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Repdigits as Sums of Four Fibonacci or Lucas Numbers

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Abstract

In this paper, we determine all base-10 repdigits expressible as sums of four Fibonacci or Lucas numbers.

1 Introduction

The Fibonacci sequence $(F_n)_n$ and the Lucas sequences $(L_n)_n$ are given, respectively, by

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 0$$

and

$$L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n \text{ for } n \geq 0.$$

Luca [2] answered the question of which repdigits can be written as sums of three Fibonacci numbers by following a general method (see [3]). Luca [2] showed that all nonnegative integer solutions (m_1, m_2, m_3, n) of the equation

$$N = F_{m_1} + F_{m_2} + F_{m_3} = d\left(\frac{10^n - 1}{9}\right) \text{ with } d \in \{1, \dots, 9\}$$

have

$$N \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 44, 55, 66, 77, 99, 111, 555, 666, 11111\}.$$

Luca, Normenyo, and Togbe [6, 7] obtained analogous results for Pell numbers and Lucas numbers.

Luca [2] conjectured that the method he employed could be used to compute all solutions of the equation

$$d\left(\frac{10^n - 1}{9}\right) = F_{m_1} + F_{m_2} + F_{m_3} + F_{m_4}$$

with $d \in \{1, \dots, 9\}$ and $m_1 \geq m_2 \geq m_3 \geq m_4$. Luca et al. [8] investigated this idea for Pell numbers. Luca et al. [8] showed that, all nonnegative integer solutions (m_1, m_2, m_3, n) of the equation

$$N = P_{m_1} + P_{m_2} + P_{m_3} + P_{m_4} = d\left(\frac{10^n - 1}{9}\right) \text{ with } d \in \{1, \dots, 9\}$$

have

$$N \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 77, 88, 99, 111, 222, 444, 888, 999\}.$$

In this paper, we compute all repdigits that can be expressed as sums of four Fibonacci or Lucas numbers. We prove Theorem 1 and Theorem 2 below.

Theorem 1. All nonnegative integer solutions (m_1, m_2, m_3, m_4, n) of the equation

$$N = F_{m_1} + F_{m_2} + F_{m_3} + F_{m_4} = d\left(\frac{10^n - 1}{9}\right) \text{ with } d \in \{1, \dots, 9\} \tag{1}$$

have

$$N \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 99, 111, 222, 333, 555, 666, 777, 999, 1111, 2222, 11111, 66666\}.$$

Theorem 2. All nonnegative integer solutions (m_1, m_2, m_3, m_4, n) of the equation

$$N = L_{m_1} + L_{m_2} + L_{m_3} + L_{m_4} = d\left(\frac{10^n - 1}{9}\right) \text{ with } d \in \{1, \ldots, 9\} \tag{2}$$

have

$$N \in \{4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 111, 222, 333, 555, 666, 999, 2222, 4444, 11111, 88888\}.$$

Here is the organization of this paper. In the next section, we recall the useful results to prove our two main results. We use them in Section 3 to prove Theorem 1. In Section 4, for the sake of completeness, we apply the same method for the entire proof of Theorem 2.

2 Preliminaries

In this section, we recall some results that are useful for the proof of Theorem 1 and Theorem 2. Let $\mathbb{K}$ be an algebraic number field of degree D over $\mathbb{Q}$, let $\alpha_1, \ldots, \alpha_n \in \mathbb{K} \setminus \{0\}$ and let $b_1, \ldots, b_n \in \mathbb{Z}$. Set

$$B = \max\{|b_1|, \ldots, |b_n|\}$$

and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1.$$

Let $A_1, \ldots, A_n$ be real numbers with

$$\max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\} \leq A_i, \qquad 1 \leq i \leq n,$$

where $h(\eta)$ is the logarithmic height of an algebraic number η which is given by the formula

$$h(\eta) = \frac{1}{d(\eta)}\left(\log|a_0| + \sum_{i=1}^{d(\eta)} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right),$$

where $d(\eta)$ is the degree of η over $\mathbb{Q}$ and

$$f(X) = a_0 \prod_{i=1}^{d(\eta)} \left(X - \eta^{(i)}\right) \in \mathbb{Z}[X]$$

the minimal polynomial of η of degree $d(\eta)$ over $\mathbb{Z}$.

Lemma 3. *([1, Theorem 9.4]) Assume that $\Lambda \neq 0$. We then have*

$$\log|\Lambda| > -3 \times 30^{n+4} \times (n+1)^{5.5} D^2 (1 + \log D)(1 + \log nB) A_1 \cdots A_n. \tag{3}$$

Furthermore, if $\mathbb{K}$ is real, we have

$$\log|\Lambda| > -1.4 \times 30^{n+3} \times n^{4.5} D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_n. \tag{4}$$

We now discuss a computational method for reducing upper bounds for solutions of Diophantine equations.

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{5}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0 be a (large) positive constant. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{6}$$

$$X \leq X_0. \tag{7}$$

When $\beta = 0$ in (5), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 4. *([9, Lemma 3.1]) (i) If (6) and (7) hold for x_1, x_2 with $X \geq X^*$, then $(-x_2, x_1) = (p_k, q_k)$ for an index k that satisfies*

$$k \leq -1 + \frac{\log\left(1 + X_0\sqrt{5}\right)}{\log\left(\frac{1+\sqrt{5}}{2}\right)} := Y_0. \tag{8}$$

Moreover, the partial quotient a_{k+1} satisfies

$$a_{k+1} > -2 + \frac{|\vartheta_2| \exp(\delta q_k)}{c q_k}. \tag{9}$$

(ii) If for some k with $q_k \geq X^*$, we have

$$a_{k+1} > \frac{|\vartheta_2| \exp(\delta q_k)}{c q_k}, \tag{10}$$

then (6) holds for $(-x_2, x_1) = (p_k, q_k)$.

Lemma 5. *([9, Lemma 3.2]) Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1}.$$

If (6) and (7) hold for x_1, x_2 and $\beta = 0$, then

$$Y < \frac{1}{\delta} \log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right). \tag{11}$$

When $\beta\vartheta_1\vartheta_2 \neq 0$ in (5), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x we define $\|x\| = \min\{|x - n|, n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 6. *([9, Lemma 3.3]) Suppose that*

$$\| q\psi \| > \frac{2X_0}{q}.$$

Then, the solutions of (6) *and* (7) *satisfy*

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

3 Proof of Theorem 1

It is well known that the Fibonacci numbers are given by

$$F_m = \frac{1}{\sqrt{5}}\left(\alpha^m - \beta^m\right) \quad \text{for} \quad m \geq 0, \quad \text{where} \quad (\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right).$$

In equation (1) we suppose that $m_1 \geq m_2 \geq m_3 \geq m_4$. A search with Maple in the range $0 \leq m_1 \leq 599$ yielded only the solutions shown in the statement of Theorem 1.

Let us suppose that solutions of equation (1) exist for $m_1 \geq 600$. For $m_1 \geq 600$, we have that

$$F_{600} \leq F_{m_1} \leq F_{m_1} + F_{m_2} + F_{m_3} + F_{m_4} = d\left(\frac{10^n - 1}{9}\right) \leq 10^n - 1,$$

and so

$$125 \leq \frac{\log(1 + F_{600})}{\log 10} \leq n.$$

That is, $n \geq 125$. Now,

$$\begin{aligned}
10^{n-1} &\leq d\left(\frac{10^n - 1}{9}\right) = F_{m_1} + F_{m_2} + F_{m_3} + F_{m_4} \\
&\leq 4F_{m_1} \\
&\leq \frac{4}{\sqrt{5}}\left(\alpha^{m_1} + |\beta|^{m_1}\right) \\
&< \frac{8}{\sqrt{5}}\alpha^{m_1} \\
&< \alpha^{m_1 + 2.7},
\end{aligned}$$

since $\frac{8}{\sqrt{5}} < \alpha^{2.7}$. This means that $10^{n-1} < \alpha^{m_1+2.7}$, and thus

$$4.78(n-1) < (n-1)\frac{\log 10}{\log \alpha} < m_1 + 2.7.$$

Consequently,

$$n < 4.78n - 7.48 < m_1,$$

as $n \geq 125$. Therefore, $125 \leq n < m_1$.

We can put equation (1) in the form

$$\frac{\alpha^{m_1}-\beta^{m_1}}{\sqrt{5}} + \frac{\alpha^{m_2}-\beta^{m_2}}{\sqrt{5}} + \frac{\alpha^{m_3}-\beta^{m_3}}{\sqrt{5}} + \frac{\alpha^{m_4}-\beta^{m_4}}{\sqrt{5}} = \frac{d\times 10^n}{9} - \frac{d}{9}. \tag{12}$$

We examine (12) in four different ways as follows.

Step 1: We express (12) in the form

$$\frac{\alpha^{m_1}}{\sqrt{5}}\left(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}+\alpha^{m_4-m_1}\right) - \frac{d\times 10^n}{9} = -\frac{d}{9} + \frac{1}{\sqrt{5}}\left(\beta^{m_1}+\beta^{m_2}+\beta^{m_3}+\beta^{m_4}\right). \tag{13}$$

It follows that

$$\left|\frac{\alpha^{m_1}}{\sqrt{5}}\left(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}+\alpha^{m_4-m_1}\right) - \frac{d\times 10^n}{9}\right| \\ \leq \frac{d}{9} + \frac{1}{\sqrt{5}}\left(|\beta|^{m_1}+|\beta|^{m_2}+|\beta|^{m_3}+|\beta|^{m_4}\right),$$

leading to

$$\left|\frac{\alpha^{m_1}}{\sqrt{5}}\left(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}+\alpha^{m_4-m_1}\right) - \frac{d\times 10^n}{9}\right| < \frac{\alpha^4}{\sqrt{5}}. \tag{14}$$

Multiplying both sides of inequality (14) by $\frac{\sqrt{5}\alpha^{-m_1}}{1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}+\alpha^{m_4-m_1}}$, we obtain

$$\left|1-\alpha^{-m_4}10^n\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}\right)\right| < \frac{\alpha^{4-m_1}}{1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}+\alpha^{m_4-m_1}},$$

and so

$$\left|1-\alpha^{-m_4}10^n\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}\right)\right| < \alpha^{4-m_1}. \tag{15}$$

Put

$$\Gamma_1 := 1-\alpha^{-m_4}10^n\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}\right). \tag{16}$$

Suppose that $\Gamma_1 = 0$. Then we have that

$$\alpha^{m_1}+\alpha^{m_2}+\alpha^{m_3}+\alpha^{m_4} = \frac{10^n\times d\sqrt{5}}{9}.$$

Conjugating in $\mathbb{Q}\left(\sqrt{5}\right)$ yields

$$\beta^{m_1} + \beta^{m_2} + \beta^{m_3} + \beta^{m_4} = -\frac{10^n \times d\sqrt{5}}{9}.$$

Thus,

$$\begin{aligned}\frac{10^{125} \times \sqrt{5}}{9} \leq \frac{10^n \times d\sqrt{5}}{9} &= |\beta^{m_1} + \beta^{m_2} + \beta^{m_3}\beta^{m_4}| \\ &\leq |\beta|^{m_1} + |\beta|^{m_2} + |\beta|^{m_3} + |\beta|^{m_4} \\ &< 4.\end{aligned}$$

This implies that $\frac{10^{125}\times\sqrt{5}}{9} < 4$, which is false. Hence, it follows that $\Gamma_1 \neq 0$.

In order to apply Lemma 3 to Γ_1, we set

$$\alpha_1 = \alpha, \quad \alpha_2 = 10, \quad \alpha_3 = \frac{d\sqrt{5}}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)},$$

$$b_1 = -m_4, \quad b_2 = n, \quad b_3 = 1,$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}\left(\sqrt{5}\right)$ and $b_1, b_2, b_3 \in \mathbb{Z}$. Thus, the degree D of $\mathbb{Q}\left(\sqrt{5}\right)$ is 2 and $B = \max\{m_4, n, 1\} \leq m_1$. The minimal polynomial of α over $\mathbb{Z}$ is $x^2 - x - 1$, and so $d(\alpha) = 2$ and $a_0(\alpha) = 1$. It follows that

$$h(\alpha) = \frac{1}{2}\log\alpha.$$

Also, the minimal polynomial of $\sqrt{5}$ over $\mathbb{Z}$ is $x^2 - 5$. Thus,

$$h\left(\sqrt{5}\right) = \frac{1}{2}\log 5.$$

We have

$$\max\{2h(\alpha_1), |\log\alpha_1|, 0.16\} = \log\alpha < 0.49 =: A_1,$$

$$\max\{2h(\alpha_2), |\log\alpha_2|, 0.16\} = 2\log 10 < 4.61 =: A_2.$$

Set

$$C_1 = 2.3 \times 10^{12} > 1.4 \times 30^6 \times 3^{4.5} \times D^2 \times (1 + \log D) \times A_1 \times A_2.$$

Next, we compute A_3. We find that,

$$\alpha_3 = \frac{d\sqrt{5}}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)} < \sqrt{5},$$

and

$$\alpha_3^{-1} = \frac{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)}{d\sqrt{5}} \leq \frac{36}{\sqrt{5}}\alpha^{m_1-m_4}.$$

Hence, $|\log\alpha_3| < 3 + (m_1 - m_4)\log\alpha$. Also, we have that

$$\begin{aligned}
h(\alpha_3) &\le h(d\sqrt{5}) + h(9) + h(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)\\
&\le h(9\sqrt{5}) + h(9) + \log 2 + h(\alpha^{m_3-m_4}(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1))\\
&\le h(9) + h(\sqrt{5}) + h(9) + 2\log 2 + h(\alpha^{m_3-m_4}) + h(\alpha^{m_2-m_3}(\alpha^{m_1-m_2} + 1))\\
&\le h(\sqrt{5}) + 2h(9) + 3\log 2 + h(\alpha^{m_3-m_4}) + h(\alpha^{m_2-m_3}) + h(\alpha^{m_1-m_2})\\
&\le h(\sqrt{5}) + 2h(9) + 3\log 2 + (m_3 - m_4)h(\alpha) + (m_2 - m_3)h(\alpha) + (m_1 - m_2)h(\alpha)\\
&= \frac{1}{2}\log 5 + 2h(9) + 3\log 2 + \frac{1}{2}(m_1 - m_4)\log\alpha.
\end{aligned}$$

Hence, $2h(\alpha_3) \le 15 + (m_1 - m_4)\log\alpha$. Therefore, we get

$$\max\{2h(\alpha_3), |\log\alpha_3|, 0.16\} \le 15 + (m_1 - m_4)\log\alpha =: A_3.$$

By applying Lemma 3 to Γ_1 given by (16), and using (15) we have that

$$\exp(-(15 + (m_1 - m_4)\log\alpha)C_1(1 + \log m_1)) < \alpha^{4-m_1}.$$

Thus,

$$m_1\log\alpha < 4\log\alpha + (15 + (m_1 - m_4)\log\alpha)C_1(1 + \log m_1). \tag{17}$$

Step 2: We have that

$$\frac{\alpha^{m_1}}{\sqrt{5}}\left(1 + \alpha^{m_2-m_1} + \alpha^{m_3-m_1}\right) - \frac{d\times 10^n}{9} = -\frac{d}{9} + \frac{1}{\sqrt{5}}(\beta^{m_1} + \beta^{m_2} + \beta^{m_3} + \beta^{m_4} - \alpha^{m_4}). \tag{18}$$

Consequently, we get

$$\left|\frac{\alpha^{m_1}}{\sqrt{5}}\left(1 + \alpha^{m_2-m_1} + \alpha^{m_3-m_1}\right) - \frac{d\times 10^n}{9}\right| \le \frac{d}{9} + \frac{1}{\sqrt{5}}(|\beta|^{m_1} + |\beta|^{m_2} + |\beta|^{m_3} + |\beta|^{m_4} + \alpha^{m_4}),$$

and so

$$\left|\frac{\alpha^{m_1}}{\sqrt{5}}\left(1 + \alpha^{m_2-m_1} + \alpha^{m_3-m_1}\right) - \frac{d\times 10^n}{9}\right| < \frac{\alpha^{m_4+5}}{\sqrt{5}}. \tag{19}$$

We multiply both sides of inequality (19) by $\frac{\sqrt{5}\alpha^{-m_1}}{1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}}$ to get

$$\left|1 - \alpha^{-m_3}10^n\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}\right)\right| < \frac{\alpha^{m_4-m_1+5}}{1 + \alpha^{m_2-m_1} + \alpha^{m_3-m_1}},$$

which gives us

$$\left|1 - \alpha^{-m_3}10^n\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}\right)\right| < \alpha^{m_4-m_1+5}. \tag{20}$$

Put

$$\Gamma_2 := 1 - \alpha^{-m_3} 10^n \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)} \right). \tag{21}$$

Suppose that $\Gamma_2 = 0$. Then, we get

$$\alpha^{m_1} + \alpha^{m_2} + \alpha^{m_3} = \frac{10^n \times d\sqrt{5}}{9}.$$

Taking the conjugate of this in $\mathbb{Q}(\sqrt{5})$, we get

$$\beta^{m_1} + \beta^{m_2} + \beta^{m_3} = -\frac{10^n \times d\sqrt{5}}{9}.$$

Consequently, we obtain

$$\frac{10^{125} \times \sqrt{5}}{9} \le \frac{10^n \times d\sqrt{5}}{9} = |\beta^{m_1} + \beta^{m_2} + \beta^{m_3}| \le |\beta|^{m_1} + |\beta|^{m_2} + |\beta|^{m_3} < 3,$$

which means that $\frac{10^{125} \times \sqrt{5}}{9} < 3$. This is false. We conclude that $\Gamma_2 \neq 0$.

To apply Lemma 3 to Γ_2 given by (21), we set

$$\alpha_1 = \alpha, \quad \alpha_2 = 10, \quad \alpha_3 = \frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}, \quad b_1 = -m_3, \quad b_2 = n, \quad b_3 = 1,$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}\left(\sqrt{5}\right)$ and $b_1, b_2, b_3 \in \mathbb{Z}$. We have $B = \max\{m_3, n, 1\} \le m_1$. We proceed to compute A_3 by first observing that

$$\alpha_3 = \frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)} < \sqrt{5}$$

and

$$\alpha_3^{-1} = \frac{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}{d\sqrt{5}} \le \frac{27}{\sqrt{5}} \alpha^{m_1-m_3}.$$

Hence, $|\log \alpha_3| < 3 + (m_1 - m_3)\log\alpha$. Additionally, we get

$$\begin{aligned} h(\alpha_3) &\le h(d\sqrt{5}) + h(9) + \log 2 + h(\alpha^{m_2-m_3}(\alpha^{m_1-m_2} + 1)) \\ &\le \frac{1}{2}\log 5 + 2h(9) + 2\log 2 + h(\alpha^{m_2-m_3}) + h(\alpha^{m_1-m_2}) \\ &\le \frac{1}{2}\log 5 + 2h(9) + 2\log 2 + (m_2 - m_3)h(\alpha) + (m_1 - m_2)h(\alpha) \\ &= \frac{1}{2}\log 5 + 2h(9) + 2\log 2 + \frac{1}{2}(m_1 - m_3)\log\alpha. \end{aligned}$$

Hence, $2h(\alpha_3) \le 14 + (m_1 - m_3)\log\alpha$. As a result, we find that

$$\max\{2h(\alpha_3), |\log\alpha_3|, 0.16\} \le 14 + (m_1 - m_3)\log\alpha =: A_3.$$

By applying Lemma 3 to Γ_2 given by (21) and using (20), we deduce that

$$\exp(-(14+(m_1-m_3)\log\alpha)C_1(1+\log m_1)) < \alpha^{m_4-m_1+5}.$$

Thus, we get

$$(m_1-m_4)\log\alpha < 5\log\alpha + (14+(m_1-m_3)\log\alpha)C_1(1+\log m_1). \tag{22}$$

Step 3: We begin with (12) written in the form

$$\frac{\alpha^{m_1}}{\sqrt{5}}\left(1+\alpha^{m_2-m_1}\right) - \frac{d\times 10^n}{9} = -\frac{d}{9} + \frac{1}{\sqrt{5}}(\beta^{m_1}+\beta^{m_2}+\beta^{m_3}+\beta^{m_4}-\alpha^{m_3}-\alpha^{m_4}). \tag{23}$$

Equation (23) leads us to

$$\begin{aligned}\left|\frac{\alpha^{m_1}}{\sqrt{5}}\left(1+\alpha^{m_2-m_1}\right) - \frac{d\times 10^n}{9}\right| &\le \frac{d}{9} + \frac{1}{\sqrt{5}}(|\beta|^{m_1}+|\beta|^{m_2}+|\beta|^{m_3}+|\beta|^{m_4}+\alpha^{m_3}+\alpha^{m_4})\\ &< 1+\frac{4}{\sqrt{5}}+\frac{2\alpha^{m_3}}{\sqrt{5}}\\ &\le \frac{1}{\sqrt{5}}\left(\sqrt{5}+6\right)\alpha^{m_3},\end{aligned}$$

from which we obtain

$$\left|\frac{\alpha^{m_1}}{\sqrt{5}}\left(1+\alpha^{m_2-m_1}\right) - \frac{d\times 10^n}{9}\right| < \frac{\alpha^{m_3+5}}{\sqrt{5}}. \tag{24}$$

Multiplying both sides of inequality (24) by $\frac{\sqrt{5}\alpha^{-m_1}}{1+\alpha^{m_2-m_1}}$ gives us

$$\left|1-\alpha^{-m_2}10^n\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\right)\right| < \frac{\alpha^{m_3-m_1+5}}{1+\alpha^{m_2-m_1}},$$

which yields

$$\left|1-\alpha^{-m_2}10^n\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\right)\right| < \alpha^{m_3-m_1+5}. \tag{25}$$

Put

$$\Gamma_3 := 1-\alpha^{-m_2}10^n\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\right). \tag{26}$$

Suppose that $\Gamma_3 = 0$. Then

$$\alpha^{m_1}+\alpha^{m_2} = \frac{10^n\times d\sqrt{5}}{9},$$

giving us

$$\beta^{m_1}+\beta^{m_2} = -\frac{10^n\times d\sqrt{5}}{9}$$

by conjugating in $\mathbb{Q}\left(\sqrt{5}\right)$. It follows that

$$\frac{10^{125}\times\sqrt{5}}{9}\le\frac{10^n\times d\sqrt{5}}{9}=|\beta^{m_1}+\beta^{m_2}|\le|\beta|^{m_1}+|\beta|^{m_2}<2,$$

which is false. Hence, $\Gamma_3\neq 0$. Using the notations in Lemma 3, we put

$$\alpha_1=\alpha,\ \ \alpha_2=10,\ \ \alpha_3=\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)},\ \ b_1=-m_2,\ \ b_2=n,\ \ b_3=1,$$

where $\alpha_1,\alpha_2,\alpha_3\in\mathbb{Q}\left(\sqrt{5}\right)$ and $b_1,b_2,b_3\in\mathbb{Z}$. We have $B=\max\{m_2,n,1\}\le m_1$. Now, we deduce that

$$\alpha_3=\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\le\sqrt{5}\ \text{ and }\ \alpha_3^{-1}=\frac{9(\alpha^{m_1-m_2}+1)}{d\sqrt{5}}\le\frac{18}{\sqrt{5}}\alpha^{m_1-m_2}.$$

So $|\log\alpha_3|<3+(m_1-m_2)\log\alpha$. Furthermore,

$$\begin{aligned}h(\alpha_3)&\le h(d\sqrt{5})+h(9)+\log 2+h(\alpha^{m_1-m_2})\\&\le h(\sqrt{5})+2h(9)+\log 2+(m_1-m_2)h(\alpha)\\&=\frac{1}{2}\log 5+2h(9)+\log 2+\frac{1}{2}(m_1-m_2)\log\alpha.\end{aligned}$$

Thus, $2h(\alpha_3)\le 12+(m_1-m_2)\log\alpha$ and so

$$\max\{2h(\alpha_3),|\log\alpha_3|,0.16\}<12+(m_1-m_2)\log\alpha=:A_3.$$

Applying Lemma 3 to Γ_3 given by (26), and using (25) we produce

$$\exp(-(12+(m_1-m_2)\log\alpha)C_1(1+\log m_1))<\alpha^{m_3-m_1+5},$$

from which we obtain

$$(m_1-m_3)\log\alpha<5\log\alpha+(12+(m_1-m_2)\log\alpha)C_1(1+\log m_1). \tag{27}$$

Step 4: Using equation (12) in the form

$$\frac{\alpha^{m_1}}{\sqrt{5}}-\frac{d\times 10^n}{9}=-\frac{d}{9}+\frac{1}{\sqrt{5}}\left(\beta^{m_1}+\beta^{m_2}+\beta^{m_3}+\beta^{m_4}-\alpha^{m_2}-\alpha^{m_3}-\alpha^{m_4}\right), \tag{28}$$

we get

$$\begin{aligned}\left|\frac{\alpha^{m_1}}{\sqrt{5}}-\frac{d\times 10^n}{9}\right|&\le\frac{d}{9}+\frac{1}{\sqrt{5}}\left(|\beta|^{m_1}+|\beta|^{m_2}+|\beta|^{m_3}+|\beta|^{m_4}+\alpha^{m_2}+\alpha^{m_3}+\alpha^{m_4}\right)\\&<1+\frac{4}{\sqrt{5}}+\frac{3}{\sqrt{5}}\alpha^{m_2}\\&\le\frac{1}{\sqrt{5}}(\sqrt{5}+7)\alpha^{m_2},\end{aligned}$$

which means that

$$\left|\frac{\alpha^{m_1}}{\sqrt{5}} - \frac{d \times 10^n}{9}\right| < \frac{\alpha^{m_2+5}}{\sqrt{5}}. \tag{29}$$

Multiplying both sides of (29) by $\sqrt{5}\alpha^{-m_1}$ yields

$$\left|1 - \alpha^{-m_1} 10^n \left(\frac{d\sqrt{5}}{9}\right)\right| < \alpha^{m_2-m_1+5}. \tag{30}$$

Put

$$\Gamma_4 := 1 - \alpha^{-m_1} 10^n \left(\frac{d\sqrt{5}}{9}\right). \tag{31}$$

Suppose that $\Gamma_4 = 0$. Then

$$\alpha^{m_1} = \frac{d\sqrt{5} \times 10^n}{9},$$

which implies that

$$\beta^{m_1} = -\frac{d\sqrt{5} \times 10^n}{9}.$$

Consequently,

$$\frac{\sqrt{5} \times 10^{125}}{9} \leq \frac{d\sqrt{5} \times 10^n}{9} = |\beta|^{m_1} < 1,$$

which is impossible. Hence, $\Gamma_4 \neq 0$. In order to apply Lemma 3 to Γ_4 given by (31), we take

$$\alpha_1 = \alpha, \quad \alpha_2 = 10, \quad \alpha_3 = \frac{\sqrt{5}d}{9}, \quad b_1 = -m_1, \quad b_2 = n, \quad b_3 = 1,$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}\left(\sqrt{5}\right)$ and $b_1, b_2, b_3 \in \mathbb{Z}$. To compute A_3, we observe that

$$\alpha_3 = \frac{d\sqrt{5}}{9} \leq \sqrt{5} \quad \text{and} \quad \alpha_3^{-1} = \frac{9}{d\sqrt{5}} \leq \frac{9}{\sqrt{5}},$$

so $|\log \alpha_3| < 1.4$. In addition, we have

$$h(\alpha_3) \leq h(d\sqrt{5}) + h(9) \leq \frac{1}{2}\log 5 + 2h(9).$$

As a result, we have that $2h(\alpha_3) < 10.4$. We see that

$$\max\{2h(\alpha_3), |\log \alpha_3|, 0.16\} < 10.4 =: A_3.$$

By applying Lemma 3 to Γ_4 given by (31) and using (30), we obtain

$$\exp(-10.4C_1(1 + \log m_1)) < \left|1 - \alpha^{-m_1} 10^n \left(\frac{d\sqrt{5}}{9}\right)\right| < \alpha^{m_2-m_1+5}.$$

This means that

$$(m_1 - m_2)\log\alpha < 5\log\alpha + 10.4C_1(1+\log m_1) < 10.5C_1(1+\log m_1). \tag{32}$$

Putting together (32) and (27) yields

$$\begin{aligned}(m_1 - m_3)\log\alpha &< 5\log\alpha + (12 + 10.5C_1(1+\log m_1))C_1(1+\log m_1)\\ &< 5\log\alpha + 10.6C_1^2(1+\log m_1)^2\\ &< 10.7C_1^2(1+\log m_1)^2.\end{aligned}$$

That is

$$(m_1 - m_3)\log\alpha < 10.7C_1^2(1+\log m_1)^2. \tag{33}$$

Combining (33) and (22), we obtain

$$\begin{aligned}(m_1 - m_4)\log\alpha &< 5\log\alpha + \left(14 + 10.7C_1^2(1+\log m_1)^2\right)C_1(1+\log m_1)\\ &< 5\log\alpha + 10.8C_1^3(1+\log m_1)^3\\ &< 10.9C_1^3(1+\log m_1)^3.\end{aligned}$$

That is

$$(m_1 - m_4)\log\alpha < 10.9C_1^3(1+\log m_1)^3. \tag{34}$$

We now combine (34) and (17) to obtain

$$\begin{aligned}m_1\log\alpha &< 4\log\alpha + (15 + 10.9C_1^3(1+\log m_1)^3)C_1(1+\log m_1)\\ &< 4\log\alpha + 11.0C_1^4(1+\log m_1)^4\\ &< 11.1C_1^4(1+\log m_1)^4\\ &< 11.1\left(2.3\times 10^{12}\right)^4(1+\log m_1)^4.\end{aligned}$$

That is

$$m_1\log\alpha < 11.1\left(2.3\times 10^{12}\right)^4(1+\log m_1)^4. \tag{35}$$

Inequality (35) gives rise to the inequality $m_1 < 2.3\times 10^{59}$. Now, we lower the bound.

Let

$$\Lambda_1 = -m_1\log\alpha + n\log 10 + \log\left(\frac{d\sqrt{5}}{9}\right). \tag{36}$$

Equation (28) leads us to

$$\begin{aligned}
\frac{\alpha^{m_1}}{\sqrt{5}} - \frac{d \times 10^n}{9} &= \frac{\alpha^{m_1}}{\sqrt{5}} \left(1 - \alpha^{-m_1} 10^n \left(\frac{d\sqrt{5}}{9}\right)\right) \\
&= \frac{\alpha^{m_1}}{\sqrt{5}} \left(1 - e^{\Lambda_1}\right) \\
&= -\frac{d}{9} + \frac{\beta^{m_1}}{\sqrt{5}} - F_{m_2} - F_{m_3} - F_{m_4} \\
&\le -\frac{1}{9} + \frac{|\beta|^{600}}{\sqrt{5}} \\
&< 0,
\end{aligned}$$

as $m_1 \ge 600$. Thus, $\Lambda_1 > 0$ and so from (30) we obtain

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = \left|1 - \alpha^{-m_1} 10^n \left(\frac{d\sqrt{5}}{9}\right)\right| < \alpha^{m_2 - m_1 + 5}.$$

This means that

$$\begin{aligned}
\log\left(\frac{d\sqrt{5}}{9}\right) + m_1(-\log\alpha) + n\log 10 &< \alpha^5 \alpha^{-(m_1 - m_2)} \\
&< \alpha^{5.1} \exp(-0.48(m_1 - m_2)),
\end{aligned}$$

which leads to

$$|\Lambda_1| < \alpha^{5.1} \exp(-0.48(m_1 - m_2)), \tag{37}$$

with $X = \max\{m_1, n\} = m_1 \le 2.3 \times 10^{59}$. We also have that

$$\frac{\Lambda_1}{\log 10} = \frac{\log(d\sqrt{5}/9)}{\log 10} - m_1 \frac{\log\alpha}{\log 10} + n.$$

Thus, we take

$$c = \alpha^{5.1}, \quad \delta = 0.48, \quad X_0 = 2.3 \times 10^{59}, \quad \psi = \frac{\log(d\sqrt{5}/9)}{\log 10}, \quad Y = m_1 - m_2,$$

$$\vartheta = \frac{\log\alpha}{\log 10}, \quad \vartheta_1 = -\log\alpha, \quad \vartheta_2 = \log 10, \quad \beta = \log(d\sqrt{5}/9).$$

The smallest value of q such that $q > X_0$ is $q = q_{125}$. We find that $q = q_{128}$ satisfies the hypothesis of Lemma 6 for $d = 1, \ldots, 9$. Applying Lemma 6, we get $m_1 - m_2 \le 310$, and hence $m_2 \ge 290$.

Taking $1 \le d \le 9$ and $0 \le m_1 - m_2 \le 310$, we let

$$\Lambda_2 = -m_2 \log\alpha + n\log 10 + \log\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\right). \tag{38}$$

We see from equation (23) that

$$\begin{aligned}\frac{\alpha^{m_1}}{\sqrt{5}}(1+\alpha^{m_2-m_1})\left(1-e^{\Lambda_2}\right) &= -\frac{d}{9} + \frac{\beta^{m_1}}{\sqrt{5}} + \frac{\beta^{m_2}}{\sqrt{5}} - F_3 - F_4 \\ &\le -\frac{1}{9} + \frac{|\beta|^{600}}{\sqrt{5}} + \frac{|\beta|^{290}}{\sqrt{5}} \\ &< 0,\end{aligned}$$

making use of $m_1 \ge 600$ and $m_2 \ge 290$. Hence, $\Lambda_2 > 0$, and so from (25) we see that

$$0 < \Lambda_2 < e^{\Lambda_2} - 1 = \left|1 - \alpha^{-m_2} 10^n \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\right)\right| < \alpha^{m_3-m_1+5}.$$

Thus, we have

$$\begin{aligned}\log\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\right) + m_2(-\log\alpha) + n\log 10 &< \alpha^{m_3-m_1+5} \\ &< \alpha^{5.1}\exp(-0.48(m_1-m_3)),\end{aligned}$$

which gives us

$$|\Lambda_2| < \alpha^{5.1}\exp(-0.48(m_1-m_3)), \tag{39}$$

where $X = \max\{m_2, n\} \le m_1 \le 2.3 \times 10^{59}$. We also have that

$$\frac{\Lambda_2}{\log 10} = \frac{1}{\log 10}\log\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\right) - m_1\frac{\log\alpha}{\log 10} + n.$$

Thus, we consider

$$c = \alpha^{5.1}, \quad \delta = 0.48, \quad X_0 = 2.3 \times 10^{59}, \quad \psi = \frac{1}{\log 10}\log\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\right),$$

$$Y = m_1 - m_3, \quad \vartheta = \frac{\log\alpha}{\log 10}, \quad \vartheta_1 = -\log\alpha, \quad \vartheta_2 = \log 10, \quad \beta = \log\left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_2}+1)}\right).$$

We find that $q = q_{132}$ satisfies the hypothesis of Lemma 6 for $d = 1, \dots, 9$ and $0 \le m_1 - m_2 \le 310$. Applying Lemma 6, we get $m_1 - m_3 \le 328$. Hence, $m_3 \ge 272$.

Taking $1 \leq d \leq 9$, $0 \leq m_2 - m_3 \leq m_1 - m_3 \leq 328$, we let

$$\Lambda_3 = -m_3 \log \alpha + n \log 10 + \log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)} \right). \tag{40}$$

Equation (18) ensures that

$$\begin{aligned} \frac{\alpha^{m_1}}{\sqrt{5}} \left(1 + \alpha^{m_2-m_1} + \alpha^{m_3-m_1}\right) \left(1 - e^{\Lambda_3}\right) &= -\frac{d}{9} + \frac{1}{\sqrt{5}} \left(\beta^{m_1} + \beta^{m_2} + \beta^{m_3}\right) - F_4 \\ &\leq -\frac{1}{9} + \frac{1}{\sqrt{5}} \left(|\beta|^{600} + |\beta|^{290} + |\beta|^{272}\right) \\ &< 0, \end{aligned}$$

where we use $m_1 \geq 600$, $m_2 \geq 290$, and $m_3 \geq 272$. Hence, $\Lambda_3 > 0$, and so from (20) we see that

$$0 < \Lambda_3 < e^{\Lambda_3} - 1 = \left| 1 - \alpha^{-m_3} 10^n \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)} \right) \right| < \alpha^{m_4-m_1+5}.$$

Hence, we have

$$\begin{aligned} \log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)} \right) + m_3(-\log \alpha) + n \log 10 &< \alpha^{m_4-m_1+5} \\ &< \alpha^{5.1} \exp(-0.48(m_1 - m_4)), \end{aligned}$$

leading to

$$|\Lambda_3| < \alpha^{5.1} \exp(-0.48(m_1 - m_4)), \tag{41}$$

where $X = \max\{m_3, n\} \leq m_1 \leq 2.3 \times 10^{59}$. Furthermore, we obtain

$$\frac{\Lambda_3}{\log 10} = \frac{1}{\log 10} \log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)} \right) - m_3 \frac{\log \alpha}{\log 10} + n.$$

Thus, we take

$$c = \alpha^{5.1}, \quad \delta = 0.48, \quad X_0 = 2.3 \times 10^{59}, \quad \psi = \frac{1}{\log 10} \log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)} \right),$$

$$Y = m_1 - m_4, \quad \vartheta = \frac{\log \alpha}{\log 10}, \quad \vartheta_1 = -\log \alpha, \quad \vartheta_2 = \log 10,$$

$$\beta = \log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)} \right).$$

We find that $q = q_{135}$ satisfies the hypothesis of Lemma 6 for $1 \le d \le 9$, $0 \le m_2 - m_3 \le m_1 - m_3 \le 328$. Applying Lemma 6, we get $m_1 - m_4 \le 335$, and hence $m_4 \ge 265$.

Taking $1 \le d \le 9$, $0 \le m_3 - m_4 \le m_2 - m_4 \le m_1 - m_4 \le 335$, we let

$$\Lambda_4 = -m_4 \log \alpha + n \log 10 + \log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)} \right). \tag{42}$$

Using equation (13), we have that

$$\begin{aligned} \frac{\alpha^{m_1}}{\sqrt{5}} \left(1 + \alpha^{m_2-m_1} + \alpha^{m_3-m_1} + \alpha^{m_4-m_1}\right) \left(1 - e^{\Lambda_4}\right) &= -\frac{d}{9} + \frac{1}{\sqrt{5}} \left(\beta^{m_1} + \beta^{m_2} + \beta^{m_3} + \beta^{m_4}\right) \\ &\le -\frac{1}{9} + \frac{1}{\sqrt{5}} \left(|\beta|^{600} + |\beta|^{290} + |\beta|^{272} + |\beta|\right) \\ &< 0 \end{aligned}$$

making use of $m_1 \ge 600$, $m_2 \ge 290$, $m_3 \ge 272$ and $m_4 \ge 265$. Hence, $\Lambda_4 > 0$, and so from (15) we see that

$$0 < \Lambda_4 < e^{\Lambda_4} - 1 = \left| 1 - \alpha^{-m_4} 10^n \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)} \right) \right| < \alpha^{4-m_1}.$$

Hence, we have

$$\log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)} \right) + m_4(-\log \alpha) + n \log 10 < \alpha^{4-m_1},$$

which implies that

$$|\Lambda_4| < \alpha^{4.1} \exp(-0.48 m_1), \tag{43}$$

where $X = \max\{m_4, n\} \le m_1 < 2.3 \times 10^{59}$. In addition,

$$\frac{\Lambda_4}{\log 10} = \frac{1}{\log 10} \log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)} \right) - m_4 \frac{\log \alpha}{\log 10} + n.$$

Thus,

$$c = \alpha^{4.1}, \quad \delta = 0.48, \quad X_0 = 2.3 \times 10^{59}, \quad Y = m_1$$

$$\psi = \frac{1}{\log 10} \log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)} \right), \quad \vartheta = \frac{\log \alpha}{\log 10},$$

$$\vartheta_1 = -\log \alpha, \ \vartheta_2 = \log 10, \ \beta = \log \left(\frac{d\sqrt{5}}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)} \right).$$

We find that $q = q_{141}$ satisfies the hypothesis of Lemma 6 for $1 \le d \le 9$, $0 \le m_3 - m_4 \le m_2 - m_4 \le m_1 - m_4 \le 335$. Applying Lemma 6, we get $m_1 \le 392$, which contradicts the assumption that $m_1 \ge 600$. This proves the result.

4 Proof of Theorem 2

Although this is similar to the proof of the previous theorem, we give the details for the convenience of the reader. We use the fact that

$$L_m = \alpha^m + \beta^m \quad \text{holds for all} \quad m \geq 0, \quad \text{where} \ (\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right).$$

In equation (2), we suppose that $m_1 \geq m_2 \geq m_3 \geq m_4$. A search with Maple in the range $0 \leq m_1 \leq 599$ yielded only the solutions shown in the statement of Theorem 2.

Let us suppose that solutions of equation (2) exist for $m_1 \geq 600$. We observe that

$$L_{600} \leq L_{m_1} \leq L_{m_1} + L_{m_2} + L_{m_3} + L_{m_4} = d\left(\frac{10^n - 1}{9}\right) \leq 10^n - 1.$$

This leads us to

$$125 \leq \frac{\log(1 + L_{600})}{\log 10} \leq n,$$

and so $n \geq 125$. We further observe that

$$10^{n-1} \leq d\left(\frac{10^n - 1}{9}\right) = L_{m_1} + L_{m_2} + L_{m_3} + L_{m_4} \leq 4\left(\alpha^{m_1} + |\beta|^{m_1}\right) < \alpha^{m_1 + 4.33}.$$

Hence, we obtain

$$4.78(n-1) < (n-1)\frac{\log 10}{\log \alpha} < m_1 + 4.33,$$

which gives us

$$n < 4.78n - 9.11 < m_1,$$

for $n \geq 125$. Therefore, $125 \leq n < m_1$.

We can put equation (2) in the form

$$\alpha^{m_1} + \beta^{m_1} + \alpha^{m_2} + \beta^{m_2} + \alpha^{m_3} + \beta^{m_3} + \alpha^{m_4} + \beta^{m_4} - \frac{d \times 10^n}{9} = -\frac{d}{9}. \tag{44}$$

Equation(44) is treated in four different ways in the steps that follow.

Step 1: We express (44) in the form

$$\alpha^{m_1}\left(1 + \alpha^{m_2 - m_1} + \alpha^{m_3 - m_1} + \alpha^{m_4 - m_1}\right) - \frac{d \times 10^n}{9} = -\frac{d}{9} - \left(\beta^{m_1} + \beta^{m_2} + \beta^{m_3} + \beta^{m_4}\right). \tag{45}$$

It follows that

$$\left|\alpha^{m_1}\left(1 + \alpha^{m_2 - m_1} + \alpha^{m_3 - m_1} + \alpha^{m_4 - m_1}\right) - \frac{d \times 10^n}{9}\right| \leq \frac{d}{9} + \left(|\beta|^{m_1} + |\beta|^{m_2} + |\beta|^{m_3} + |\beta|^{m_4}\right),$$

leading to

$$\left|\alpha^{m_1}\left(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}+\alpha^{m_4-m_1}\right)-\frac{d\times 10^n}{9}\right|<\alpha^{3.35}. \tag{46}$$

Multiplication of both sides of (46) by $\frac{\alpha^{-m_1}}{1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}+\alpha^{m_4-m_1}}$ gives us

$$\left|1-\alpha^{-m_4}10^n\left(\frac{d}{9(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}\right)\right|<\frac{\alpha^{3.35-m_1}}{1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}+\alpha^{m_4-m_1}},$$

from which we get

$$\left|1-\alpha^{-m_4}10^n\left(\frac{d}{9(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}\right)\right|<\alpha^{3.35-m_1}. \tag{47}$$

Put

$$\Gamma_1:=1-\alpha^{-m_4}10^n\left(\frac{d}{9(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}\right). \tag{48}$$

Suppose that $\Gamma_1=0$. Then, we have that

$$\alpha^{m_1}+\alpha^{m_2}+\alpha^{m_3}+\alpha^{m_4}=\frac{10^n\times d}{9}.$$

Conjugating in $\mathbb{Q}\left(\sqrt{5}\right)$ yields

$$\beta^{m_1}+\beta^{m_2}+\beta^{m_3}+\beta^{m_4}=\frac{10^n\times d}{9}.$$

Thus,

$$\frac{10^{125}}{9}\leq\frac{10^n\times d}{9}=|\beta^{m_1}+\beta^{m_2}+\beta^{m_3}\beta^{m_4}|\leq|\beta|^{m_1}+|\beta|^{m_2}+|\beta|^{m_3}+|\beta|^{m_4}<4.$$

This implies that $\frac{10^{125}}{9}<4$, which is false. Hence, it follows that $\Gamma_1\neq 0$.

In the notation of Lemma 3, we set

$$\alpha_1=\alpha,\ \ \alpha_2=10,\ \ \alpha_3=\frac{d}{9(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)},$$

$$b_1=-m_4,\ \ b_2=n,\ \ b_3=1,$$

where $\alpha_1,\alpha_2,\alpha_3\in\mathbb{Q}\left(\sqrt{5}\right)$ and $b_1,b_2,b_3\in\mathbb{Z}$. We get $B=\max\{m_4,n,1\}\leq m_1$. The minimal polynomial of α over $\mathbb{Z}$ is x^2-x-1, and so $d(\alpha)=2$ and $a_0(\alpha)=1$. It is known that

$$h(\alpha)=\frac{1}{2}\log\alpha.$$

We have

$$\max\{2h(\alpha_1),|\log\alpha_1|,0.16\}=\log\alpha<0.49=:A_1,$$

$$\max\{2h(\alpha_2), |\log\alpha_2|, 0.16\} = 2\log 10 < 4.61 =: A_2.$$

Set

$$C_1 = 2.4\times 10^{12} > 1.4\times 30^6\times 3^{4.5}\times D^2\times(1+\log D)\times A_1\times A_2.$$

Next, we compute A_3. We find that,

$$\alpha_3 = \frac{d}{9(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)} < 1,$$

and

$$\alpha_3^{-1} = \frac{9(\alpha^{m_1-m_4}+\alpha^{m_2-m_4}+\alpha^{m_3-m_4}+1)}{d} \le 36\alpha^{m_1-m_4},$$

hence, $|\log\alpha_3| < 4+(m_1-m_4)\log\alpha$. Also, we have that

$$\begin{aligned}
h(\alpha_3) &\le h(d)+h(9)+\log 2+h(\alpha^{m_3-m_4}(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1))\\
&\le 2h(9)+2\log 2+h(\alpha^{m_3-m_4})+h(\alpha^{m_2-m_3}(\alpha^{m_1-m_2}+1))\\
&\le 2h(9)+3\log 2+h(\alpha^{m_3-m_4})+h(\alpha^{m_2-m_3})+h(\alpha^{m_1-m_2})\\
&\le 2h(9)+3\log 2+(m_3-m_4)h(\alpha)+(m_2-m_3)h(\alpha)+(m_1-m_2)h(\alpha)\\
&= 2h(9)+3\log 2+\frac{1}{2}(m_1-m_4)\log\alpha.
\end{aligned}$$

Hence, $2h(\alpha_3)\le 13+(m_1-m_4)\log\alpha$. Therefore, we get

$$\max\{2h(\alpha_3), |\log\alpha_3|, 0.16\} \le 13+(m_1-m_4)\log\alpha =: A_3.$$

By applying Lemma 3 to Γ_1 given by (48), and using (47) we have that

$$\exp(-(13+(m_1-m_4)\log\alpha)C_1(1+\log m_1)) < \alpha^{3.35-m_1}.$$

Thus,

$$m_1\log\alpha < 3.35\log\alpha+(13+(m_1-m_4)\log\alpha)C_1(1+\log m_1). \tag{49}$$

Step 2: Writing equation (44) as

$$\alpha^{m_1}\left(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}\right)-\frac{d\times 10^n}{9} = -\frac{d}{9}-\alpha^{m_4}-(\beta^{m_1}+\beta^{m_2}+\beta^{m_3}+\beta^{m_4}), \tag{50}$$

we get

$$\left|\alpha^{m_1}\left(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}\right)-\frac{d\times 10^n}{9}\right| \le \frac{d}{9}+\alpha^{m_4}+|\beta|^{m_1}+|\beta|^{m_2}+|\beta|^{m_3}+|\beta|^{m_4},$$

and so

$$\left|\alpha^{m_1}\left(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}\right)-\frac{d\times 10^n}{9}\right| < \alpha^{m_4+3.73}. \tag{51}$$

By multiplying both sides of inequality (51) by $\frac{\alpha^{-m_1}}{1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}}$ we obtain

$$\left|1-\alpha^{-m_3}10^n\left(\frac{d}{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}\right)\right|<\frac{\alpha^{m_4-m_1+3.73}}{1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}},$$

which leads to

$$\left|1-\alpha^{-m_3}10^n\left(\frac{d}{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}\right)\right|<\alpha^{m_4-m_1+3.73}. \tag{52}$$

Put

$$\Gamma_2:=1-\alpha^{-m_3}10^n\left(\frac{d}{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}\right). \tag{53}$$

Suppose that $\Gamma_2=0$. Then, we get

$$\alpha^{m_1}+\alpha^{m_2}+\alpha^{m_3}=\frac{10^n\times d}{9}.$$

Taking the conjugate of this in $\mathbb{Q}(\sqrt{5})$, we get

$$\beta^{m_1}+\beta^{m_2}+\beta^{m_3}=\frac{10^n\times d}{9},$$

which implies that

$$\frac{10^{125}}{9}\leq\frac{10^n\times d}{9}=|\beta^{m_1}+\beta^{m_2}+\beta^{m_3}|\leq|\beta|^{m_1}+|\beta|^{m_2}+|\beta|^{m_3}<3.$$

Thus, $\frac{10^{125}}{9}<3$, which is false. We conclude that $\Gamma_2\neq 0$.

To apply Lemma 3 to Γ_2 given by (53), we set

$$\alpha_1=\alpha,\quad \alpha_2=10,\quad \alpha_3=\frac{d}{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)},\quad b_1=-m_3,\quad b_2=n,\quad b_3=1,$$

where $\alpha_1,\alpha_2,\alpha_3\in\mathbb{Q}\left(\sqrt{5}\right)$ and $b_1,b_2,b_3\in\mathbb{Z}$. Also, we obtain $B=\max\{m_3,n,1\}\leq m_1$. We proceed to compute A_3 by first observing that

$$\alpha_3=\frac{d}{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}<1,$$

and

$$\alpha_3^{-1}=\frac{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}{d}\leq 27\alpha^{m_1-m_3}.$$

Hence, $|\log\alpha_3|<4+(m_1-m_3)\log\alpha$. Additionally, we get

$$\begin{aligned}h(\alpha_3)&\leq h(d)+h(9)+\log 2+h(\alpha^{m_2-m_3}(\alpha^{m_1-m_2}+1))\\&\leq 2h(9)+2\log 2+h(\alpha^{m_2-m_3})+h(\alpha^{m_1-m_2})\\&\leq 2h(9)+2\log 2+(m_2-m_3)h(\alpha)+(m_1-m_2)h(\alpha)\\&=2h(9)+2\log 2+\frac{1}{2}(m_1-m_3)\log\alpha.\end{aligned}$$

Hence, $2h(\alpha_3) \leq 12 + (m_1 - m_3)\log\alpha$. As a result, we find that

$$\max\{2h(\alpha_3), |\log\alpha_3|, 0.16\} \leq 12 + (m_1 - m_3)\log\alpha =: A_3.$$

By applying Lemma 3 to Γ_2 given by (53) and using (52), we deduce that

$$\exp(-(12 + (m_1 - m_3)\log\alpha)C_1(1 + \log m_1)) < \alpha^{m_4 - m_1 + 3.73}.$$

Thus, we get

$$(m_1 - m_4)\log\alpha < 3.73\log\alpha + (12 + (m_1 - m_3)\log\alpha)C_1(1 + \log m_1). \tag{54}$$

Step 3: Writing (44) as

$$\alpha^{m_1}\left(1 + \alpha^{m_2 - m_1}\right) - \frac{d \times 10^n}{9} = -\frac{d}{9} - (\beta^{m_1} + \beta^{m_2} + \beta^{m_3} + \beta^{m_4}) - (\alpha^{m_3} + \alpha^{m_4}), \tag{55}$$

gives us

$$\left|\alpha^{m_1}\left(1 + \alpha^{m_2 - m_1}\right) - \frac{d \times 10^n}{9}\right| \leq \frac{d}{9} + |\beta|^{m_1} + |\beta|^{m_2} + |\beta|^{m_3} + |\beta|^{m_4} + \alpha^{m_3} + \alpha^{m_4} \leq 7\alpha^{m_3},$$

which leads to

$$\left|\alpha^{m_1}\left(1 + \alpha^{m_2 - m_1}\right) - \frac{d \times 10^n}{9}\right| < \alpha^{m_3 + 4.05}. \tag{56}$$

Multiplying both sides of (56) by $\frac{\alpha^{-m_1}}{1+\alpha^{m_2-m_1}}$ gives us

$$\left|1 - \alpha^{-m_2}10^n\left(\frac{d}{9(\alpha^{m_1 - m_2} + 1)}\right)\right| < \frac{\alpha^{m_3 - m_1 + 4.05}}{1 + \alpha^{m_2 - m_1}},$$

which yields

$$\left|1 - \alpha^{-m_2}10^n\left(\frac{d}{9(\alpha^{m_1 - m_2} + 1)}\right)\right| < \alpha^{m_3 - m_1 + 4.05}. \tag{57}$$

Put

$$\Gamma_3 := 1 - \alpha^{-m_2}10^n\left(\frac{d}{9(\alpha^{m_1 - m_2} + 1)}\right). \tag{58}$$

Suppose that $\Gamma_3 = 0$. Then

$$\alpha^{m_1} + \alpha^{m_2} = \frac{10^n \times d}{9},$$

giving us

$$\beta^{m_1} + \beta^{m_2} = \frac{10^n \times d}{9}$$

by conjugating in $\mathbb{Q}\left(\sqrt{5}\right)$. We see that

$$\frac{10^{125}}{9} \leq \frac{10^n \times d}{9} = |\beta^{m_1} + \beta^{m_2}| \leq |\beta|^{m_1} + |\beta|^{m_2} < 2,$$

which is false. Hence, $\Gamma_3 \neq 0$. Using the notations in Lemma 3, we put

$$\alpha_1 = \alpha, \quad \alpha_2 = 10, \quad \alpha_3 = \frac{d}{9(\alpha^{m_1-m_2}+1)}, \quad b_1 = -m_2, \quad b_2 = n, \quad b_3 = 1,$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}\left(\sqrt{5}\right)$ and $b_1, b_2, b_3 \in \mathbb{Z}$. We get $B = \max\{m_2, n, 1\} \leq m_1$. It is easily seen that

$$\alpha_3 = \frac{d}{9(\alpha^{m_1-m_2}+1)} \leq 1 \text{ and } \alpha_3^{-1} = \frac{9(\alpha^{m_1-m_2}+1)}{d} \leq 18\alpha^{m_1-m_2}.$$

So $|\log\alpha_3| < 3 + (m_1 - m_2)\log\alpha$. Additionally, we have

$$\begin{aligned} h(\alpha_3) &\leq h(d) + h(9) + \log 2 + h(\alpha^{m_1-m_2}) \\ &\leq 2h(9) + \log 2 + (m_1 - m_2)h(\alpha) \\ &= 2h(9) + \log 2 + \frac{1}{2}(m_1 - m_2)\log\alpha. \end{aligned}$$

Thus, $2h(\alpha_3) \leq 11 + (m_1 - m_2)\log\alpha$ and so

$$\max\{2h(\alpha_3), |\log\alpha_3|, 0.16\} < 11 + (m_1 - m_2)\log\alpha =: A_3.$$

Applying Lemma 3 to Γ_3 given by (58), and using (57) we produce

$$\exp(-(11 + (m_1 - m_2)\log\alpha)C_1(1 + \log m_1)) < \alpha^{m_3-m_1+4.05},$$

from which we obtain

$$(m_1 - m_3)\log\alpha < 4.05\log\alpha + (11 + (m_1 - m_2)\log\alpha)C_1(1 + \log m_1). \tag{59}$$

Step 4: Writing equation (44) as

$$\alpha^{m_1} - \frac{d \times 10^n}{9} = -\frac{d}{9} - (\beta^{m_1} + \beta^{m_2} + \beta^{m_3} + \beta^{m_4}) - (\alpha^{m_2} + \alpha^{m_3} + \alpha^{m_4}), \tag{60}$$

we get

$$\left|\alpha^{m_1} - \frac{d \times 10^n}{9}\right| \leq \frac{d}{9} + |\beta|^{m_1} + |\beta|^{m_2} + |\beta|^{m_3} + |\beta|^{m_4} + \alpha^{m_2} + \alpha^{m_3} + \alpha^{m_4} \leq 8\alpha^{m_2},$$

which means that

$$\left|\alpha^{m_1} - \frac{d \times 10^n}{9}\right| < \alpha^{m_2+4.33}. \tag{61}$$

Multiplying both sides of (61) by α^{-m_1} yields

$$\left|1 - \alpha^{-m_1}10^n\left(\frac{d}{9}\right)\right| < \alpha^{m_2-m_1+4.33}. \tag{62}$$

Put

$$\Gamma_4 := 1 - \alpha^{-m_1} 10^n \left(\frac{d}{9}\right). \tag{63}$$

Suppose that $\Gamma_4 = 0$. Then

$$\alpha^{m_1} = \frac{d \times 10^n}{9},$$

and by conjugation

$$\beta^{m_1} = \frac{d \times 10^n}{9}.$$

Consequently,

$$\frac{10^{125}}{9} \leq \frac{d \times 10^n}{9} = |\beta|^{m_1} < 1,$$

which is impossible. Hence, $\Gamma_4 \neq 0$. In order to apply Lemma 3 to Γ_4 given by (63), we take

$$\alpha_1 = \alpha, \quad \alpha_2 = 10, \quad \alpha_3 = \frac{d}{9}, \quad b_1 = -m_1, \quad b_2 = n, \quad b_3 = 1,$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}\left(\sqrt{5}\right)$ and $b_1, b_2, b_3 \in \mathbb{Z}$. To compute A_3, we observe that

$$\alpha_3 = \frac{d}{9} \leq 1 \quad \text{and} \quad \alpha_3^{-1} = \frac{9}{d} \leq 9,$$

so $|\log \alpha_3| < 2.2$. In addition, we have

$$h(\alpha_3) \leq h(d) + h(9) \leq 2h(9).$$

This gives us $2h(\alpha_3) < 8.79$. And so we have

$$\max\{2h(\alpha_3), |\log \alpha_3|, 0.16\} < 8.79 =: A_3.$$

By applying Lemma 3 to Γ_4 given by (63) and using (62), we obtain

$$\exp(-8.79 C_1 (1 + \log m_1)) < \left|1 - \alpha^{-m_1} 10^n \left(\frac{d}{9}\right)\right| < \alpha^{m_2 - m_1 + 4.33}.$$

This means that

$$(m_1 - m_2) \log \alpha < 4.33 \log \alpha + 8.79 C_1 (1 + \log m_1) < 8.80 C_1 (1 + \log m_1). \tag{64}$$

Putting together (64) and (59) yields

$$\begin{aligned}(m_1 - m_3) \log \alpha &< 4.05 \log \alpha + (11 + 8.80 C_1 (1 + \log m_1)) C_1 (1 + \log m_1) \\ &= 4.05 \log \alpha + 11 C_1 (1 + \log m_1) + 8.80 C_1^2 (1 + \log m_1)^2 \\ &< 8.81 C_1^2 (1 + \log m_1)^2,\end{aligned}$$

since $4.05\log\alpha + 11C_1(1+\log m_1) < 0.01C_1^2(1+\log m_1)^2$. Hence,

$$(m_1 - m_3)\log\alpha < 8.81C_1^2(1+\log m_1)^2. \tag{65}$$

Combining (65) and (54), we obtain

$$\begin{aligned}(m_1 - m_4)\log\alpha &< 3.73\log\alpha + \left(12 + 8.81C_1^2(1+\log m_1)^2\right)C_1(1+\log m_1)\\ &= 3.73\log\alpha + 12C_1(1+\log m_1) + 8.81C_1^3(1+\log m_1)^3\\ &< 8.82C_1^3(1+\log m_1)^3.\end{aligned}$$

since $3.73\log\alpha + 12C_1(1+\log m_1) < 0.01C_1^3(1+\log m_1)^3$. Thus,

$$(m_1 - m_4)\log\alpha < 8.82C_1^3(1+\log m_1)^3. \tag{66}$$

We now combine (66) and (49) to obtain

$$\begin{aligned}m_1\log\alpha &< 3.35\log\alpha + (13 + 8.82C_1^3(1+\log m_1)^3)C_1(1+\log m_1)\\ &= 3.35\log\alpha + 13C_1(1+\log m_1) + 8.82C_1^4(1+\log m_1)^4\\ &< 8.83C_1^4(1+\log m_1)^4\\ &< 8.83\left(2.4\times 10^{12}\right)^4(1+\log m_1)^4.\end{aligned}$$

That is

$$m_1\log\alpha < 8.83\left(2.4\times 10^{12}\right)^4(1+\log m_1)^4. \tag{67}$$

Inequality (67) gives rise to the inequality $m_1 < 2.2\times 10^{59}$. Now, we need to lower the bound.

Let

$$\Lambda_1 = -m_1\log\alpha + n\log 10 + \log\left(\frac{d}{9}\right). \tag{68}$$

Making use of equation (60), we have that

$$\begin{aligned}\alpha^{m_1} - \frac{d\times 10^n}{9} &= \alpha^{m_1}\left(1 - \alpha^{-m_1}10^n\left(\frac{d}{9}\right)\right) = \alpha^{m_1}\left(1 - e^{\Lambda_1}\right)\\ &= -\frac{d}{9} - \beta^{m_1} - L_{m_2} - L_{m_3} - L_{m_4}\\ &\le -\frac{1}{9} + |\beta|^{600}\\ &< 0,\end{aligned}$$

as $m_1 \ge 600$. Thus, $\Lambda_1 > 0$ and so from (62) we obtain

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = \left|1 - \alpha^{-m_1}10^n\left(\frac{d}{9}\right)\right| < \alpha^{m_2 - m_1 + 4.33}.$$

This means that

$$\log\left(\frac{d}{9}\right) + m_1(-\log\alpha) + n\log 10 < \alpha^{4.33}\alpha^{-(m_1-m_2)} < \alpha^{4.34}\exp(-0.48(m_1-m_2)),$$

which leads to

$$|\Lambda_1| < \alpha^{4.34}\exp(-0.48(m_1-m_2)), \tag{69}$$

where $X = \max\{m_1, n\} = m_1 \leq 2.2\times 10^{59}$. We also have that

$$\frac{\Lambda_1}{\log 10} = \frac{\log(d/9)}{\log 10} - m_1\frac{\log\alpha}{\log 10} + n.$$

Hence, we set

$$c = \alpha^{4.34},\ \ \delta = 0.48,\ \ X_0 = 2.2\times 10^{59},\ \ \psi = \frac{\log(d/9)}{\log 10},\ \ Y = m_1 - m_2$$

$$\vartheta = \frac{\log\alpha}{\log 10},\ \ \vartheta_1 = -\log\alpha,\ \ \vartheta_2 = \log 10,\ \ \beta = \log(d/9).$$

When $d = 9$, $\beta = 0$. Substituting $X_0 = 2.2\times 10^{59}$ into inequality (8) yields $0 \leq k \leq 284$. In the notation of Lemma 5 we find that $A = a_{138} = 770$, from the continued fraction expansion of $\frac{\log\alpha}{\log 10}$. Applying Lemma 5, we get $m_1 - m_2 \leq 301$. We now consider the case $\beta \neq 0$. The smallest value of q such that $q > X_0$ is $q = q_{125}$. We find that $q = q_{127}$ satisfies the hypothesis of Lemma 6 for $d = 1, \ldots, 8$. Applying Lemma 6, we get $m_1 - m_2 \leq 309$. We see that $m_1 - m_2 \leq 309$ for $d = 1, \ldots, 9$ and hence $m_2 \geq 291$.

Taking $1 \leq d \leq 9$ and $0 \leq m_1 - m_2 \leq 309$, we let

$$\Lambda_2 = -m_2\log\alpha + n\log 10 + \log\left(\frac{d}{9(\alpha^{m_1-m_2}+1)}\right). \tag{70}$$

We use equation (55) to arrive at

$$\begin{aligned}\alpha^{m_1}(1+\alpha^{m_2-m_1})\left(1-e^{\Lambda_2}\right) &= -\frac{d}{9} - \beta^{m_1} - \beta^{m_2} - L_3 - L_4\\ &\leq -\frac{1}{9} + |\beta|^{600} + |\beta|^{291}\\ &< 0,\end{aligned}$$

making use of $m_1 \geq 600$ and $m_2 \geq 291$. Hence, $\Lambda_2 > 0$, and so from (57) we see that

$$0 < \Lambda_2 < e^{\Lambda_2} - 1 = \left|1 - \alpha^{-m_2}10^n\left(\frac{d}{9(\alpha^{m_1-m_2}+1)}\right)\right| < \alpha^{m_3-m_1+4.05}.$$

Thus, we have

$$\begin{aligned}\log\left(\frac{d}{9(\alpha^{m_1-m_2}+1)}\right) + m_2(-\log\alpha) + n\log 10 &< \alpha^{m_3-m_1+4.05}\\ &< \alpha^{4.06}\exp(-0.48(m_1-m_3)),\end{aligned}$$

which gives us

$$|\Lambda_2| < \alpha^{4.06} \exp(-0.48(m_1 - m_3)), \tag{71}$$

where $X = \max\{m_2, n\} \le m_1 \le 2.2 \times 10^{59}$. In addition, we have

$$\frac{\Lambda_2}{\log 10} = \frac{1}{\log 10} \log\left(\frac{d}{9(\alpha^{m_1-m_2}+1)}\right) - m_1 \frac{\log\alpha}{\log 10} + n.$$

So, we take

$$c = \alpha^{4.06}, \quad \delta = 0.48, \quad X_0 = 2.2 \times 10^{59}, \quad \psi = \frac{1}{\log 10} \log\left(\frac{d}{9(\alpha^{m_1-m_2}+1)}\right),$$

$$Y = m_1 - m_3, \quad \vartheta = \frac{\log\alpha}{\log 10}, \quad \vartheta_1 = -\log\alpha, \quad \vartheta_2 = \log 10, \quad \beta = \log\left(\frac{d}{9(\alpha^{m_1-m_2}+1)}\right).$$

We find that $q = q_{132}$ satisfies the hypothesis of Lemma 6 for $d = 1, \ldots, 9$ and $0 \le m_1 - m_2 \le 309$. Applying Lemma 6, we get $m_1 - m_3 \le 327$. Hence, $m_3 \ge 273$.

Taking $1 \le d \le 9$, $0 \le m_2 - m_3 \le m_1 - m_3 \le 327$, we let

$$\Lambda_3 = -m_3 \log\alpha + n \log 10 + \log\left(\frac{d}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}\right). \tag{72}$$

Equation (50) allows us to write

$$\begin{aligned}
\alpha^{m_1}\left(1 + \alpha^{m_2-m_1} + \alpha^{m_3-m_1}\right)\left(1 - e^{\Lambda_3}\right) &= -\frac{d}{9} - (\beta^{m_1} + \beta^{m_2} + \beta^{m_3}) - L_4 \\
&\le -\frac{1}{9} + |\beta|^{600} + |\beta|^{291} + |\beta|^{273} \\
&< 0,
\end{aligned}$$

where we use $m_1 \ge 600$, $m_2 \ge 291$, and $m_3 \ge 273$. Hence, $\Lambda_3 > 0$, and so from (52) we see that

$$0 < \Lambda_3 < e^{\Lambda_3} - 1 = \left|1 - \alpha^{-m_3} 10^n \left(\frac{d}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}\right)\right| < \alpha^{m_4-m_1+3.73}.$$

Hence, we have

$$\begin{aligned}
\log\left(\frac{d}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}\right) + m_3(-\log\alpha) + n\log 10 &< \alpha^{m_4-m_1+3.73} \\
&< \alpha^{3.74} \exp(-0.48(m_1 - m_4)),
\end{aligned}$$

leading to

$$|\Lambda_3| < \alpha^{3.74} \exp(-0.48(m_1 - m_4)), \tag{73}$$

where $X = \max\{m_3, n\} \leq m_1 \leq 2.2 \times 10^{59}$. Furthermore, we obtain

$$\frac{\Lambda_3}{\log 10} = \frac{1}{\log 10} \log\left(\frac{d}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}\right) - m_3 \frac{\log \alpha}{\log 10} + n.$$

Thus, we take

$$c = \alpha^{3.74}, \quad \delta = 0.48, \quad X_0 = 2.2 \times 10^{59}, \quad Y = m_1 - m_4,$$

$$\psi = \frac{1}{\log 10} \log\left(\frac{d}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}\right), \quad \vartheta = \frac{\log \alpha}{\log 10}, \quad \vartheta_1 = -\log \alpha,$$

$$\vartheta_2 = \log 10, \quad \beta = \log\left(\frac{d}{9(\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1)}\right).$$

We find that $q = q_{138}$ satisfies the hypothesis of Lemma 6 for $1 \leq d \leq 9$, $0 \leq m_2 - m_3 \leq m_1 - m_3 \leq 327$. Applying Lemma 6, we get $m_1 - m_4 \leq 371$, and hence $m_4 \geq 229$.

Taking $1 \leq d \leq 9$, $0 \leq m_3 - m_4 \leq m_2 - m_4 \leq m_1 - m_4 \leq 371$, we let

$$\Lambda_4 = -m_4 \log \alpha + n \log 10 + \log\left(\frac{d}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)}\right). \tag{74}$$

Using equation (45), we get

$$\begin{aligned}
\alpha^{m_1}\left(1 + \alpha^{m_2-m_1} + \alpha^{m_3-m_1} + \alpha^{m_4-m_1}\right)\left(1 - e^{\Lambda_4}\right) &= -\frac{d}{9} - \left(\beta^{m_1} + \beta^{m_2} + \beta^{m_3} + \beta^{m_4}\right) \\
&\leq -\frac{1}{9} + \left(|\beta|^{600} + |\beta|^{291} + |\beta|^{273} + |\beta|^{229}\right) \\
&< 0
\end{aligned}$$

making use of $m_1 \geq 600$, $m_2 \geq 291$, $m_3 \geq 273$ and $m_4 \geq 229$. Hence, $\Lambda_4 > 0$, and so from (47) we see that

$$0 < \Lambda_4 < e^{\Lambda_4} - 1 = \left|1 - \alpha^{-m_4} 10^n \left(\frac{d}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)}\right)\right| < \alpha^{3.35-m_1}.$$

Hence, we have

$$\log\left(\frac{d}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)}\right) + m_4(-\log \alpha) + n \log 10 < \alpha^{3.35-m_1},$$

which implies that

$$|\Lambda_4| < \alpha^{3.36} \exp(-0.48 m_1), \tag{75}$$

where $X = \max\{m_4, n\} \leq m_1 < 2.2 \times 10^{59}$. In addition,

$$\frac{\Lambda_4}{\log 10} = \frac{1}{\log 10} \log\left(\frac{d}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)}\right) - m_4 \frac{\log \alpha}{\log 10} + n.$$

Thus,

$$c = \alpha^{3.36}, \quad \delta = 0.48, \quad X_0 = 2.2 \times 10^{59}, \quad Y = m_1,$$

$$\psi = \frac{1}{\log 10} \log\left(\frac{d}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)}\right), \quad \vartheta = \frac{\log\alpha}{\log 10},$$

$$\vartheta_1 = -\log\alpha, \quad \vartheta_2 = \log 10, \quad \beta = \log\left(\frac{d}{9(\alpha^{m_1-m_4} + \alpha^{m_2-m_4} + \alpha^{m_3-m_4} + 1)}\right).$$

We find that $q = q_{145}$ satisfies the hypothesis of Lemma 6 for $1 \leq d \leq 9$, $0 \leq m_3 - m_4 \leq m_2 - m_4 \leq m_1 - m_4 \leq 371$. Applying Lemma 6, we get $m_1 \leq 403$, which contradicts the assumption that $m_1 \geq 600$. This proves the result.

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References

[1] Y. Bugeaud, M. Mignotte, and S. Siksek, Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers, *Ann. of Math.* **163** (2006), 969–1018.

[2] F. Luca, Repdigits as sums of three Fibonacci numbers, *Math. Commun.* **17** (2012), 1–11.

[3] F. Luca, Distinct digits in base b expansions of linear recurrences, *Quaest. Math.* **23** (2000), 389–404.

[4] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, *II, Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 125–180. English translation in *Izv. Math.* **64** (2000), 1217–1269.

[5] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math.* **49** (1998), 291–306.

[6] B. V. Normenyo, F. Luca, and A. Togbé, Repdigits as sums of three Pell numbers, *Periodica Math. Hung.*, to appear.

[7] B. V. Normenyo, F. Luca, and A. Togbé, Repdigits as sums of three Lucas numbers, *Colloq. Math.*, to appear.

[8] B. V. Normenyo, F. Luca, and A. Togbé, Repdigits as sums of four Pell numbers, *Bol. Soc. Mat. Mexicana*, to appear.

[9] B. M. M. de Weger, *Algorithms for Diophantine Equations*, Stichting Mathematisch Centrum, 1989.

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