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On Two New Classes of B-q-bonacci Polynomials

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Abstract

In this paper we define two new classes of polynomials associated with generalized Fibonacci polynomials. We call them $h(x)$-B-q-bonacci polynomials and incomplete $h(x)$-B-q-bonacci polynomials. We present some identities for the two classes of polynomials, and the convolution property of $h(x)$-B-q-bonacci polynomials and its applications.

1 Introduction

The Fibonacci sequence, polynomials associated with the Fibonacci sequence, and their extended forms produce interesting and fascinating properties. For details see [8, 14]. Arolkar and Valaulikar introduced the B-tribonacci sequence [2] and B-tribonacci polynomials [1].

The B-tribonacci sequence [2] and B-tribonacci polynomials [1] are further extended to q^{th} order recurrence relations in [5] and [6] respectively. Arolkar and Valaulikar extended and studied the $h(x)$-Fibonacci polynomials [9] to $h(x)$-B-tribonacci polynomials [3]. Filipponi [7] introduced the incomplete Fibonacci and Lucas numbers. Ramírez [11] studied various identities related to the incomplete k-Fibonacci and k-Lucas numbers. Ramírez [13] introduced interesting classes of polynomials, namely, the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials. Arolkar and Valaulikar [4] extended the incomplete $h(x)$-Fibonacci and $h(x)$-Lucas polynomials. Ramírez and Sirvent [12] defined and studied identities related to the incomplete tribonacci numbers and polynomials. Yilmaz and Taskara [10] obtained identities for the incomplete tribonacci-Lucas numbers and polynomials.

The aim of this paper is to extend two classes of polynomials, namely, the $h(x)$-B-tribonacci polynomials [3] and incomplete $h(x)$-B-tribonacci polynomials of [4] to the q^{th} order relations. We call them the $h(x)$-B-q-bonacci polynomials and incomplete $h(x)$-B-q-bonacci polynomials. We study some properties of these polynomials.

2 $h(x)$-B-q-bonacci polynomials

We first define the class of $h(x)$-B-q-bonacci polynomials.

Definition 1. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-B-q-bonacci polynomials, denoted by $({}^qB)_{h,n}(x), n \in \mathbb{N} \cup \{0\}, q \geq 2$, are defined by

$$({}^qB)_{h,n+q-1}(x) = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \; h^{q-1-r}(x)({}^qB)_{h,n+q-2-r}(x), \forall n \geq 1, \tag{1}$$

with $({}^qB)_{h,i}(x) = 0,\ i = 0, 1, 2, 3, \ldots, q-2$ and $({}^qB)_{h,q-1}(x) = 1$, where the coefficients of the terms on the right-hand side are the terms of the binomial expansion of $(h(x)+1)^{q-1}$ and $({}^qB)_{h,n}(x)$ is the n^{th} polynomial.

For simplicity, henceforth we denote $({}^qB)_{h,n}(x)$ by $({}^qB)_{h,n}$ and $h(x)$ by h. We have the following identities for $({}^qB)_{h,n}$.

(1) The n^{th} term $({}^qB)_{h,n}$ of (1) is given by

$$({}^qB)_{h,n} = \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1)-r)-r}, \tag{2}$$

$n \geq q-1$, where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. We prove the identity using induction on n.

For $n = q-1$, (2) implies

$$({}^{q}B)_{h,q-1} = \sum_{r=0}^{0} \frac{((q-1)\,(-r))^{\underline{r}}}{r!} \quad h^{(q-1)(-r)-r} = 1.$$

Hence (2) is true for $n = q-1$. Assume that (2) is true for $n \leq m$. We divide the result into q cases, namely, $m = qk, qk+1, qk+2, \ldots, qk+(q-1)$, for some $k \geq 1$.

Case (i): Let $m = qk$ and $t = \left\lfloor \frac{(q-1)(qk-s-(q-1))}{q} \right\rfloor$. Then

$$\begin{aligned}
&\sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!}\ h^{q-1-s}({}^{q}B)_{h,qk-s} \\
&= \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \sum_{r=0}^{t} \frac{((q-1)\,(qk-(q-1)-(r+s)))^{\underline{r}}}{r!}\ h^{(q-1)(qk+1-(q-1)-(r+s))-(r+s)} \\
&= \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \sum_{p=s}^{(q-1)k-(q-2)} \frac{((q-1)\,(qk-(q-1)-p))^{\underline{p-s}}}{(p-s)!}\ h^{(q-1)(qk+1-(q-1)-p)-p} \\
&= \Big(\frac{(q-1)^{\underline{0}}}{0!} \sum_{p=0}^{(q-1)k-(q-2)} \frac{((q-1)\,(qk-(q-1)-p))^{\underline{p}}}{p!} \\
&+ \frac{(q-1)^{\underline{1}}}{1!} \sum_{p=1}^{(q-1)k-(q-2)} \frac{((q-1)\,(qk-(q-1)-p))^{\underline{p-1}}}{(p-1)!} + \cdots \\
&+ \frac{(q-1)^{\underline{(q-1)}}}{(q-1)!} \sum_{p=q-1}^{(q-1)k-(q-2)} \frac{((q-1)\,(qk-(q-1)-p))^{\underline{p-(q-1)}}}{(p-(q-1))!} \Big)\ h^{(q-1)(qk+1-(q-1)-p)-p} \\
&= \Bigg(\frac{((q-1)\,(qk-(q-1)))^{\underline{0}}}{0!} \\
&+ \left(\frac{((q-1)\,(qk-(q-1)-1))^{\underline{1}}}{1!} + \frac{(q-1)^{\underline{1}}}{1!} \frac{((q-1)\,(qk-(q-1)-1))^{\underline{0}}}{0!} \right) + \cdots \\
&+ \sum_{p=q-1}^{(q-1)k-(q-2)} \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \frac{((q-1)\,(qk-(q-1)-p))^{\underline{p-s}}}{(p-s)!} \Bigg)\ h^{(q-1)(qk+1-(q-1)-p)-p}.
\end{aligned}$$

Therefore, using $\sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \frac{n^{\underline{p-s}}}{(p-s)!} = \frac{(n+(q-1))^{\underline{p}}}{p!}$, we have

$$\begin{aligned}
&\sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} h^{q-1-s} ({}^qB)_{h,qk-s} \\
&= \sum_{p=0}^{(q-1)k-(q-2)} \frac{((q-1)(qk+1-(q-1)-p))^{\underline{p}}}{p!} h^{(q-1)(qk+1-(q-1)-p)-p} \\
&= ({}^qB)_{h,qk+1}.
\end{aligned}$$

Thus, assuming the result for $m = qk$, we have proved it for $m = qk+1$. Similarly, we can prove the other cases.

We conclude that $\sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} h^{q-1-s} ({}^qB)_{h,m-s} = ({}^qB)_{h,m+1}$.

Hence, by induction, the result follows. ☐

(2) The sum of the first $n+1$ terms of (1) is given by

$$\sum_{r=0}^{n} ({}^qB)_{h,r} = \frac{({}^qB)_{h,n+1} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} \; ({}^qB)_{h,n-i} - 1}{(h+1)^{q-1} - 1}, \tag{3}$$

provided $\begin{cases} h \neq 0, & \text{if } q \text{ is even;} \\ h \neq 0, -2, & \text{if } q \text{ is odd.} \end{cases}$

Proof. We obtain the result by induction on n. For $n = q-1$, $\sum_{r=0}^{q-1} ({}^qB)_{h,r} = ({}^qB)_{h,q-1} = 1$. Also,

$$\begin{aligned}
&\frac{({}^qB)_{h,q} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} \; ({}^qB)_{h,q-1-i} - 1}{(h+1)^{q-1} - 1} \\
&= \frac{h^{q-1} + \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} \; ({}^qB)_{h,q-1} - 1}{(h+1)^{q-1} - 1} = 1.
\end{aligned}$$

Thus, (3) is true for $n = q-1$.

Assume that (3) is true for $n \leq m$. Then

$$\begin{aligned}
\sum_{r=0}^{m+1} ({}^qB)_{h,r} &= \sum_{r=0}^{m} ({}^qB)_{h,r} + ({}^qB)_{h,m+1} \\
&= \frac{({}^qB)_{h,m+1} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} \; ({}^qB)_{h,m-i} - 1}{(h+1)^{q-1} - 1} + ({}^qB)_{h,m+1} \\
&= \frac{({}^qB)_{h,m+2} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{q-1-r} \; ({}^qB)_{h,m+1-i} - 1}{(h+1)^{q-1} - 1}.
\end{aligned}$$

Thus, the result is true for $n = m+1$. Hence by induction the result follows. ☐

(3) The generating function for (1) is given by

$$({}^{q}G_{(B)})_h(z) = \frac{1}{1 - z(h+z)^{q-1}}, \tag{4}$$

provided $|(z(h+z)^{q-1})| < 1$.

Proof. Let $t = \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor$

$$\begin{aligned}
({}^{q}G_{(B)})_h(z) &= \sum_{n=0}^{\infty} ({}^{q}B)_{h,n}\ z^{n-(q-1)} \\
&= \sum_{n=0}^{q-2} ({}^{q}B)_{h,n}\ z^{n-(q-1)} + \sum_{n=q-1}^{\infty} ({}^{q}B)_{h,n}\ z^{n-(q-1)} \\
&= \sum_{n=q-1}^{\infty} ({}^{q}B)_{h,n}\ z^{n-(q-1)},\ \text{since } ({}^{q}B)_{h,i} = 0,\ i = 0, 1, 2, 3, \ldots, q-2 \\
&= \sum_{n=q-1}^{\infty} \sum_{r=0}^{t} \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ \ h^{(q-1)(n-(q-1)-r)-r}\ \ z^{n-(q-1)} \\
&= 1 + h^{q-1} z + \left(h^{2(q-1)} + \frac{(q-1)^{\underline{1}}}{1!} h^{q-2} \right) z^2 + \ldots \\
&= 1 + z(h+z)^{q-1} + z^2 (h+z)^{2(q-1)} + \ldots \\
&= \frac{1}{1 - z(h+z)^{q-1}},\ \text{provided } |(z(h+z)^{q-1})| < 1.
\end{aligned}$$

□

We now obtain the following property.

Theorem 2. (***Convolution property for*** $({}^{q}B)_{h,n}$)
For all $n \geq q-1$, *we have*

$$\frac{d}{dx}\left(({}^{q}B)_{h,n}\right) = (q-1)\frac{dh}{dx}\left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!}\ h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^{q}B)_{h,i}\ ({}^{q}B)_{h,n+q-2-r-i} \right). \tag{5}$$

Proof. Equation (4) implies

$$\sum_{n=0}^{\infty} ({}^{q}B)_{h,n}\ z^{n-(q-1)} = \frac{1}{1 - z(h+z)^{q-1}}.$$

Differentiating both sides with respect to x, we get

$$\sum_{n=0}^{\infty} \frac{d}{dx}\left(({}^{q}B)_{h,n}\right) z^{n-(q-1)} = z(q-1)(h+z)^{q-2} \; \frac{1}{\left(1-z(h+z)^{q-1}\right)^2} \; \frac{dh}{dx}$$

$$= \left((q-1) \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} z^{r+1} \; \left(\sum_{n=0}^{\infty} ({}^{q}B)_{h,n} \; z^{n-(q-1)} \right)^2 \right) \frac{dh}{dx}$$

$$= \left((q-1) \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \; z^{-2(q-1)+r+1} \; \left(\sum_{n=0}^{\infty} ({}^{q}B)_{h,n} \; z^{n} \right)^2 \right) \frac{dh}{dx}$$

$$= (q-1)\frac{dh}{dx} \sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} ({}^{q}B)_{h,i} ({}^{q}B)_{h,n-i} \; z^{n-2(q-1)+r+1} \right).$$

Comparing the coefficients of $z^{n-(q-1)}$, we get

$$\frac{d}{dx}\left(({}^{q}B)_{h,n}\right) = (q-1)\frac{dh}{dx} \left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^{q}B)_{h,i} ({}^{q}B)_{h,n+q-2-r-i} \right).$$

$\square$

We now give an application of the convolution property. It is required to prove an identity in the next section.

Theorem 3. *For $n \geq q-1$,*

$$\sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} r \; \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1))-qr}$$

$$= \frac{(q-1)\,(n-(q-1))}{q} \; ({}^{q}B)_{h,n}$$

$$-\frac{h}{q} \; (q-1) \left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^{q}B)_{h,i} \; ({}^{q}B)_{h,n+q-2-r-i} \right). \tag{6}$$

Proof. Equation (2) implies

$$({}^{q}B)_{h,n} = \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1))-qr}.$$

Differentiating both sides with respect to x and simplifying, we get

$$\frac{d}{dx}\big(({}^{q}B)_{h,n}\big)h = ((q-1)(n-(q-1)))\ ({}^{q}B)_{h,n}\ \frac{dh}{dx}$$
$$-\ q\ \frac{dh}{dx} \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} r\ \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1))-qr}.$$

Thus,

$$\frac{dh}{dx} \sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} r\ \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1))-qr}$$
$$= \frac{(q-1)(n-(q-1))}{q}\ ({}^{q}B)_{h,n}\ \frac{dh}{dx} - \frac{h}{q}\ \frac{d}{dx}\left(({}^{q}B)_{h,n}\right).$$

Hence (5) implies

$$\sum_{r=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} r\ \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ \ h^{(q-1)(n-(q-1))-qr}$$
$$= \frac{(q-1)(n-(q-1))}{q}\ ({}^{q}B)_{h,n}$$
$$-\ \frac{h}{q}\ (q-1)\left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!}\ h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^{q}B)_{h,i}\ ({}^{q}B)_{h,n+q-2-r-i}\right).$$

□

3 Incomplete $h(x)$-B-q-bonacci Polynomials

In this section we define the class of incomplete $h(x)$-B-q-bonacci polynomials and discuss some of its properties.

Definition 4. The incomplete $h(x)$-B-q-bonacci polynomials are defined by

$$({}^{q}B)^{l}_{h,n}(x) = \sum_{r=0}^{l} \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ \ h^{(q-1)(n-(q-1)-r)-r}(x), \tag{7}$$

$$\forall\ 0 \le l \le \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor \text{ and } n \ge q-1.$$

Note that $({}^{q}B)_{h,n}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}(x) = ({}^{q}B)_{h,n}(x)$.

For simplicity, we use $({}^{q}B)_{h,n}^{l}(x) = ({}^{q}B)_{h,n}^{l}$, $({}^{q}B)_{h,n}(x) = ({}^{q}B)_{h,n}$ and $h(x) = h$. We prove identities related to the recurrence relation for $({}^{q}B)_{h,n}^{l}$.

Theorem 5. *The recurrence relation for $({}^{q}B)_{h,n}^{l}$ is given by*

$$({}^{q}B)_{h,n+q}^{l+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} ({}^{q}B)_{h,n+q-1-r}^{l+q-1-r}, \ 0 \le l \le \left\lfloor \frac{(q-1)(n-q)}{q} \right\rfloor, \forall n \ge q. \tag{8}$$

Proof. Consider,

$$\begin{aligned}
&\sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} \ ({}^{q}B)_{h,n+q-1-r}^{l+q-1-r} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} \\
&\sum_{i=0}^{l+q-1-r} \frac{((q-1)(n+q-1-r-(q-1)-i))^{\underline{i}}}{i!} \ h^{(q-1)(n+q-1-r-(q-1)-i)-i} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ h^{q-1-r} \ \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n-r-i))^{\underline{i}}}{i!} \ h^{(q-1)(n-r)-qi} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n-r-i))^{\underline{i}}}{i!} \ h^{(q-1)(n+1)-qr-qi} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n-(r+i)))^{\underline{i}}}{i!} \ h^{(q-1)(n+1)-q(r+i)}.
\end{aligned}$$

Taking $j = i + r$, we get

$$\begin{aligned}
&\sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ ({}^{q}B)_{h,n+q-1-r}^{l+q-1-r} \ h^{q-1-r} \\
&= \sum_{r=0}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} \ \sum_{j=r}^{l+q-1} \frac{((q-1)(n-j))^{\underline{j-r}}}{(j-r)!} \ h^{(q-1)(n+1)-qj} \\
&= \sum_{j=0}^{l+q-1} \frac{((q-1)(n+1-j))^{\underline{j}}}{j!} \ h^{(q-1)(n+1)-qj} \\
&= ({}^{q}B)_{h,n+q}^{l+q-1}.
\end{aligned}$$

$\square$

Theorem 6. *For $s \geq 1$,*

$$({}^qB)^{l+(q-1)s}_{h,n+qs} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^{\underline{i}}}{i!} \; ({}^qB)^{l+i}_{h,n+i} \; h^i, \tag{9}$$

$0 \leq l \leq \left\lfloor \frac{(q-1)(n-s-(q-1))}{q} \right\rfloor$.

Proof. Follows using induction. □

Theorem 7. *For $n \geq \left\lfloor \frac{ql+2(q-1)}{q-1} \right\rfloor$, $({}^qB)^{l+(q-1)}_{h,n+(q-1)+s} - h^{(q-1)s}({}^qB)^{l+q-1}_{h,n+q-1}$*

$$= \sum_{i=0}^{s-1} \sum_{r=1}^{q-1} \frac{(q-1)^{\underline{r}}}{r!} h^{(q-1)s-(q-1)i-r} \; ({}^qB)^{l+(q-1)-r}_{h,n+(q-1)+i-r}. \tag{10}$$

Proof. Follows using induction. □

The next theorem is related to the sum of incomplete $h(x)$-B-q-bonacci polynomials $({}^qB)^l_{h,n}$.

Theorem 8. *For all $n \geq q-1$,*

$$\sum_{l=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} ({}^qB)^l_{h,n} = \left(\left\lfloor \frac{(q-1)\,(n-(q-1))}{q} \right\rfloor + \frac{q-(q-1)\,(n-(q-1))}{q} \right) ({}^qB)_{h,n}$$

$$+ \frac{h}{q} \; (q-1) \left(\sum_{r=0}^{q-2} \frac{(q-2)^{\underline{r}}}{r!} \; h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,i} \; ({}^qB)_{h,n+q-2-r-i} \right). \tag{11}$$

Proof. $\sum_{l=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} ({}^qB)^l_{h,n}$

$$= ({}^qB)^0_{h,n} + ({}^qB)^1_{h,n} + \cdots + ({}^qB)^r_{h,n} + \cdots + ({}^qB)^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor}_{h,n}$$
$$= \frac{((q-1)(n-(q-1)))^{\underline{0}}}{0!} \; h^{(q-1)(n-(q-1))}$$
$$+ \left(\frac{((q-1)(n-(q-1)))^{\underline{0}}}{0!} h^{(q-1)(n-(q-1))} + \frac{(q-1)(n-(q-1)-1))^{\underline{1}}}{1!} \; h^{(q-1)(n-(q-1))-q} \right)$$
$$+ \cdots +$$
$$\left(\frac{((q-1)(n-(q-1)))^{\underline{0}}}{0!} \; h^{(q-1)(n-(q-1))} + \cdots + \frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!} \; h^{(q-1)(n-(q-1))-qr} \right)$$
$$+ \cdots$$

$$
\begin{aligned}
&+\left(\frac{((q-1)(n-(q-1)))^{\underline{0}}}{0!}\ h^{(q-1)(n-(q-1))}+\cdots\right.\\
&+\frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1))-qr}+\cdots\\
&\left.+\frac{\left((q-1)\left(n-(q-1)-\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor\right)\right)^{\underline{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}}}{(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor)!}\ h^{(q-1)(n-(q-1))-q\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}\right)\\
&=\sum_{r=0}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}\left(\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor+1-r\right)\frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1))-qr}\\
&=\sum_{r=0}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}\left(\left\lfloor\frac{(q-1)(n-(q-1))}{q}\right\rfloor+1\right)\frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1))-qr}\\
&-\sum_{r=0}^{\lfloor\frac{(q-1)(n-(q-1))}{q}\rfloor}r\frac{((q-1)(n-(q-1)-r))^{\underline{r}}}{r!}\ h^{(q-1)(n-(q-1))-qr}\\
&=\left(\left\lfloor\frac{(q-1)(n-(q-1))}{q}\right\rfloor+1-\frac{(q-1)(n-(q-1))}{q}\right)({}^{q}B)_{h,n}\\
&+\frac{h}{q}(q-1)\left(\sum_{r=0}^{q-2}\frac{(q-2)^{\underline{r}}}{r!}\ h^{(q-2)-r}\sum_{i=0}^{n+q-2-r}({}^{q}B)_{h,i}\ ({}^{q}B)_{h,n+q-2-r-i}\right).
\end{aligned}
$$

Using (6) of Theorem 3 in Section 2, the result follows. □

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