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Enumeration of Border-Strip Decompositions and Weil-Petersson Volumes

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Abstract

We describe an injection from border-strip decompositions of certain diagrams to permutations. This allows us to provide enumeration results as well as q-analogues of enumeration formulas. Finally, we use this injection to prove a connection between the number of border-strip decompositions of the $n \times 2n$ rectangle and the Weil-Petersson volume of the moduli space of an n-punctured Riemann sphere.

1 Introduction and overview of results

Border-strip *tableaux* have a rich history, originating with the celebrated Murnaghan-Nakayama rule [5, 6], which provides a combinatorial formula for computing character values in the symmetric group. It is given as a signed sum over border-strip tableaux, but the sign only depends on the border-strip *decomposition*, i.e., the "unlabeled version" of the tableaux. This motivates us to study border-strip decompositions. Note that there is a hook-formula for enumerating border-strip tableaux given by Fomin and Lulov [2] but less study has been devoted to enumerating border-strip decompositions. Even determining if a region can be tiled by n-strips is non-trivial and sometimes NP-complete, see the papers by Beauquier et al., and Pak [1, 7].

We introduce a family of diagrams called *simple diagrams*, which have particularly nice properties with respect to enumeration. These diagrams are parameterized by a binary sequence and the strip size. In particular, we show that certain normalized enumerations grow as a polynomial in n — the size of the strips.

We show that border-strip tableaux and border-strip decompositions of simple diagrams are in bijection with certain classes of permutations, see Proposition 13 and Corollary 19. This allows us to study a certain q-analogue of border-strip decompositions which generalize the classical inversion-statistic on permutations. For example, in Corollary 28, we give the formula

$$\sum_{\mathbf{w}\in\{\mathbf{r},\mathbf{c}\}^k} \sum_{D\in \mathrm{BSD}(\mathbf{w},n)} q^{\mathrm{inv}(D)} = [n+1]_q^k [n]_q!$$

where the first sum is over all binary sequences of length k (defining a simple diagram), and $\mathrm{BSD}(\mathbf{w}, n)$ is the set of border-strip decompositions with strips of size n and diagram determined by $(\mathbf{w}, n)$.

In Proposition 21 we give an efficient way to compute the number of border-strip decompositions of simple diagrams, as a function of n. This allows us to prove that "straighter" simple diagrams admit a larger number of border-strip decompositions, see Theorem 29 and the maximum is attained for rectangles. In contrast, in Proposition 14 we show that whenever $n \geq k$, all these diagrams admit *the same number of border-strip tableaux*.

Finally, we give a new interpretation of the sequence A115047 in the OEIS. We show that these numbers count tilings of a $n \times 2n$-rectangle with strips of size n, thus providing a new simple combinatorial interpretation of certain Weil-Petersson volumes. We cannot give an intuitive explanation for this curious connection and it invites for further research.

2 Enumeration of border-strip decomposition

We first give simplified definitions of border strip tableaux and decompositions which are sufficient for the purpose of this article — for a thorough background see Richard Stanley's book [9].

2.1 Preliminaries

A *diagram* is formally defined as a set of integer coordinates

$$\{(i,j) \in \mathbb{N}^2 : \mu_i \leq j \leq \lambda_i\},$$

where λ and μ are integer partitions such that $\mu_i \leq \lambda_i$ for all $i = 1, 2, \ldots$. We refer the coordinates in the diagram as *boxes*. For example, $\lambda = (4, 3, 2, 2, 2)$, $\mu = (2, 1, 1)$ is illustrated as

		13	14
	22	23	
	32		
41	42		
51	52		

where ij is written in the box (i, j). A *border-strip* or *strip* for short of a diagram is a subset of boxes

$$(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$$

such that for every $i = 1, \ldots, n-1$,

$$a_i = a_{i+1} \text{ and } b_i - 1 = b_{i+1} \qquad \text{or} \qquad a_i + 1 = a_{i+1} \text{ and } b_i = b_{i+1}.$$

A *tableau* T is map $T : D \to \mathbb{N}$ from a diagram to the set of natural numbers. A tableau is illustrated by writing $T(i, j)$ in the box (i, j). A *border-strip tableau*[1] is a tableau T such that rows and columns are weakly increasing and for all i, the set of boxes $T^{-1}(i)$ is a border-strip. In this paper, we shall only consider border-strip tableaux such that for some m and n,

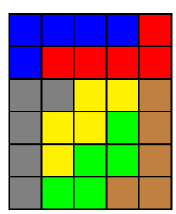
$$|T^{-1}(1)| = |T^{-1}(2)| = \cdots = |T^{-1}(m)| = n \text{ and } |T^{-1}(j)| = 0 \text{ for all } j > m.$$

Hence, each strip of T has size n and T is a border-strip tableau with strip size n. Let $\mathrm{BST}(D, n)$ denote the set of border-strip tableaux with strip size n and underlying diagram D. For our purposes, D could be any parameterization of a diagram, in our case we are going to use binary words — see Definition 2 below.

A *border-strip decomposition* with strip size n is a set-partition of a diagram into border-strips where all the border-strip are of size n. Analogously, $\mathrm{BSD}(D, n)$ denotes the set of border-strip decompositions of the diagram D. Observe that every border strip tableau T in $\mathrm{BST}(D, n)$ defines a border-strip decomposition in $\mathrm{BSD}(D, n)$, as we shall see in the following example.

Example 1. The following tableau is an element in $\mathrm{BST}(6 \times 5, 5)$, where 6×5 denotes a rectangle of dimension 6×5. The corresponding border-strip decomposition with the strips indicated by the colors is shown on the right.

1	1	1	1	2
1	2	2	2	2
3	3	4	4	6
3	4	4	5	6
3	4	5	5	6
3	5	5	6	6



(1)

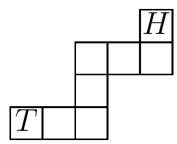
[1]Also known as rim-hook tableau.

It is straightforward to verify that for each border-strip decomposition there is at least one border-strip tableau, so we get $|\mathrm{BST}(D,n)| \geq |\mathrm{BSD}(D,n)|$ for any diagram D and natural number n. On the other hand, we obtain a border-strip decomposition from a border-strip tableau by "forgetting the numbers" so we have that $\mathrm{BST}(D,n) = \emptyset \iff \mathrm{BSD}(D,n) = \emptyset$.

Let B be a border-strip, which is a subset of boxes (i,j) (row-index, column-index). It is easy to verify that the set

$$\{j - i : (i,j) \in B\}$$

is of the form $a, a+1, \ldots, b$ for some $a < b$ where $b - a + 1$ is the number of boxes in B. Define the *head*, $\mathcal{H}(B)$, of a border-strip to be the unique box where $j - i$ is maximal, and its *tail*, $\mathcal{T}(B)$, to be the unique box which minimizes $j - i$. In (2), the head and tail boxes have been marked.



(2)

2.2 Simple diagrams and permutations

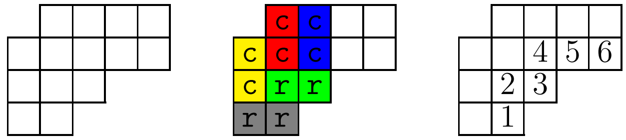
In this subsection, we introduce a natural family of diagrams with particularly nice properties. We first describe a bijection between border-strip decompositions of simple diagrams and certain permutations.

Definition 2. A *simple diagram* is parameterized by $(\mathbf{w}, n)$ where $\mathbf{w}$ is a finite sequence of elements in $\{\mathtt{r}, \mathtt{c}\}$ and n is a natural number. The family of simple diagrams is defined recursively as follows:

- If $\mathbf{w} = \emptyset$, then $(\mathbf{w}, n)$ is the $n \times n$-square.
- The diagram $(\mathtt{c}\mathbf{w}, n)$ is obtained from $(\mathbf{w}, n)$ by adding an additional column of size n on the left, such that the bottom-most square of the new column is in the bottom-most row of $(\mathbf{w}, n)$.
- The diagram $(\mathtt{r}\mathbf{w}, n)$ is obtained from $(\mathbf{w}, n)$ by adding an additional row of size n on the bottom such that the left-most square of the new row is in the leftmost column of $(\mathbf{w}, n)$.

Let $\mathrm{BSD}(\mathbf{w}, n)$ denote the set of border-strip decompositions of $(\mathbf{w}, n)$ with strips of size n, and $\mathrm{BST}(\mathbf{w}, n)$ denote the set of border-strip tableaux of $(\mathbf{w}, n)$ with strips of size n.

Example 3. The simple diagram determined by $(\mathtt{rcrcc}, 2)$ is the diagram on the left below. We can see how $(\mathtt{rcrcc}, 2)$ is constructed from the 2×2 square by adding successively the blue, red, green, yellow, and gray boxes to a 2×2 square.



(3)

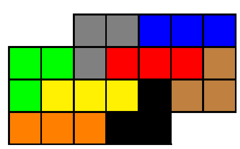
Given a simple diagram D, the shape $(\mathbf{w}, n)$ can be recovered as follows. First, n is given by the smallest number of boxes in a single row, or column. Secondly, k can be found by using the fact that $n^2 + kn$ is the total number of boxes in the diagram. We then start in the nth square in the bottom-most row of D and follow the outline of the diagram by going right if possible and up otherwise, until $k+1$ squares have been visited. The path in the example above is *up, right, up right right*, which correspond to the word `rcrcc`.

Definition 4. In a fixed border-strip decomposition, a border-strip B_a is *above* a border-strip B_b if there is a box a of B_a, and a box b of B_b such that the column index of a is smaller or equal to the column index of b, and the row index of a is smaller or equal to the row index of b. If B_a is above B_b, we say that B_b is *below* B_a.

A border-strip B_a is *inner* to a border-strip B_b if there exists a sequence $B_a = B_1, B_2, \dots, B_k = B_b$ such that B_i is above B_{i+1} for all $i = 1, 2, \dots, k-1$. It means the relation *inner* is the transitive closure of the relation *above*. If B_a is inner to B_b, then B_b is *outer* to B_a. Two border strips B_a and B_b are *comparable*, if B_a is inner or outer to B_b.

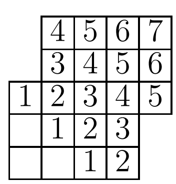
Remark 5. If B_1 is above B_2, it implies B_1 must contain a smaller number than B_2 in any border-strip tableau. This observation proves that the transitive closure of "above" is well-defined.

Example 6. An example of a border-strip decomposition in BSD(`ccrcc`, 3) is given in (4). The blue strip is above the red one and the red strip is above the yellow one. Hence the blue strip is inner to the yellow strip and the blue and yellow strips are comparable. Note that the blue strip is neither above nor below the yellow strip.



(4)

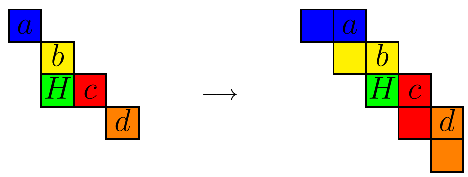
Definition 7 (Diagonal labeling)**.** For a sequence $\mathbf{w}$ of length k, we label the diagonals of the simple diagram $(\mathbf{w}, n)$ from $n+k$ to 1, starting from the top right corner as shown in the example below for (`crrc`, 3):



Lemma 8. *Let* $\mathbf{w}$ *be a sequence of length* k*. Then for any decomposition* $B \in \mathrm{BSD}(\mathbf{w}, n)$ *there is a unique head in each diagonal from* 1 *to* $n+k$ *and the positions of the heads uniquely determine* B*.*

Proof. We show that the positions of the heads uniquely determine the decomposition by processing the diagonals one by one and iteratively prolonging the strips, starting from diagonal $n + k$.

First, the single box in diagonal $n + k$ must be the head of some strip. For diagonal i with $k < i < n+k$ there is one box more in diagonal i compared to diagonal $i+1$. All strips we already encountered have fewer than n boxes and must continue in diagonal $i+1$. Hence, there is exactly one head in diagonal i. Moreover, the position of the head H in diagonal i determines the continuation of the strips encountered as shown in the following figure:



(5)

For $i \leq k$ there is exactly one strip ending in diagonal $i + 1$. Furthermore, diagonals i and $i + 1$ have the same size, therefore there must be exactly one head in diagonal i. Once the heads have been placed there are $n - 1$ boxes left in diagonal i and $n - 1$ strips which must have a box in diagonal i. Consequently there is at most one way to extend the strips to diagonal i.

The diagonals below diagonal 1 decrease in size by one in each step and it is easy to see that there cannot be any heads below diagonal 1. There is a unique way to extend the partial border-strip decomposition from diagonal 1 to the entire diagram. □

Given a border-strip decomposition of a simple diagram, the unique strip with head in diagonal i is referred to as *strip* i.

Proposition 9. *Let* $(\mathbf{w}, n)$ *be a simple diagram with* $\mathbf{w}$ *of length* k *and* $B \in \mathrm{BSD}(\mathbf{w}, n)$*. If* $1 \leq i < j \leq n + k$ *and* $j - i \leq n$ *then strip* i *and strip* j *in* B *are comparable.*

Proof. The tail of j is at most one diagonal higher than the head of i. From this observation, it follows that the strips must be comparable. □

We notice in Lemma 8 that the positions of the heads of the strips determine the border-strip decomposition uniquely. The next definition and proposition encodes the placements of the heads as a permutation with certain restrictions, giving an alternative description of *border-strip tableaux* of simple diagrams. Further down, we add additional restrictions so that the resulting set of permutations is in bijection with the set of *border-strip decompositions* of simple diagrams.

Definition 10. Define $\psi : \mathrm{BST}(\mathbf{w}, n) \to S_{n+k}$ so that $\psi(T) = \sigma$ where $\sigma(j)$ is the diagonal containing the head of the strip labeled with j. Let $\mathrm{BSP}(\mathbf{w}, n) \subseteq S_{n+k}$ denote the image of $\mathrm{BST}(\mathbf{w}, n)$ under ψ.

Example 11. Consider the following $T \in \mathrm{BST}(\mathtt{ccc}, n)$. The strip labeled 1 has its head in diagonal 3, thus $\psi(T)(1) = 3$. The strip labeled 2 has its head in diagonal 2, so $\psi(T)(2) = 2$.

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & \mathbf{1} & 3 & \mathbf{3} & \mathbf{4} \\ \hline 2 & 2 & \mathbf{2} & 3 & 4 & 4 \\ \hline 5 & 5 & \mathbf{5} & 6 & 6 & \mathbf{6} \\ \hline \end{array} \qquad \text{and} \qquad \psi(T) = [3, 2, 5, 6, 1, 4] \in S_6. \tag{6}$$

Proposition 12. *The map ψ is injective. In particular, $\psi : \mathrm{BST}(\mathbf{w}, n) \to \mathrm{BSP}(\mathbf{w}, n)$ is a bijection.*

Proof. A permutation $\sigma \in \mathrm{BSP}(\mathbf{w}, n)$ determines the values of the heads, and thus the set of values of all the boxes in each diagonal. As the entries in diagonals have to be increasing, σ uniquely determine the corresponding border-strip tableau. $\square$

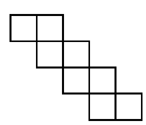
Proposition 13. *Let $\mathbf{w} = (w_1, \dots, w_k)$ be a sequence of length k. A permutation $\sigma \in S_{n+k}$ is in $\mathrm{BSP}(\mathbf{w}, n)$ if and only if for all i with $1 \le i \le k$ we have*

(a) $\sigma^{-1}(i) < \sigma^{-1}(n + i)$ whenever $w_i = \mathtt{c}$, and

(b) $\sigma^{-1}(i) > \sigma^{-1}(n + i)$ whenever $w_i = \mathtt{r}$.

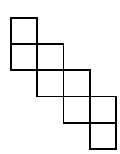
Proof. We construct the tableau T from $\sigma \in \mathrm{BSP}(\mathbf{w}, n)$ by starting from the last diagonal. By definition, the unique head in diagonal i must have value $\sigma^{-1}(i)$ in T. If $k < i \le n + k$ then diagonal i has one box more than diagonal $i + 1$ and it is always possible to extend the partial border-strip tableau one diagonal further in a unique fashion as in Lemma 8.

However, when $1 \le i \le k$ we need to take w_i into consideration. If $w_i = \mathtt{c}$ then diagonals i and $i + 1$ are arranged as follows.



We observe that the new strip starting in diagonal i must be *above* the strip ending in diagonal $i + 1$ (and starting in diagonal $n + i$). Hence, it must be filled with a smaller number in T, so we must have $\sigma^{-1}(i) < \sigma^{-1}(n + i)$.

If $w_i = \mathtt{r}$ diagonals i and $i + 1$ are arranged as follows.



Analogously to the previous argument, the new strip starting in diagonal i must be *below* strip $n + i$ and thus $\sigma^{-1}(i) > \sigma^{-1}(n + i)$. $\square$

2.3 Counting border-strip tableaux and decompositions

We now have the necessary setup to obtain some enumerative results.

Proposition 14. *For a sequence* $\mathbf{w}$ *of length* $k \leq n$*,*

$$|\mathrm{BST}(\mathbf{w}, n)| = 2^{-k}(n+k)!.$$

In particular, the number of border-strip tableaux only depends on the length of the sequence for $n \geq k$*.*

Proof. Recall from Proposition 12 that $|\mathrm{BST}(\mathbf{w}, n)| = |\mathrm{BSP}(\mathbf{w}, n)|$. From conditions (a) and (b) in Proposition 13 we know that the $\mathbf{w}$ determines the relative order of the strips i and $n+i$ for every $i = 1, 2, \ldots, k$. Whenever $n \geq k$, these pairs are disjoint. As there are no further restrictions on the $(n+k)!$ possible permutations, $|\mathrm{BSP}(\mathbf{w}, n)| = 2^{-k}(n+k)!$. □

Proposition 14 cannot be strengthened to the case $k > n$, as the count $|\mathrm{BST}(\mathbf{w}, n)|$ in this case might depend on $\mathbf{w}$.

Corollary 15. *For every permutation* $\sigma \in S_{n+k}$ *there is exactly one sequence* $\mathbf{w}$ *of length* k *such that* $\sigma \in \mathrm{BSP}(\mathbf{w}, n)$*. In particular,* ψ *determines a bijection between* $\{\mathrm{BST}(\mathbf{w}, n) : \mathbf{w} \in \{\mathtt{r}, \mathtt{c}\}^k\}$ *and* S_{n+k}*. Consequently,*

$$\sum_{\mathbf{w} \in \{\mathtt{r}, \mathtt{c}\}^k} |\mathrm{BST}(\mathbf{w}, n)| = (n+k)!.$$

Proof. From Proposition 12 we know ψ restricted to one sequence $\mathbf{w}$ is injective. Proposition 13 shows that a permutation uniquely determines the sequence $\mathbf{w}$, ψ must be injective over the set of all sequences of length k.

On the other hand for any permutation $\sigma \in S_{n+k}$ there is always one sequence $\mathbf{w} \in \{\mathtt{r}, \mathtt{c}\}^k$ such that $\sigma \in \mathrm{BSP}(\mathbf{w}, n)$ as we can find appropriate values for w_i from $\sigma^{-1}(i)$ and $\sigma^{-1}(n+i)$ together with Proposition 13. Hence ψ is also surjective. □

Definition 16 (k-descents)**.** Let $\sigma \in S_n$. We say that $i \in \{1, \ldots, n-1\}$ is a *k-descent* of σ if $\sigma(i) - k > \sigma(i+1)$. Let $\mathrm{DES}_k(\sigma)$ denote the set of k-descents of σ and let $\mathrm{des}_k(\sigma)$ be the number of such k-descents. For example, the 3-descents of the permutation

$$[2, 4, \mathbf{10}, 5, 6, 3, \mathbf{8}, 1, 7, 9]$$

are 3 and 7 (marked in bold).

Lemma 17 (Technical lemma)**.** *Let* $T = \psi^{-1}(\sigma)$ *for some* $\sigma \in \mathrm{BSP}(\mathbf{w}, n)$*. The following three statements are equivalent:*

(a) The permutation σ *has an* n*-descent.*

(b) There is an m such that the strips $B_m \coloneqq T^{-1}(m)$ and $B_{m+1} \coloneqq T^{-1}(m+1)$ are not comparable in the sense of Definition 4 and $\sigma(m) > \sigma(m+1)$.

(c) There are two strips $B_x \coloneqq T^{-1}(x)$ and $B_y \coloneqq T^{-1}(y)$ in T such that B_x is not comparable to B_y in the sense of Definition 4, where $x < y$ and $\sigma(x) > \sigma(y)$.

Proof. Recall that $\sigma(x)$ is simply the diagonal containing the head of B_x.

Case $(a) \implies (b)$. We have that $\sigma(m)-\sigma(m+1) > n$ for some m. Since $\sigma(m) > \sigma(m+1)+n$, we cannot have that B_m is adjacent to B_{m+1}. Furthermore, there cannot be a longer sequence of strips $B_m = B_{i_1}, \dots, B_{i_\ell} = B_{m+1}$ such that B_{i_j} is above $B_{i_{j+1}}$ as the values of these strips must then lie strictly between m and $m+1$, which is impossible. Hence, B_m is not inner to B_{m+1} and this implies that they are not comparable.

Case $(b) \implies (c)$. Just take $x = m$ and $y = m+1$.

Case $(c) \implies (a)$. Consider the interval of values $x, x+1, \dots, y$ and choose $x \leq m < y$ such that $\sigma(m) - \sigma(m+1)$ is maximized. Suppose $\sigma(m) - \sigma(m+1) \leq n$. Then there is a sequence of values $i_1, \dots, i_s$ such that

$$x = i_1 < i_2 < \cdots < i_s = y, \qquad \sigma(x) = \sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_s) = \sigma(y),$$

and for all j we have $\sigma(i_j) - \sigma(i_{j+1}) \leq n$. But Proposition 9 implies that the strip with value i_j is above the strip with value i_{j+1}. By transitivity, B_x must be inner to B_y which violates one of the conditions. It follows that $\sigma(m) - \sigma(m+1) > n$ and σ has an n-descent. □

We are now ready to characterize a subset of $\mathrm{BSP}(\mathbf{w}, n)$ which is in bijection with the border-strip decomposition $\mathrm{BSD}(\mathbf{w}, n)$. Let $s_1, \dots, s_{n-1}$ denote the simple transpositions in S_n.

Proposition 18. *Suppose $\mathbf{w}$ is a sequence of length k, $\sigma \in \mathrm{BSP}(\mathbf{w}, n)$, and $i \in \mathrm{DES}_n(\sigma)$. Then the border-strip tableaux $\psi^{-1}(s_i\sigma)$ and $\psi^{-1}(\sigma)$ have the same border-strip decomposition. Moreover, the sets*

$$\mathrm{BSD}(\mathbf{w}, n) \quad \textit{and} \quad \{\sigma \in \mathrm{BSP}(\mathbf{w}, n) : \mathrm{des}_n(\sigma) = 0\}$$

are in bijection.

Proof. Let $\tau \coloneqq s_i\sigma$, $T_\sigma \coloneqq \psi^{-1}(\sigma)$, and $T_\tau \coloneqq \psi^{-1}(\tau)$ be the corresponding border-strip tableaux. First we need to show that $\tau \in \mathrm{BSP}(\mathbf{w}, n)$. The only places where τ^{-1} differ from σ^{-1} are $\tau(i)$ and $\tau(i+1)$. Now, if $j \in \{1, \dots, k\}$ then at most one of $\sigma^{-1}(j)$ and $\sigma^{-1}(n+j)$ is different from τ^{-1} and the quantities

$$\tau^{-1}(j) - \tau^{-1}(n+j) \text{ and } \sigma^{-1}(j) - \sigma^{-1}(n+j) \tag{7}$$

are either the same or differ by 1. In particular, the quantities in (7) cannot have opposite signs. Since $\sigma \in \mathrm{BSP}(\mathbf{w}, n)$, the corresponding inequalities in Proposition 13 are also fulfilled by τ. Hence $\tau \in \mathrm{BSP}(\mathbf{w}, n)$ as well.

It remains to show that T_τ and T_σ have the same border-strip decomposition. The only strips in T_σ and T_τ which have a different number are strip $\tau(i)$ and strip $\tau(i+1)$, with a ± 1 difference. Therefore, the only pair of strips that has a different relative ordering in T_τ is the pair $(\tau(i), \tau(i+1))$. However, since i is an n-descent it does not affect the construction in the proof of Proposition 13 and it follows that T_σ and T_τ have the same border-strip decomposition.

For the second statement, we use Lemma 17. In order for σ to be free of n-descents, the relative order of the values of all pairs of *non-comparable* strips is uniquely determined. On the other hand, the relative order of pairs of *comparable strips* is determined by the shape of the diagram. Since a permutation is uniquely determined by the relative order of all pairs of entries, there is at most one permutation without n-descents in $\mathrm{BSP}(\mathbf{w}, n)$ with a given border-strip decomposition.

On the other hand, we can always find such a σ by starting from any permutation in $\mathrm{BSP}(\mathbf{w}, n)$ and repeatedly decrease the number of n-descents until a permutation without n-descents is obtained. This is done by interchanging the values in the non-comparable strips B_m and B_{m+1} in Lemma 17 (b). □

Corollary 19. *Let $\mathbf{w} \in \{\mathtt{c}, \mathtt{r}\}^k$. The set of border-strip decompositions of the simple diagram $(\mathbf{w}, n)$ is in bijection with the set of permutations in S_{n+k} such that for each $i \in [k]$*

- $w_i = \mathtt{c} \quad \Longrightarrow \quad \sigma^{-1}(i) < \sigma^{-1}(n+i)$,
- $w_i = \mathtt{r} \quad \Longrightarrow \quad \sigma^{-1}(i) > \sigma^{-1}(n+i)$ *and*
- $\sigma(j) - \sigma(j+1) \leq n$ *for all* $j \in [n+k-1]$.

Let $\mathrm{BSP}^*(\mathbf{w}, n)$ be the subset of permutations in S_{n+k} that fulfill the three conditions in Corollary 19. Hence,

$$\mathrm{BSP}^*(\mathbf{w}, n) = \{\sigma \in \mathrm{BSP}(\mathbf{w}, n) : \mathrm{des}_n(\sigma) = 0\}.$$

Note that ψ^{-1} gives a bijection from $\mathrm{BSP}^*(\mathbf{w}, n)$ to $\mathrm{BSD}(\mathbf{w}, n)$.

Definition 20. For a sequence $\mathbf{w} \in \{\mathtt{r}, \mathtt{c}\}^k$ let

$$\hat{f}_{\mathbf{w}}(n) := |\mathrm{BSD}(\mathbf{w}, n)| \frac{(2k)!}{(n-k)!}. \tag{8}$$

Proposition 21. *Whenever $n \geq 2k$, the function $\hat{f}_{\mathbf{w}}(n)$ is equal to*

$$f_{\mathbf{w}}(n) := \sum_{\tau \in \mathrm{BSP}(\mathbf{w}, k)} (n + k - \mathrm{des}_k(\tau))_{2k}. \tag{9}$$

As a consequence, $\hat{f}_{\mathbf{w}}(n)$ is a polynomial in n of degree $2k$ with integer coefficients when restricted to values $n \geq 2k$. Moreover, $f_{\mathbf{w}}(n)$ is divisible by the falling factorial $(n+1)_{k+1}$.

Proof. We start by assuming that $n \geq 2k$. Interpreting permutations in S_{n+k} as sequences of $n+k$ numbers, we note that the first two conditions in Corollary 19 only apply to the relative order of the first and last k elements. Thus, in order to construct a permutation σ in S_{n+k} satisfying the three conditions in Corollary 19, we proceed in three steps.

1. Choose an ordering of the entries $\{1, 2, \dots, k\} \cup \{n+1, n+2, \dots, n+k\}$.

2. Choose the positions of the entries $\{1, 2, \dots, k\} \cup \{n+1, n+2, \dots, n+k\}$.

3. Choose an ordering of the entries $\{k+1, \dots, n\}$.

Note that as $n \geq 2k$, the unions in the first two steps above are disjoint. Not every choice here will fulfill the conditions in Corollary 19 — we shall see below which ones are valid. For a choice in the first step, two things might happen.

a) There is some pair $(i, i+n)$ in the wrong order — violating one of the first two conditions. In this case we do not have a border-strip tableau, and thus no border-strip composition for this choice.

b) All pairs $(i, i+n)$ have the correct order. In this case, the ordering of the $2k$ entries

$$\{1, 2, \dots, k\} \cup \{n+1, n+2, \dots, n+k\}$$

fulfills the conditions (after standardization!) of being a permutation $\tau \in \mathrm{BSP}(\mathbf{w}, k)$. Notice the use of k and not n here as strip-size.

Now we need to ensure that there are no n-descents in the final permutation. If there are no k-descents in τ in step $b)$ above, then there are no n-descents either. Otherwise, we need to insert another number after every k-descent of τ. This means we only have $\binom{n+k-\mathrm{des}_k(\tau)}{2k}$ valid choices in step (2). The last step always has $(n-k)!$ valid choices as the order on $\{k+1, \dots, n\}$ does not matter. It follows that whenever $n \geq 2k$, the function $\hat{f}_{\mathbf{w}}(n)$ is given by

$$\begin{aligned}
\hat{f}_{\mathbf{w}}(n) &= \frac{(2k)!}{(n-k)!} \sum_{\tau \in \mathrm{BSP}(\mathbf{w},k)} \binom{n+k-\mathrm{des}_k(\tau)}{2k} (n-k)! \\
&= \sum_{\tau \in \mathrm{BSP}(\mathbf{w},k)} (n+k-\mathrm{des}_k(\tau))_{2k}.
\end{aligned}$$

This function is obviously a polynomial of degree $2k$. Furthermore, since $\mathrm{des}_k(\tau)$ is between 0 and $k-1$ it follows that $(n+k-\mathrm{des}_k(\tau))_{2k}$ is divisible by $(n+1)_{k+1}$. $\square$

Corollary 22. *We have the enumeration*

$$|\mathrm{BSD}(\mathtt{rc}, n)| = (n+1)!(3n+2)/12 \textit{ whenever } n \geq 2.$$

Proof. Using Proposition 21, we know that $|\mathrm{BSD}(\mathtt{rc}, n)|$ can be expressed as $(n-2)!\hat{f}_{\mathtt{rc}}(n)/4!$. Since we know that $\hat{f}_{\mathtt{rc}}(n)$ is a polynomial in n for $n \geq 4$, it suffices to verify the formula for the first few values of n. $\square$

The sequence $a_{n+1} = (n+1)!(3n+2)/12$ appears A227404 in the OEIS, where a_n count the total number of inversions in all permutations in S_n consisting of a single cycle. For example, the permutations (123) and (132) have four inversions in total, giving $a_3 = 4$. We have not managed to find an explicit bijection that shows this correspondence.

2.4 Inversions and q-analogues

Definition 23 (Inversions)**.** Two border strips B_1 and B_2 in a decomposition form an *inversion* if and only if the following three conditions are fulfilled:

- The intersection
$$\{j - i : (i, j) \in B_1\} \cap \{j - i : (i, j) \in B_2\}$$
is non-empty. Equivalently, there is a diagonal as in Definition 7 that intersects both B_1 and B_2.
- B_1 is inner to B_2, and
- $\mathcal{H}(B_1) > \mathcal{H}(B_2)$.

We prove in Corollary 25 that this definition generalizes the notion of inversions of permutations in a natural manner.

Lemma 24. *Let $\sigma \in \mathrm{BSP}^*(\mathbf{w}, n)$, with $B_\sigma \coloneqq \psi^{-1}(\sigma)$ being the corresponding border-strip decomposition. Then the strips i and j in B_σ with $i < j$ form an inversion if and only if $j - i < n$ and $\sigma^{-1}(i) > \sigma^{-1}(j)$.*

Proof. If $j - i \geq n$ they do not have an element on the same diagonal, and by definition do not form an inversion. If $j - i < n$ they share an element on the same diagonal and if $\sigma^{-1}(i) > \sigma^{-1}(j)$ then strip j is above strip i and they form an inversion. $\square$

Given a border-strip decomposition B, let $\mathrm{inv}(B)$ denote the total number of inversions in B. Furthermore, for $\sigma \in S_{n+k}$ let

$$\mathrm{inv}_n(\sigma) \coloneqq \{(i, j) : 0 < j - i < n \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$

The q-analogue of $\mathrm{BSD}(\mathbf{w}, n)$ is defined as

$$|\mathrm{BSD}(\mathbf{w}, n)|_q \coloneqq \sum_{B \in \mathrm{BSD}(\mathbf{w}, n)} q^{\mathrm{inv}(B)} \tag{10}$$

and by Lemma 24 we have that

$$|\mathrm{BSD}(\mathbf{w}, n)|_q = \sum_{\sigma \in \mathrm{BSP}^*(\mathbf{w}, n)} q^{\mathrm{inv}_n(\sigma)}. \tag{11}$$

Corollary 25. *The q-analogue of the $n \times n$-square, $|\mathrm{BSD}(\emptyset, n)|_q$, satisfies the identity*

$$\sum_{B \in \mathrm{BSD}(\emptyset,n)} q^{\mathrm{inv}(B)} = [n]_q!.$$

Proof. From Proposition 13 we know every permutation in S_n is in $\mathrm{BSP}^*(\emptyset, n)$. From Proposition 18, we know every border-strip tableau correspond to a unique border-strip decomposition. From Lemma 24 we can then deduce that the q-analogue is given by $[n]_q!$. $\square$

Corollary 26. *We have the following q-analogue for* $\mathrm{BSD}(\mathtt{c}, n)$*:*

$$\sum_{B \in \mathrm{BSD}(\mathtt{c},n)} q^{\mathrm{inv}(B)} = [n-1]_q! \sum_{i=1}^{n} i q^{i-1}.$$

Proof. We get a permutation corresponding to a decomposition by placing 1 and $n+1$ (i.e., choose $\sigma^{-1}(1)$ and $\sigma^{-1}(n+1)$), and then choose the order of $2, \dots, n$. This choice gives $[n-1]_q!$ and the possible positions of 1 and $n+1$ give $\sum_{i=1}^{n} i q^{i-1}$, as 1 has to be before $n+1$ for it to be a border-strip tableau. Note that there cannot be any n-descents and therefore the number of border-strip tableaux is equal to the number of decompositions. $\square$

Proposition 27. *If* $\mathbf{w}$ *is a sequence of a simple diagram, then*

$$|\mathrm{BSD}(\mathtt{c}\mathbf{w}, n)| + |\mathrm{BSD}(\mathtt{r}\mathbf{w}, n)| = (n+1)|\mathrm{BSD}(\mathbf{w})|.$$

Furthermore, this relation extends to the following q-analogue:

$$|\mathrm{BSD}(\mathtt{c}\mathbf{w}, n)|_q + |\mathrm{BSD}(\mathtt{r}\mathbf{w}, n)|_q = [n+1]_q|\mathrm{BSD}(\mathbf{w})|_q.$$

Proof. If we fix the positions of the heads in $(\mathbf{w}, n)$, the new head in $(\mathtt{c}\mathbf{w}, n)$ must be above the strip it replaces, where as in $(\mathtt{r}\mathbf{w}, n)$ it must be below. Together, this gives $n+1$ choices to complete the partial decomposition to a BSD of $(\mathbf{w}, n)$. If, in $(\mathtt{c}\mathbf{w}, n)$ or in $(\mathtt{r}\mathbf{w}, n)$, we place the new head in position i of the diagonal, the new strip forms an inversion with all $i-1$ strips above it. This proves the q-analogue. $\square$

Corollary 28. *We can enumerate the total number of border-strip decompositions for all sequences of length k:*

$$\sum_{\mathbf{w} \in \{\mathtt{r},\mathtt{c}\}^k} |\mathrm{BSD}(\mathbf{w}, n)| = (n+1)^k n!$$

and this relation extends to the q-analogue

$$\sum_{\mathbf{w} \in \{\mathtt{r},\mathtt{c}\}^k} \sum_{B \in \mathrm{BSD}(\mathbf{w},n)} q^{\mathrm{inv}(B)} = [n+1]_q^k [n]_q!. \tag{12}$$

Proof. It suffices to show the q-analogue. We proceed by induction over k. The base case $k = 0$ is given by Corollary 25. We then have

$$\begin{aligned}
\sum_{\mathbf{w}\in\{\mathtt{r},\mathtt{c}\}^k} |\mathrm{BSD}(\mathbf{w},n)|_q &= \sum_{\mathbf{w}\in\{\mathtt{r},\mathtt{c}\}^{k-1}} |\mathrm{BSD}(\mathtt{r}\mathbf{w},n)|_q + \sum_{\mathbf{w}\in\{\mathtt{r},\mathtt{c}\}^{k-1}} |\mathrm{BSD}(\mathtt{c}\mathbf{w},n)|_q \\
&= [n+1]_q \sum_{\mathbf{w}\in\{\mathtt{r},\mathtt{c}\}^{k-1}} |\mathrm{BSD}(\mathbf{w},n)|_q \\
&= [n+1]_q \left([n+1]_q^{k-1}[n]_q!\right) \\
&= [n+1]_q^k [n]_q!
\end{aligned}$$

where the second equality is due to Proposition 27, and the third equality is our induction hypothesis. This completes the proof. □

For $n = k-1$, (12) gives the sequence $a(n) = (n+1)^{n-1}n!$, which is A066319. This sequence also show up in the work of Weist [10, Thm. 5.4]. Let $K_{n,n+1}$ be the complete bipartite graph with n sources and $n+1$ sinks. There are $a(n)$ spanning trees such that every source has exactly 2 incident edges. As $a(n)$ is also related to computing the Euler characteristic of certain moduli spaces, we ask if it perhaps is related to what we discuss in Section 3 below.

For a sequence $\mathbf{w}$, let $C_{\mathbf{w}}$ be the total number of $\mathtt{c}$'s in $\mathbf{w}$, $R_{\mathbf{w}}$ be the total number of $\mathtt{r}$'s in $\mathbf{w}$ and $\mathrm{hor}(\mathbf{w}) := C_{\mathbf{w}} - R_{\mathbf{w}}$. For example,

$$C_{\mathtt{rcrcc}} = 3, \quad R_{\mathtt{rcrcc}} = 1 \text{ and } \mathrm{hor}(\mathtt{rcrcc}) = 3 - 2 = 1.$$

Intuitively, $\mathrm{hor}(\mathbf{w})$ measures how "horizontal" the diagram is, and $|\mathrm{hor}(\mathbf{w})|$ is smaller for "straigher" shapes. The following theorem shows that straighter diagrams admit a larger number of decompositions, in a precise sense:

Theorem 29. *If* $\mathbf{v}$ *and* $\mathbf{w}$ *are sequences of length* k *and* $|\mathrm{hor}(\mathbf{v})| < |\mathrm{hor}(\mathbf{w})|$, *then*

$$|\mathrm{BSD}(\mathbf{v},n)| < |\mathrm{BSD}(\mathbf{w},n)| \textit{ for } n \textit{ sufficiently large.}$$

In fact,

$$\frac{|\mathrm{BSD}(\mathbf{w},n)| - |\mathrm{BSD}(\mathbf{v},n)|}{(n-k)!} = O(n^{2k-1}).$$

Proof. From Proposition 21, we know that for sufficiently large n, we have that

$$\hat{f}_{\mathbf{v}}(n) := |\mathrm{BSD}(\mathbf{w},n)|\frac{(2k)!}{(n-k)!} = \sum_{\sigma\in\mathrm{BSP}(\mathbf{v},k)} (n+k-\mathrm{des}_k(\sigma))_{2k}.$$

From Proposition 14 we know that $|\mathrm{BSP}(\mathbf{v},k)| = (2k)!/2^{2k}$. It then follows that

$$\hat{f}_{\mathbf{v}}(n) = \frac{(2k)!}{2^k}n^{2k} + \alpha n^{2k-1} + \text{l.o.t and } \hat{f}_{\mathbf{w}}(n) = \frac{(2k)!}{2^k}n^{2k} + \beta n^{2k-1} + \text{l.o.t.}$$

Our goal is to prove that $\alpha < \beta$. For a fixed permutation $\sigma \in \mathrm{BSP}(\mathbf{v}, k)$, its contribution to α is given by

$$\sum_{i=0}^{2k-1}(k - \mathrm{des}_k(\sigma) - i) = 2k^2 - k(2k-1) - 2k\,\mathrm{des}_k(\sigma) = k - 2k\,\mathrm{des}_k(\sigma).$$

Hence, by summing over all permutations,

$$\alpha = k|\mathrm{BSP}(\mathbf{v}, k)| - 2k \sum_{\sigma \in \mathrm{BSP}(\mathbf{v},k)} \mathrm{des}_k(\sigma).$$

As $|\mathrm{BSP}(\mathbf{v}, k)|$ does not depend on $\mathbf{v}$, the only part depending on $\mathbf{v}$ is

$$J_{\mathbf{v}} := \sum_{\sigma \in \mathrm{BSP}(\mathbf{v},k)} \mathrm{des}_k(\sigma),$$

and it suffices to prove that $J_{\mathbf{v}}$ increases as $|\operatorname{hor}(\mathbf{v})|$ decreases.

To do this, we count the number of permutations where $b+k$ is a $k-$descent with a, for $1 \leq a < b \leq k$ fixed (i.e., we have $\ldots, b+k, a, \ldots$ in the permutation). To create such a permutation, we can choose the order of all elements different from $a, b, a+k, b+k$ in any way respecting the orders of pairs $\sigma(i), \sigma(i+k)$, which gives $(2k-4)!/2^{k-2}$ choices. Then we must choose the order of the three blocks $a+k, (b+k)a, b$. If $a+k$ and b are on the same side of $(b+k)a$, this gives two possibilities, otherwise there is only one way. We observe $a+k$ and b are on the same side if and only if $\mathbf{v}_a \neq \mathbf{v}_b$. Finally, we can chose the position of the three blocks $a+k, b, (b+k)a$, which gives $\binom{2k}{3}$ choices. So the number of permutations where $b+k$ is a k-descent with a is exactly

$$\begin{cases} 2\binom{2k}{3}(2k-4)!/2^{k-2}, & \text{if } \mathbf{v}_a \neq \mathbf{v}_b; \\ \binom{2k}{3}(2k-4)!/2^{k-2}, & \text{otherwise.} \end{cases}$$

Recall $C_{\mathbf{v}}$ is the number of $\mathtt{c}$'s in $\mathbf{v}$ and $R_{\mathbf{v}}$ is the number of $\mathtt{r}$'s in $\mathbf{v}$. The previous argument implies

$$J_{\mathbf{v}} = \binom{2k}{3}\frac{(2k-4)!}{2^{k-2}}\left[\binom{C_{\mathbf{v}}}{2} + 2C_{\mathbf{v}}R_{\mathbf{v}} + \binom{R_{\mathbf{v}}}{2}\right].$$

Since $R_{\mathbf{v}} = k - C_{\mathbf{v}}$ it follows that

$$\binom{C_{\mathbf{v}}}{2} + 2C_{\mathbf{v}}R_{\mathbf{v}} + \binom{R_{\mathbf{v}}}{2} = \binom{k}{2} + C_{\mathbf{v}}R_{\mathbf{v}}$$

which increases as $|\operatorname{hor}(\mathbf{v})|$ decreases. This implies the theorem. □

Conjecture 30. The function $f_{\mathbf{w}}(n)$ uniquely determines $\mathbf{w}$ up to isometry of the diagram $\mathbf{w}$, i.e., up to exchanging $\mathtt{r}$ and $\mathtt{c}$ and reversing the sequence.

Note that for fixed k, the polynomials (in n) $\{(n+k-i)_{2k}\}_{i=0}^{k-1}$ are linearly independent. They span the same space as

$$\left\{\frac{(n+i)_{2k}}{(2k)!}\right\}_{i=1}^{k} \quad = \quad \left\{\binom{n+i}{2k}\right\}_{i=1}^{k}$$

and the latter collection of polynomials can be seen to be linearly independent.

As a consequence, given $f_{\mathbf{w}}(n)$, which is a sum over permutations in $\mathrm{BSP}(\mathbf{w},k)$, for any i we can extract the number of permutations $\sigma \in \mathrm{BSP}(\mathbf{w},k)$ with $\mathrm{des}_k(\sigma) = i$. Hence, the conjecture is reduced to determining if the multi-set of des_k-values of the elements in $\mathrm{BSP}(\mathbf{w},k)$ uniquely determines $\mathbf{w}$ up to isometry.

In particular, if the number of terms without k-descents is different, the polynomial is also different, so we can formulate the stronger conjecture that $|\mathrm{BSD}(\mathbf{w},k)|$ uniquely determines a sequence $\mathbf{w}$ of length k up to isometry.

3 A connection with the Weil-Petersson volume

We let c^n denote the simple shape which is a rectangle of width $2n$ and height n. It follows from Corollary 19 that the set $\mathrm{BSD}(\mathsf{c}^n,n)$ is in bijection with the set of permutations of $\{x_1,\dots,x_n,y_1,\dots,y_n\}$ such that x_i appears before y_i for all i, and we do not have $\dots,x_i,y_j,\dots$ (consecutive), such that $i>j$.

Lemma 31 (Adaptation of Xiang [11]). *The cardinality of* $\mathrm{BSD}(\mathsf{c}^n,n)$ *is given by the formula*

$$|\mathrm{BSD}(\mathsf{c}^n,n)| = \sum_{p\vdash n}(-1)^{|p-1|}\frac{1}{m!}\binom{|p|}{p}\binom{|p+1|}{p+1}, \tag{13}$$

where $m=(m_1,m_2,\dots,m_k)$ *and* m_i *is the multiplicity of* i *in* p, *and we use the notation* $p\pm 1 := (p_1\pm 1,\dots,p_k\pm 1)$ *and* $|p| = p_1+\dots+p_k$. *Note that* $\binom{|p|}{p}$ *and* $\binom{|p+1|}{p+1}$ *are multinomial coefficients.*

Proof. For a permutation $\sigma\in S_{2n}$ corresponding to a *border-strip tableau*, let Γ_σ be the graph on the vertex set $[n]$ with edge set

$$\{(\sigma(i)-n,\sigma(i+1)) : \sigma(i)-n > \sigma(i+1)\}.$$

Let G be the set of graphs obtained from such border-strip tableaux. Let E be $\binom{[n]}{2}$, that is, the set of all possible edges on the vertex set $[n]$ and let $G(e_1,\dots,e_r)\subseteq G$ be the set of graphs that include the edges $\{e_1,\dots,e_r\}\subseteq E$. We have that elements in $\mathrm{BSP}^*(\mathsf{c}^n,n)$ are in bijection with $G(\emptyset)$, and Proposition 18 tells us that

$$|\mathrm{BSD}(\mathsf{c}^n,n)| = G\setminus\left(\bigcup_{r=1}^{n}\bigcup_{e_1,\dots,e_r\in E} G(e_1,\dots,e_r)\right).$$

Using the inclusion-exclusion principle, it follows that

$$|\mathrm{BSD}(\mathsf{c}^n, n)| = |\mathrm{BST}(\mathsf{c}^n, n)| - \sum_{e_1 \in E} |G(e_1)| + \sum_{e_1, e_2 \in E} |G(e_1, e_2)| - \cdots .$$

We then observe that these graphs are characterized by the connected components induced by the forced edges $e_1, \ldots, e_r$, determining an integer partition p of n. Furthermore, the sign in the above formula only depends on the number of forced edges, which is equal to $|p - 1|$, so we can transform this into a sum over all partitions of n. Given a partition $p \vdash n$, the number of graphs with component sizes $p_1, p_2, \ldots, p_k$ is given by $\frac{1}{m!}\binom{|p|}{p}$, with m given as above.

Claim: Let $e_1, e_2, \ldots, e_r$ be fixed edges such that the component sizes are given by p. Then

$$G(e_1, \ldots, e_r) = \binom{|p+1|}{p+1}.$$

Proof: Suppose $\Gamma_\sigma \in G(e_1, \ldots, e_r)$ has a component $(i_1, i_2, \ldots, i_j)$, in increasing order. From Proposition 13, we know $\sigma^{-1}(i_s) < \sigma^{-1}(i_s + n)$ for all $1 \leq s \leq j$, and for $(i_1, i_2, \ldots, i_j)$ to be connected, we need $i_s + n$ to form an n-descent with i_{s-1} for all $1 < s \leq j$ i.e., $\sigma^{-1}(i_s + n) = \sigma^{-1}(i_{s-1}) - 1$.

Together these two statements imply that σ, has the following structure:

$$\begin{aligned} \sigma^{-1}(i_j) < \sigma^{-1}(i_j + n) \lessdot \sigma^{-1}(i_{j-1}) < \sigma^{-1}(i_{j-1} + n) \lessdot \cdots \\ \cdots < \sigma^{-1}(i_3 + n) \lessdot \sigma^{-1}(i_2) < \sigma^{-1}(i_2 + n) \lessdot \sigma^{-1}(i_1) < \sigma^{-1}(i_1 + n) \end{aligned}$$

where $a \lessdot b$ means that $a + 1 = b$. Thus, we have $j + 1$ *blocks*,

$$[\sigma^{-1}(i_j)],\ [\sigma^{-1}(i_{s+1} + n) \lessdot \sigma^{-1}(i_s)] \text{ for } s = 1, \ldots, j-1 \text{ and } [\sigma^{-1}(i_1 + n)]$$

which need to appear in order, but there is no further restriction. The number of Γ_σ with component sizes determined by p is therefore $\binom{|p+1|}{p+1}$, which concludes the proof. ☐

Let us now dive into a completely different part of mathematics. We shall not use the following definition and we omit many details. The important part is Theorem 32 further down.

The *Weil-Petersson volume*, $\mathrm{Vol}_{WP}(\cdot)$, is defined as

$$\mathrm{Vol}_{WP}(M) := \frac{1}{(n-3)!} \int_M \wedge^{n-3}(\omega_M),$$

where ωM is the Weil-Petersson symplectic form on the manifold M. A special type of manifold is $M_{0,n}$, which denotes the moduli space of an n-punctured Riemann sphere, that is

$$M_{0,n} := \{(z_1, \ldots, z_n) \in \hat{\mathbb{C}}^n \ : z_i \neq z_j\} / S_n \times \mathrm{PSL}(2, \mathbb{C})$$

and S_n acts by permuting variables, and $\mathrm{PSL}(2, \mathbb{C})$ acts as a linear fractional transformation.

Theorem 32 (Zograf [12]). *The Weil-Petersson volume of the moduli space of an n-punctured Riemann sphere $M_{0,n}$ is given by the formula*

$$\mathrm{Vol}_{WP}(M_{0,n}) = \frac{\pi^{2(n-3)} v_n}{n!(n-3)!}, \text{ for } n \geq 4,$$

where the sequence $(v_n)_{n=3}^{\infty}$ is defined via the recursion

$$v_3 = 1, \qquad v_n = \frac{1}{2} \sum_{i=1}^{n-3} \frac{i(n-i-2)}{n-1} \binom{n-4}{i-1} \binom{n}{i+1} v_{i+2} v_{n-i}, \qquad n \geq 4. \tag{14}$$

This sequence shows up as A115047 in the OEIS [8], see Kaufmann, Manin, and Zagier [3] and Matone [4] for more background. In the paper by Kauffman et al., [3] the following relationship is shown.

Proposition 33. *Let the sequence v_n be defined as in (14). Then*

$$v_n = \sum_{k=1}^{n-3} \frac{(-1)^{n-3-k}}{k!} \sum_{\substack{m_1,\dots,m_k>0 \\ m_1+\cdots+m_k=n-3}} \binom{n-3}{m_1,\dots,m_k} \binom{n-3+k}{m_1+1,\dots,m_k+1}.$$

We are now ready to prove the following connection between the sequence $v(n)$ and border-strip decompositions.

Theorem 34. *Let $a(n)$ be defined inductively by $a(0) = 1$ and*

$$a(n) = \frac{1}{2} \sum_{i=1}^{n} \frac{i(n-i+1)}{(n+2)} \binom{n-1}{i-1} \binom{n+3}{i+1} a(i-1) a(n-i) \tag{15}$$

and let v_n be given as in (14). Then $a(n) = v_{n+3} = |\mathrm{BSD}(\mathsf{c}^n, n)|$.

Proof. The first equality, $v_{n+3} = a(n)$ follows from comparing (14) and (15). It is a straightforward calculation to verify that they are equal. To get the second identity, note that we can get the formula in Proposition 33 from Equation (13) by replacing integer partitions with integer compositions, and then refining the sum over the number of parts, denoted by k in Proposition 33. □

4 Further directions

Given the connection with Euler characteristics of moduli spaces mentioned after Corollary 28, and the connection with moduli spaces in Theorem 34, is there a generalization of this mysterious connection? For example, there are formula for the volumes of surfaces of other genus; see Matone [4].

Another interesting direction is to consider the q-analogue of border-strip tableaux rather than decompositions.

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