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On Engel's Inequality for Bell Numbers

Horst Alzer
Morsbacher Straße 10
51545 Waldbröl
Germany
h.alzer@gmx.de

Abstract

We prove that for all integers $n \geq 2$ the expression $B_{n-1}B_{n+1} - B_n^2$ can be represented as an infinite series with nonnegative terms. Here B_k denotes the k-th Bell number. It follows that the sequence $(B_n)_{n\geq 0}$ is strictly log-convex. This result refines Engel's inequality $B_n^2 \leq B_{n-1}B_{n+1}$.

1 Introduction

A partition of a set S with n elements is a collection of nonempty, pairwise disjoint subsets whose union is equal to S. The Bell number B_n, named after the British mathematician Eric T. Bell, is the number of partitions of S. The first few numbers are

$$B_0 = 1, \quad B_1 = 1, \quad B_2 = 2, \quad B_3 = 5, \quad B_4 = 15, \quad B_5 = 52.$$

The Bell numbers are given by the exponential generating function

$$\exp(e^x - 1) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

and they satisfy the recurrence relation

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \quad (n \geq 0).$$

Moreover, the Bell numbers can be expressed in terms of the Stirling numbers of the second kind,

$$B_n = \sum_{k=1}^{n} S(n,k) \quad (n \geq 1).$$

The following remarkable series representation is due to Dobiński [6],

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!} \quad (n \geq 0). \tag{1}$$

In 1994, Engel [7] proved that the sequence $(B_n)_{n\geq 0}$ is log-convex, that is,

$$B_n^2 \leq B_{n-1} B_{n+1} \quad (n \geq 1). \tag{2}$$

Canfield [5], Asai, Kubo, and Kuo [1] and Bouroubi [3] published further proofs of (2). For more information and references on Bell numbers we refer to [4] and [8].

Richard E. Bellman (1920–1984), who was one of the leading mathematicians in the field of inequalities, pointed out that "every inequality should come from an equality which makes the inequality obvious" [2, p. 449]. In view of this statement it is natural to ask whether Engel's inequality is a consequence of an equality. It is the aim of this note to give an affirmative answer to this question. In particular, we prove that for all $n \geq 1$ strict inequality holds in (2). We show that an application of Dobiński's formula leads to the following result.

2 The main result

Theorem 1. *For all natural numbers $n \geq 2$ we have*

$$B_{n-1}B_{n+1} - B_n^2 = \frac{1}{2e^2} \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} \frac{j^{n-1}(k-j)^{n-1}}{j!(k-j)!} (k-2j)^2. \tag{3}$$

Proof. Applying (1) and the Cauchy product for infinite series we obtain for $n \geq 2$,

$$\begin{aligned} e^2\big(B_{n-1}B_{n+1} - B_n^2\big) &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{j^{n-1}(k-j)^{n+1}}{j!(k-j)!} - \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{j^n (k-j)^n}{j!(k-j)!} \\ &= \sum_{k=2}^{\infty} \big(L_n(k) - R_n(k)\big) \end{aligned} \tag{4}$$

with

$$L_n(k) = \sum_{j=1}^{k-1} \frac{j^{n-1}(k-j)^{n+1}}{j!(k-j)!} \quad \text{and} \quad R_n(k) = \sum_{j=1}^{k-1} \frac{j^n (k-j)^n}{j!(k-j)!}.$$

Since

$$\sum_{j=1}^{k-1} \frac{j^{n-1}(k-j)^{n+1}}{j!(k-j)!} = \sum_{j=1}^{k-1} \frac{j^{n+1}(k-j)^{n-1}}{j!(k-j)!},$$

we get

$$\begin{aligned} L_n(k) - R_n(k) &= \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{j!(k-j)!} \big(j^{n-1}(k-j)^{n+1} + j^{n+1}(k-j)^{n-1} - 2j^n(k-j)^n \big) \\ &= \frac{1}{2} \sum_{j=1}^{k-1} \frac{j^{n-1}(k-j)^{n-1}}{j!(k-j)!}(k-2j)^2. \end{aligned} \tag{5}$$

From (4) and (5) we conclude that (3) holds. □

Remark 2. Using $B_0 B_2 - B_1^2 = 1$ and (3) reveals that the sequence $(B_n)_{n \geq 0}$ is not only log-convex, but even strictly log-convex,

$$B_n^2 < B_{n-1} B_{n+1} \quad (n \geq 1). \tag{6}$$

Remark 3. Asai, Kubo, and Kuo [1] used Engel's inequality to prove that

$$B_m B_n \leq B_{m+n} \quad (m, n \geq 0).$$

An application of (6) gives for $m, n \geq 1$,

$$B_m = \prod_{\nu=1}^{m} \frac{B_\nu}{B_{\nu-1}} < \prod_{\nu=1}^{m} \frac{B_{n+\nu}}{B_{n+\nu-1}} = \frac{B_{m+n}}{B_n}.$$

Thus,

$$B_m B_n < B_{m+n} \quad (m, n \geq 1).$$

References

[1] N. Asai, I. Kubo, and H.-H. Kuo, Bell numbers, log-concavity, and log-convexity, *Acta Appl. Math.* **6**3 (2000), 79–87.

[2] R. Bellman, Why study inequalities?, in E. F. Beckenbach, ed., *General Inequalities 2*, Int. Ser. Numer. Math. **47**, Birkhäuser, 1980, p. 449.

[3] S. Bouroubi, Bell numbers and Engel's conjecture, *Rostock. Math. Kolloq.* **6**2 (2007), 61–70.

[4] D. Branson, Stirling numbers and Bell numbers: their role in combinatorics and probability, *Math. Scientist* **2**5 (2000), 1–31.

[5] E. R. Canfield, Engel's inequality for Bell numbers, *J. Combin. Th. Ser. A* **7**2 (1995), 184–187.

[6] G. Dobiński, Summierung der Reihe $\sum n^m/n!$ für $m = 1, 2, 3, 4, 5, \ldots$, *Archiv Math. Phys.* **6**1 (1877), 333–336.

[7] K. Engel, On the average rank of an element in a filter of the partition lattice, *J. Combin. Th. Ser. A* **6**5 (1994), 67–78.

[8] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, OEIS Foundation, Inc., 2019. Available at https://oeis.org.

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