply Lemma 25. $\square$

Proposition 29. *Let $B_5 = 1.993$. Then we have $H_5(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$. To proof the claim, we define $f_5(x) = 3.986xe^x - 2x^4 + 5x^3 - 47.15x^2 + 60x - 48.28$. Since $f_5'''(x) \geq 0$ for every $x \geq x_0$ and $f_5''(x_0) \geq 0$, we obtain $f_5''(x) \geq 0$ for every $x \geq x_0$. Combined with $f_5'(x_0) \geq 0$, it turns out that $f_5'(x) \geq 0$ for every $x \geq x_0$. Together with $f_5(x_0) \geq 0.203$, we conclude that $f_5(\log\log n) \geq 0$, and thus $H_5(n) \geq 0$, for every integer $n \geq M_0$. $\square$

Adding the constants $B_1, \ldots, B_5$ given in Proposition 21 and Propositions 26-29, we get $12.85 - B_1 - B_2 - B_3 - B_4 - B_5 = 6.571$. Now we set $B_6 = 0.055$ to obtain the following result.

Proposition 30. *Let $B_6 = 0.055$. Then $H_6(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$. Furthermore, let $r(x,t) = (0.118e^x + 4.116)x\Phi(x) + 3.15xe^x - 3.15(t^2+1)e^t$ and let $f_6(x) = r(x,x)$. If $t_0 \leq x \leq t_1$, then $f_6(x) \geq r(t_0, t_1)$. We check with a computer that $r(3.6 + i \cdot 10^{-3}, 3.6 + (i+1) \cdot 10^{-3}) \geq 0$ for every integer i such that $0 \leq i \leq 599$. Hence $f_6(x) \geq 0$ for every x such that $3.6 \leq x \leq 4.2$. To show that $f_6(x) \geq 0$ for every $x \geq 4.2$, we set

$$g(x) = (0.055(xe^x + e^x) + 6.571)(e^x + x) + (0.055e^x + 6.571)xe^x - 3.15xe^x(1+x).$$

Then $g'(x) = h(x)e^x + 6.571$ where $h(x) = 0.22(1+x)e^x - 3.095x^2 - 2.714x + 10.047$. Since $h(x) \geq 0$ for every $x \geq 4.2$, we get $g'(x) \geq 0$ for every $x \geq 4.2$. Together with $g(4.2) \geq 0$, we see that $g(x) \geq 0$ for every $x \geq 4.2$. Using Lemmata 23 and 24, we obtain $f_6'(x) \geq g(x) \geq 0$ for every $x \geq 4.2$. Combined with $f_6(4.2) \geq 17.047$, we have $f_6(x) \geq 0$ for every $x \geq 4.2$. Hence $f_6(x) \geq 0$ for every $x \geq x_0 \geq 3.6$. Now we apply Lemma 25 to get $H_6(n) \geq 0$ for every integer $n \geq M_0$. $\square$

Proposition 31. *If $B_7 = 0.0006$, then we have $H_7(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$. Substituting the definition of $P_5(x)$, we get

$$H_7(n) = \frac{0.0006w}{y^2 z} - \frac{38.55w^2 - 77.1w + 66.82}{2y^3 z^3}.$$

To show that $H_7(n) \geq 0$ for every integer $n \geq M_0$, we first consider the function $f_7(x) = 0.0012xe^x\Phi^2(x) - 38.55x^2 + 77.1x - 66.82$. We have $f_7(x_0) \geq 31.88$. Additionally, we use Lemmata 23 and 24 to get $f_7'(x) \geq (0.0012(e^x + x)^2(1 + e^x) + 0.0024e^{2x}(e^x + x) - 77.1)x \geq 0$ for every $x \geq x_0$. Hence, $f_7(\log\log n) \geq 0$ for every integer $n \geq M_0$. Finally, it suffices to apply Lemma 25. ☐

Proposition 32. *Let $B_8 = 0.0199$. Then $H_8(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$. We set $f_8(x) = 0.0199x\,\Phi^2(x) - 12.85(x^2 - x + 1)$. We have $f_8(x_0) \geq 0.906$. By Lemmata 23 and 24, we obtain $f_8'(x) \geq (0.0199(e^x + x) + 0.0398(e^x + x)e^x - 25.7)x \geq 0$ for every $x \geq x_0$. Hence $f_8(\log\log n) \geq 0$ for every integer $n \geq M_0$. Finally, we use Lemma 25. ☐

Proposition 33. *If $B_9 = 0.055$, then $H_9(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$. We define $f_9(x) = 0.055x\,\Phi^4(x) - 463.2275e^{2x}$. By Lemmata 23 and 24, we have $f_9'(x) \geq ((0.055 + 0.22x)(e^x + x)^2 - 926.455)e^{2x} \geq 0$ for every $x \geq x_0$. Combined with $f_9(x_0) \geq 2263.343$, we get $f_9(x) \geq 0$ for every $x \geq x_0$. Substituting $x = \log\log n$ in $f_9(x)$, we apply Lemma 25 to see that $H_9(n) \geq 0$ for every integer $n \geq M_0$. ☐

Finally, we set $B_{10} = 0.0125$ and check that $H_{10}(n) \geq 0$ for every integer $n \geq M_0$.

Proposition 34. *Let $B_{10} = 0.0125$. Then we have $H_{10}(n) \geq 0$ for every integer $n \geq M_0$.*

Proof. Let $x_0 = \log\log M_0$ and let $f_{10}(x) = 0.0125x\,\Phi^5(x) - 4585e^{2x}$. Applying Lemmata 23 and 24, we get $f_{10}'(x) \geq (0.4x(e^x + x)^3 - 9170)e^{2x} \geq 0$ for every $x \geq x_0$. Together with $f_{10}(x_0) \geq 55867.822$, we see that $f_{10}(\log\log n) \geq 0$ for every integer $n \geq M_0$. Now, we use Lemma 25 to conclude that $H_{10}(n) \geq 0$ for every integer $n \geq M_0$. ☐

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