Journal of Integer Sequences, Vol. 21 (2018),
Article 18.9.8

Planar Additive Bases for Rectangles

Jukka Kohonen
Department of Computer Science
University of Helsinki
P.O. Box 68, FI-00014 University of Helsinki
Finland
jukka.kohonen@helsinki.fi

Visa Koivunen and Robin Rajamäki
Department of Signal Processing and Acoustics
Aalto University
P.O. Box 15400, FI-00076 Aalto
Finland
visa.koivunen@aalto.fi, robin.rajamaki@aalto.fi

Abstract

We study a generalization of additive bases into a planar setting. A planar additive basis is a set of non-negative integer pairs whose vector sumset covers a given rectangle. Such bases find applications in active sensor arrays used in, for example, radar and medical imaging.

We propose two algorithms for finding the minimal bases of small rectangles: one in the unrestricted case where the basis elements can be anywhere in the rectangle, and another in the restricted case, where the elements are confined to the lower left quadrant. We present numerical results from such searches, including the minimal cardinalities and number of unique solutions for all rectangles up to $[0, 11] \times [0, 11]$ in the unrestricted case, and up to $[0, 26] \times [0, 26]$ in the restricted case. For squares we list the minimal basis cardinalities up to $[0, 13] \times [0, 13]$ in the unrestricted case, and up to $[0, 46] \times [0, 46]$ in the restricted case. Furthermore, we prove asymptotic upper and lower bounds on the minimal basis cardinality for large rectangles.

1 Introduction

An *additive basis* for an interval of integers $[0, n] = \{0, 1, 2, \ldots, n\}$ is a set of non-negative integers A such that $A + A \supseteq [0, n]$. By extension we define that a *planar additive basis* for a rectangle of integers $R = [0, s_x] \times [0, s_y]$ is a set of points with non-negative integer coordinates

$$A = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}, \quad \text{such that } A + A \supseteq R.$$

The *sumset* (or *sum co-array*) is defined in terms of vector addition, that is

$$A + A' = \{(x + x',\ y + y') \ :\ (x, y) \in A,\ (x', y') \in A'\}.$$

Additive bases for integer intervals have been widely studied since Rohrbach [22]. Often one seeks to maximize n when the basis cardinality $|A| = k$ is given. For small k this has been approached with computations [1, 12, 17, 21], and for large k with asymptotic bounds [11, 25].

Less is known about planar additive bases. Kozick and Kassam discussed them in an application context of signal processing, and proposed some simple designs [13]. In a rather different line of work, sumsets in vector spaces and abelian groups have been studied with the interest in how *small* the sumset can be [2, 3, 4]. Boundary effects in planar sumsets have also been studied by e.g., Han [6].

We now aim to minimize the cardinality k of a planar additive basis, when the target rectangle $R = [0, s_x] \times [0, s_y]$ is given. To the best of our knowledge, this combinatorial optimization problem has not been addressed before.

Planar bases have an application in signal processing, when an array of sensor elements is deployed on a plane to be used in active imaging or radar surveillance [19]. Here “active” means that the sensors both transmit a signal towards objects such as radar targets or human tissue, and receive the reflections. In this context, the pairwise vector sums of the sensor locations make up a virtual sensor array, called the *sum co-array*, which may be used to improve imaging resolution, or to reduce the number of sensors without significant performance loss [8].

An important special case is that of *restricted* bases. A basis A for $[0, n]$ is restricted if $A \subseteq [0, n/2]$. Analogously we define that a basis A for $[0, s_x] \times [0, s_y]$ is restricted if $A \subseteq [0, s_x/2] \times [0, s_y/2]$. Apart from practical motivations related to the physical placing of sensors, there are other reasons to study restricted bases. Computationally, we can study much larger instances in the restricted case, at least with our current algorithms. Numerically, restricted bases often attain the same minimum cardinality as unrestricted bases. Also, the minimal restricted bases often exhibit interesting geometric structure.

We introduce here the following results. First, a search algorithm is proposed for finding all bases of a given size for a given rectangle; and the minimum basis sizes are determined for all rectangles with $s_x, s_y \leq 11$ or $s_x = s_y \leq 13$. Secondly, a meet-in-the-middle method is developed that constructs a restricted planar basis by concatenating together four smaller bases, one in each corner; and the minimum restricted basis sizes are determined for all even $s_x, s_y \leq 26$ or even $s_x = s_y \leq 46$. Thirdly, some asymptotic upper and lower bounds on the minimum basis size for large rectangles are established.

2 Definitions and preliminary observations

The target rectangle is $R = [0, s_x] \times [0, s_y]$. If R is square, we call it the *s-square*, with $s = s_x = s_y$. A basis containing k elements is a *k-basis*. The size of the smallest basis for $[0, s_x] \times [0, s_y]$ is denoted by $k(s_x, s_y)$.

If s_x and s_y are even, we set $h_x = s_x/2$ and $h_y = s_y/2$. Then a basis A is *restricted* if $A \subseteq [0, h_x] \times [0, h_y]$. Note that it follows that $A + A = R$. The size of the smallest restricted basis is $k^*(s_x, s_y)$.

If A is a basis for R such that $A \subseteq R$, we say that A is *admissible*. If not, then it cannot be minimal, since one can simply drop the elements that are outside the target. So we confine our attention to admissible bases.

We relate the basis size $k = |A|$ to the number of target elements $N = |R| = (s_x+1)(s_y+1)$, which may be understood as the *target area* measured in grid points. The *efficiency* of a basis is defined as

$$c = N/k^2. \tag{1}$$

The shape of the target is characterized by its *aspect ratio* $\rho = (s_y + 1)/(s_x + 1)$.

Two simple basis constructions were proposed by Kozick and Kassam in the context of sensor arrays [13]. For any rectangle, the *L-shaped basis* is

$$([0, s_x] \times \{0\}) \;\cup\; (\{0\} \times [0, s_y]), \tag{2}$$

which has $s_x + s_y + 1$ elements. If $s_x, s_y \geq 2$ are even, the *boundary basis* is

$$([0, h_x]\times\{0, h_y\}) \;\cup\; (\{0, h_x\}\times[0, h_y]), \tag{3}$$

which has s_x+s_y elements and is restricted. These two provide a minimal basis for most small squares (boundary basis if $s \geq 2$ is even, L-shaped otherwise). The smallest counterexample is the 7-square, whose minimal bases have only 14 elements, one less than the L-shaped basis (see Figure 1c). However, for non-square rectangles, (2) and (3) are generally not minimal. Examples of this will be presented in Section 5, and an asymptotic result in Section 6.

The following observations about the corners and the horizontal edges of planar additive bases will be useful. Corresponding results in the vertical direction can be proven by transposing x and y.

Lemma 1 (Origin corner)**.** *If A is a basis for a rectangle with $s_x \geq 1$, then $(0, 0), (1, 0) \in A$.*

Proof. The only way to represent $(1, 0)$ as a sum of two pairs of non-negative integers is $(0, 0) + (1, 0)$, so those elements must be in the basis. □

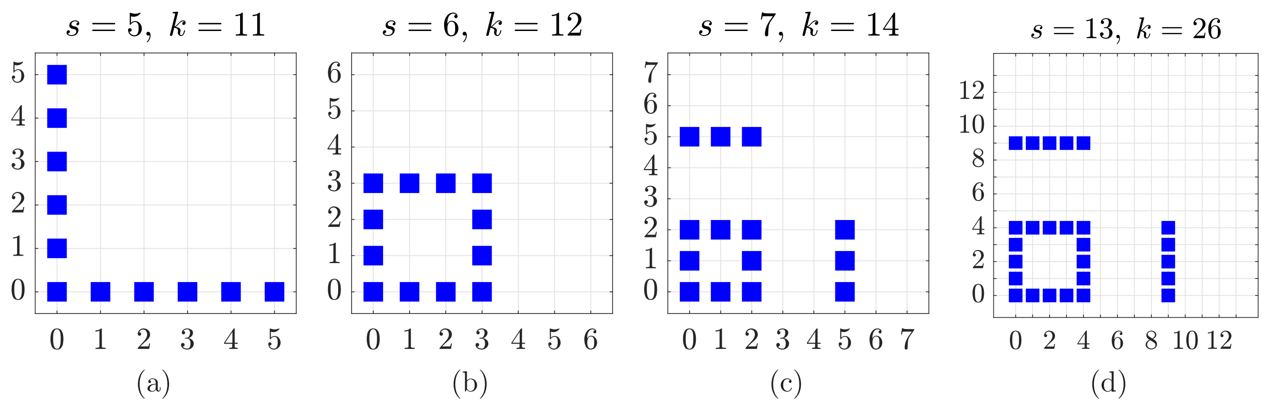


Figure 1: (a) The L-shaped basis for the 5-square. (b) The boundary basis for the 6-square. (c) A minimal basis for the 7-square. (d) A minimal basis for the 13-square.

Lemma 2 (Restricted edges)**.** *If A is a restricted basis for $[0, s_x] \times [0, s_y]$, then its bottom edge $\{x : (x, 0) \in A\}$ and top edge $\{x : (x, h_y) \in A\}$ are (one-dimensional) restricted bases for $[0, s_x]$.*

Proof. Consider first the bottom edge. Since the y coordinates in A are non-negative, for any $x \in [0, s_x]$ the point $(x, 0)$ must be the sum of some $(x', 0), (x'', 0) \in A$. Since A is restricted, we have $x', x'' \leq h_x$.

Consider next the top edge. Since the y coordinates in A are at most h_y, for any $x \in [0, s_x]$ the point (x, s_y) must be the sum of some $(x', h_y), (x'', h_y) \in A$. Since A is restricted, we have $x', x'' \leq h_x$. □

Lemma 3 (Two rows)**.** *For any even $s_x \geq 0$, we have $k^*(s_x, 2) = 2k^*(s_x, 0)$.*

Proof. Let A be a restricted basis for $[0, s_x] \times [0, 2]$. By Lemma 2 its bottom and top edges are restricted bases for $[0, s_x]$, so each has at least $k^*(s_x, 0)$ elements. Thus $|A| \geq 2k^*(s_x, 0)$.

To see that $k^*(s_x, 2) \leq 2k^*(s_x, 0)$, let A^* be a restricted basis for $[0, s_x]$. Then $A^* \times [0, 1]$ is a restricted basis for $[0, s_x] \times [0, 2]$. □

3 Search algorithm for admissible bases

In this section we develop a method to find all admissible k-bases for a given rectangle. Then we can also establish the minimum value of k. For example, the L-shaped basis suffices to show that $k(9, 9) \leq 19$, but to prove that $k(9, 9) = 19$ we must ascertain that there is no 18-basis for the 9-square. Trying out the $\binom{100}{18} \approx 3 \cdot 10^{19}$ ways of placing 18 elements in $[0, 9] \times [0, 9]$ is obviously impractical.

Our Algorithm 1 is a relatively straightforward generalization of Challis's algorithm, which finds one-dimensional bases [1]. Assume for simplicity that $s_x \geq 2$. By Lemma 1 the points $(0, 0)$ and $(1, 0)$ must be included in the basis. Next we branch on the decision

Algorithm 1 Find all admissible k-bases for $[0, s_x] \times [0, s_y]$

1: **procedure** FINDBASES(k, s_x, s_y)
2: EXTEND($k, s_x, s_y, \{(0,0), (1,0)\}, 1, 0$)
3: **procedure** EXTEND(k, s_x, s_y, A, x, y)
4: ▷ (x, y) is the latest location considered (either filled or left empty).
5: $j \leftarrow |A|$ ▷ Number of elements
6: $G \leftarrow |[0, s_x] \times [0, s_y] \setminus (A + A)|$ ▷ Number of gaps
7: **if** $(j = k) \wedge (G = 0)$ **then** PRINT(A) ▷ Found a basis
8: **if** $j = k$ **then return** ▷ Reached full size
9: $M \leftarrow (k + j + 1)(k - j)/2$ ▷ Max. sums to expect
10: **if** $M < G$ **then return** ▷ Too many gaps
11: **if** $x < s_x$ **then**
12: $x \leftarrow x + 1$ ▷ Proceed right
13: **else if** $y < s_y$ **then**
14: $x \leftarrow 0$ ▷ Begin next row
15: $y \leftarrow y + 1$
16: **else**
17: **return** ▷ Reached the top right
18: **if** $(x, y) \in A + A$ **then** ▷ Already covered?
19: EXTEND(k, s_x, s_y, A, x, y) ▷ Branch without (x, y)
20: EXTEND($k, s_x, s_y, A \cup \{(x, y)\}, x, y$) ▷ Branch with (x, y)

whether $(2, 0)$ is included. We proceed to the right and rowwise, branching at each location on whether that point is included, until we have k elements or reach the top right corner.

During the search, two tests prune unfruitful branches. One of them (line 18) concerns unfillable holes in the sumset. Suppose that we are currently at (x, y). Because of the way how the search proceeds, any location (x', y') considered deeper in the search will have $x' > x$ or $y' > y$ (or both). Thus any such elements will not generate the sum (x, y), by the non-negativity of coordinates. If (x, y) has not already been covered, then (x, y) has to be included in the basis.

The other test (line 10) is based on a counting argument. Suppose that after placing j elements there are G gaps, or target points not covered by the current sumset. No matter where the remaining $k - j$ elements are placed, they will generate at most $M = (j + 1) + (j + 2) + \cdots + k = (k + j + 1)(k - j)/2$ more sums. If $M < G$, then the current search branch cannot lead to any solutions.

This algorithm is quite simple, and there may be several ways to speed it up it by exploiting the geometry of the problem. For example, instead of proceeding rowwise, the target rectangle can be explored in a different order: after completing the bottom edge ($y = 0$), do next all of the left edge ($x = 0$), then second row, second column, and so on.

The idea is to introduce necessary conditions from both the left and bottom edges early on. This change does not affect the validity of the algorithm. Empirically we observed that it saves about 37% of the running time in the example case of 19-bases of the 9-square.

As is typical for a combinatorial branch-and-bound method, the time requirement of this algorithm grows rapidly as k increases. We implemented the algorithm in C++ and ran it on Intel Xeon E7-8890 processors (nominal clock frequency 2.2 GHz). For 19-bases of the 9-square the search took 0.44 hours of processor time; for 23-bases of the 11-square it took 1058 hours. Results are summarized in Table 1 (squares) and Table 2 (rectangles).

s	k	m	m_{u}
0	1	1	1
1	3	1	1
2	4	1	1
3	7	15	10
4	8	8	5
5	11	137	76
6	12	24	14
7	14	14	9
8	16	103	54
9	19	3531	1792
10	20	360	182
11	23	26857	13465
12	24	1585	797
13	26		

Table 1: Minimal bases for squares.

4 Meet-in-the-middle method for restricted bases

In one dimension, i.e., for integer intervals, Kohonen proposed a meet-in-the-middle (MIM) method to find optimal restricted bases [10]. In its simplest form the MIM method splits a restricted basis at its midpoint into two components, a prefix and a suffix, which are then sought separately among the admissible bases of a smaller interval. It is much faster to consider all pairs of these components than to search directly for restricted bases by a method similar to Algorithm 1. The largest known optimal restricted bases for integer intervals have been computed by this method, with $k^*(734, 0) = 48$ [9].

s_x	s_y	k	Δk	$m_{\mathbf{u}}$
0	0	1	0	1
1	0	2	0	1
	1	3	0	1
2	0	2	0	1
	1	4	0	3
	2	4	0	1
3	0	3	0	2
	1	5	0	6
	2	6	0	16
	3	7	0	10
4	0	3	0	2
	1	5	-1	3
	2	6	0	6
	3	8	0	75
	4	8	0	5
5	0	4	0	5
	1	6	-1	10
	2	7	-1	1
	3	9	0	86
	4	10	0	283
	5	11	0	76
6	0	4	0	5
	1	6	-2	4
	2	8	0	101
	3	9	-1	1
	4	10	0	16
6	5	12	0	660
	6	12	0	14
7	0	4	-1	2
	1	7	-2	28
	2	8	-2	5
	3	10	-1	25
	4	11	-1	50
	5	13	0	924
	6	14	0	3576
	7	14	-1	9
8	0	4	-1	1
	1	7	-3	6
	2	8	-2	1
	3	11	-1	325
	4	11	-1	4
	5	13	-1	3
	6	14	0	73
	7	15	-1	16
	8	16	0	54
9	0	5	-1	11
	1	8	-3	70
	2	10	-2	647
	3	12	-1	1940
	4	13	-1	920
	5	15	0	11479
	6	15	-1	2
9	7	17	0	5433
	8	18	0	9171
	9	19	0	1792
10	0	5	-1	8
	1	8	-4	19
	2	10	-2	174
	3	12	-2	203
	4	13	-1	64
	5	15	-1	267
	6	16	0	357
	7	17	-1	81
	8	18	0	212
	9	20	0	17076
	10	20	0	182
11	0	5	-2	1
	1	9	-4	258
	2	10	-4	3
	3	13	-2	1368
	4	14	-2	109
	5	16	-1	534
	6	17	-1	96
	7	18	-1	92
	8	19	-1	12
	9	21	0	13860
	10	22	0	42862
	11	23	0	13465

Table 2: Minimal bases for rectangles.

The MIM method is extended to the planar setting as follows. We want to find all k-bases for $R = [0, s_x] \times [0, s_y]$, subject to the restriction $A \subseteq R_h = [0, h_x] \times [0, h_y]$, where $h_x = s_x/2 > 0$ and $h_y = s_y/2 > 0$. First we divide R_h into four disjoint rectangles by choosing breaking points $a_x \in [0, h_x - 1]$ and $a_y \in [0, h_y - 1]$ arbitrarily, and defining

$$\begin{aligned} R_{\mathrm{I}} &= [0, a_x] \times [0, a_y], \\ R_{\mathrm{II}} &= [a_x + 1, h_x] \times [0, a_y], \\ R_{\mathrm{III}} &= [a_x + 1, h_x] \times [a_y + 1, h_y], \\ R_{\mathrm{IV}} &= [0, a_x] \times [a_y + 1, h_y]. \end{aligned}$$

These are the colored rectangles in the left part of Figure 2. Now split a basis A into components $A_{\mathrm{I}}, A_{\mathrm{II}}, A_{\mathrm{III}}, A_{\mathrm{IV}}$ so that $A_{\mathrm{I}} = A \cap R_{\mathrm{I}}$, and similarly with the others. By the non-negativity of all coordinates, any sumset involving A_{II}, A_{III} or A_{IV} is completely outside the lower left corner R_{I}. So in order to have $A + A \supseteq R$ we need $A_{\mathrm{I}} + A_{\mathrm{I}} \supseteq R_{\mathrm{I}}$. That is, A_{I} must be an admissible basis for R_{I}. All candidates for A_{I} can be listed by Algorithm 1.

A similar argument applies in the lower right corner of the target, with some necessary coordinate transformations. Let $C_{\mathrm{II}} = [h_x + a_x + 1, s_x] \times [0, a_y]$. Then we need $A_{\mathrm{II}} + A_{\mathrm{II}} \supseteq C_{\mathrm{II}}$, since all the other component sumsets are outside C_{II}. Consider the "mirror image" of A_{II}, namely $B_{\mathrm{II}} = \{(h_x - x,\ y) : (x, y) \in A_{\mathrm{II}}\}$. By construction, we have $B_{\mathrm{II}} \subseteq [0, b_x] \times [0, a_y]$, where for convenience we have written $b_x = h_x - a_x - 1$. Now the condition $A_{\mathrm{II}} + A_{\mathrm{II}} \supseteq C_{\mathrm{II}}$ implies that $B_{\mathrm{II}} + B_{\mathrm{II}} \supseteq [0, b_x] \times [0, a_y]$. So B_{II} must be an admissible basis for $[0, b_x] \times [0, a_y]$, and again all candidates can be found by Algorithm 1.

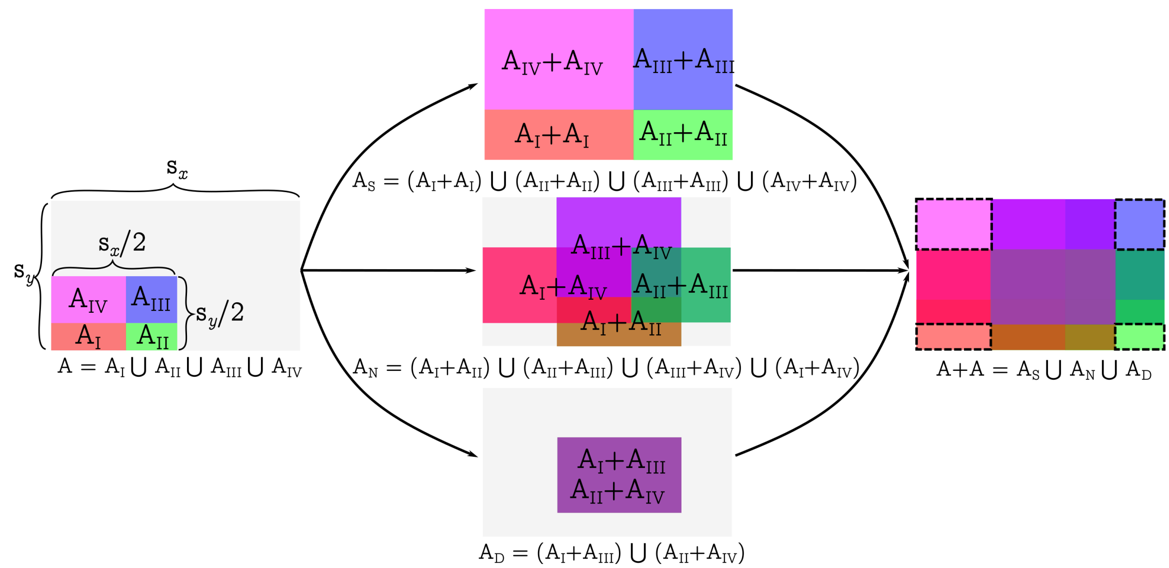


Figure 2: MIM decomposition of a restricted basis A. The four components $A_{\mathrm{I}}, \ldots, A_{\mathrm{IV}}$ are contained in the colored rectangles (left). Consequently, $A + A$ (right) is the union of A_{S}, A_{N} and A_{D} (center), which are the self, neighboring and diagonal sums of the components. The extreme corners of $A + A$ (areas within dashed rectangles) are covered only by the self sumsets, so $A_{\mathrm{I}}, \ldots, A_{\mathrm{IV}}$ must be admissible bases for those rectangles (up to suitable coordinate transformations).

Similar conditions for A_{III} and A_{IV} apply in the remaining two corners. Consequently, A must be the union of four components, which are (mirror images of) admissible bases of suitable rectangles. Since we have so far only dealt with necessary conditions, we have not lost any possible solutions. The conditions guarantee only that the four extreme corner regions are covered; for any candidate solution $A = A_{\mathrm{I}} \cup A_{\mathrm{II}} \cup A_{\mathrm{III}} \cup A_{\mathrm{IV}}$ we must finally check whether in fact $A + A \supseteq R$.

Algorithm 2 gives a formal description of the MIM method. We choose $a_x = \lfloor h_x/2 \rfloor$ and $a_y = \lfloor h_y/2 \rfloor$ so the components have roughly equal dimensions. The final ingredient of the algorithm, on lines 8–14, concerns how the overall budget of k elements is allocated to the

Algorithm 2 Find all restricted k-bases for $[0, s_x] \times [0, s_y]$

procedure MIM(k, s_x, s_y)
  $h_x \leftarrow s_x/2$ ▷ dimensions of rectangle containing A
  $h_y \leftarrow s_y/2$
  $a_x \leftarrow \lfloor h_x/2 \rfloor$ ▷ dimensions of rectangle containing A_{I}
  $a_y \leftarrow \lfloor h_y/2 \rfloor$
  $b_x \leftarrow h_x - a_x - 1$ ▷ dimensions of other rectangles
  $b_y \leftarrow h_y - a_y - 1$
  $k_{\mathrm{I}}^{\min} \leftarrow k(a_x, a_y)$ ▷ look up minimum sizes of the components
  $k_{\mathrm{II}}^{\min} \leftarrow k(b_x, a_y)$
  $k_{\mathrm{III}}^{\min} \leftarrow k(b_x, b_y)$
  $k_{\mathrm{IV}}^{\min} \leftarrow k(a_x, b_y)$
  ▷ Iterate feasible ways of allocating k among the four quadrants
  for $(k_{\mathrm{I}}, k_{\mathrm{II}}, k_{\mathrm{III}}, k_{\mathrm{IV}})$ such that $k_{\mathrm{I}} + k_{\mathrm{II}} + k_{\mathrm{III}} + k_{\mathrm{IV}} = k$ **do**
    if $k_{\mathrm{I}} \geq k_{\mathrm{I}}^{\min} \wedge k_{\mathrm{II}} \geq k_{\mathrm{II}}^{\min} \wedge k_{\mathrm{III}} \geq k_{\mathrm{III}}^{\min} \wedge k_{\mathrm{IV}} \geq k_{\mathrm{IV}}^{\min}$ **then**
      ▷ Compute or look up admissible component bases
      $\mathcal{B}_{\mathrm{I}} \leftarrow$ output from FindBases$(k_{\mathrm{I}}, a_x, a_y)$
      $\mathcal{B}_{\mathrm{II}} \leftarrow$ output from FindBases$(k_{\mathrm{II}}, b_x, a_y)$
      $\mathcal{B}_{\mathrm{III}} \leftarrow$ output from FindBases$(k_{\mathrm{III}}, b_x, b_y)$
      $\mathcal{B}_{\mathrm{IV}} \leftarrow$ output from FindBases$(k_{\mathrm{IV}}, a_x, b_y)$
      for $(B_{\mathrm{I}}, B_{\mathrm{II}}, B_{\mathrm{III}}, B_{\mathrm{IV}}) \in \mathcal{B}_{\mathrm{I}} \times \mathcal{B}_{\mathrm{II}} \times \mathcal{B}_{\mathrm{III}} \times \mathcal{B}_{\mathrm{IV}}$ **do**
        $A_{\mathrm{I}} \leftarrow B_{\mathrm{I}}$
        $A_{\mathrm{II}} \leftarrow \{(h_x - x,\ y) : (x, y) \in B_{\mathrm{II}}\}$ ▷ Mirror x coordinates
        $A_{\mathrm{III}} \leftarrow \{(h_x - x,\ h_y - y) : (x, y) \in B_{\mathrm{III}}\}$ ▷ Mirror x, y coordinates
        $A_{\mathrm{IV}} \leftarrow \{(x,\ h_y - y) : (x, y) \in B_{\mathrm{IV}}\}$ ▷ Mirror y coordinates
        $A \leftarrow A_{\mathrm{I}} \cup A_{\mathrm{II}} \cup A_{\mathrm{III}} \cup A_{\mathrm{IV}}$ ▷ Concatenate components
        if $A + A = R$ **then** Print(A) ▷ Found a basis

four components. Note that A_{I} need not be a minimal basis for R_{I}. It may have more than $k(a_x, a_y)$ elements, and indeed this may be necessary to find any solutions for $A + A \supseteq R$. The same goes for the other three components.

In order to determine the value of $k^*(s_x, s_y)$, just run Algorithm 2 repeatedly, beginning with $k = k_{\mathrm{I}}^{\min} + k_{\mathrm{II}}^{\min} + k_{\mathrm{III}}^{\min} + k_{\mathrm{IV}}^{\min}$ since certainly there are no solutions below that size, and increase k in steps of 1 until some solutions are found.

Example 4. A restricted basis A for $R = [0, 10] \times [0, 10]$ satisfies $A \subseteq R_h = [0, 5] \times [0, 5]$. The first quadrant of R_h is $R_{\mathrm{I}} = [0, 2] \times [0, 2]$, and the other quadrants have the same size. Since $k(2, 2) = 4$, we have necessarily $|A| \geq 4 + 4 + 4 + 4 = 16$. There is only one 4-basis for $[0, 2] \times [0, 2]$, so for $k = 16$ there is only one combination to check in the innermost loop of Algorithm 2. But this combination does not give a basis for $[0, 10] \times [0, 10]$, so more than 16 elements are needed.

It turns out that $k = 20$ is enough. After some simple pruning conditions (not shown in Algorithm 2) we find that the only possible allocations of 20 elements are $(k_{\mathrm{I}}, k_{\mathrm{II}}, k_{\mathrm{III}}, k_{\mathrm{IV}}) = (4, 6, 4, 6)$ and $(5, 5, 5, 5)$. There are nine 5-bases and eighteen 6-bases for $[0, 2] \times [0, 2]$, so the first allocation leads to $1 \cdot 18 \cdot 1 \cdot 18 = 324$ combinations to be checked, and the second gives $9 \cdot 9 \cdot 9 \cdot 9 = 6561$ combinations. Out of these, we find 17 restricted solutions. This is less than one second of computation. In comparison, finding *all* 20-bases for the 10-square with our implementation of Algorithm 1 takes more than an hour.

There are a few ways to significantly prune the number of candidate solutions that need to be checked. Firstly, the complete sumset of a candidate restricted basis does not have to be calculated immediately. A necessary condition for a restricted basis is that any two neighboring quadrants form a restricted basis along one of the coordinate axes. It therefore suffices to first check whether this condition is satisfied for all four neighboring quadrant pairs. Only if the condition is met does the full sumset need to be checked.

Secondly, often some of the component pieces have the same dimensions (indeed all of them if h_x, h_y are both odd). If the pieces also have the same cardinality, then the set of candidate solutions is the same for all of them, up to suitable coordinate transformations.

Example 5. Consider a restricted basis A for the square $[0, s] \times [0, s]$, with $s/2 = 2a + 1$ odd and $a \geq 0$. Each quadrant has the same dimensions $a_x = b_x = a_y = b_y = a$. If all component sets also have equal cardinality, then the candidates for A_{II}, A_{III} and A_{IV} are the same as for A_{I}, up to suitable mirroring. Furthermore, if the sumset $(A_{\mathrm{I}} \cup A_{\mathrm{II}}) + (A_{\mathrm{I}} \cup A_{\mathrm{II}})$ does not cover $[0, s] \times [0, a]$, then all candidate solutions containing any rotation of this pair can be pruned.

Thirdly, when components have different cardinalities, the order in which they are concatenated matters. One possible strategy is to first concatenate component pairs of low cardinality, not only because they usually have fewer component solutions to try out, but also because they are less likely to produce feasible concatenations than pairs of higher cardinality. Occasionally, an infeasible concatenation rules out all potential solutions containing high cardinality components. Then these components do not even have to be computed in the first place.

Example 6. Consider the square restricted basis A in Example 5. Let the cardinality of this basis be $4k^{\min} + \tilde{k} = 4k^{\min} + (\tilde{k}_{\mathrm{I}} + \tilde{k}_{\mathrm{II}} + \tilde{k}_{\mathrm{III}} + \tilde{k}_{\mathrm{IV}})$, where $\tilde{k}_{\mathrm{I}}, \ldots, \tilde{k}_{\mathrm{IV}}$ represent the number of extra elements in each quadrant, which are now of equal dimensions. Let $\tilde{k} = 3$. After accounting for rotational and mirror symmetries of A, it turns out that there are four unique ways to distribute the extra elements: $(\tilde{k}_{\mathrm{I}}, \tilde{k}_{\mathrm{II}}, \tilde{k}_{\mathrm{III}}, \tilde{k}_{\mathrm{IV}}) = (0, 0, 0, 3)$, $(0, 0, 1, 2)$, $(0, 1, 0, 2)$, or $(0, 1, 1, 1)$. If the concatenation of pair $(\tilde{k}_{\mathrm{I}}, \tilde{k}_{\mathrm{II}}) = (0, 0)$ gives no solutions, then the candidate solutions containing pairs $(\tilde{k}_{\mathrm{III}}, \tilde{k}_{\mathrm{IV}}) = (0, 3)$ and $(\tilde{k}_{\mathrm{III}}, \tilde{k}_{\mathrm{IV}}) = (1, 2)$ are discarded. Consequently, we avoid computing the potentially many solutions of the $(k^{\min} + 3)$-basis altogether.

5 Numerical results

We now describe some results obtained for small rectangles with Algorithms 1 and 2. Examples of minimal bases are shown in Figures 3 and 4. We note that especially the restricted solutions in Figure 4 exhibit regular structure that can perhaps be generalized to larger bases.

In the result listings, m is the number of all minimal bases, and m_{u} is the number of "unique" bases after taking into account rotation and mirror symmetries. Each basis may have up to 8 symmetric variants if the target is square, and up to 4 variants otherwise.

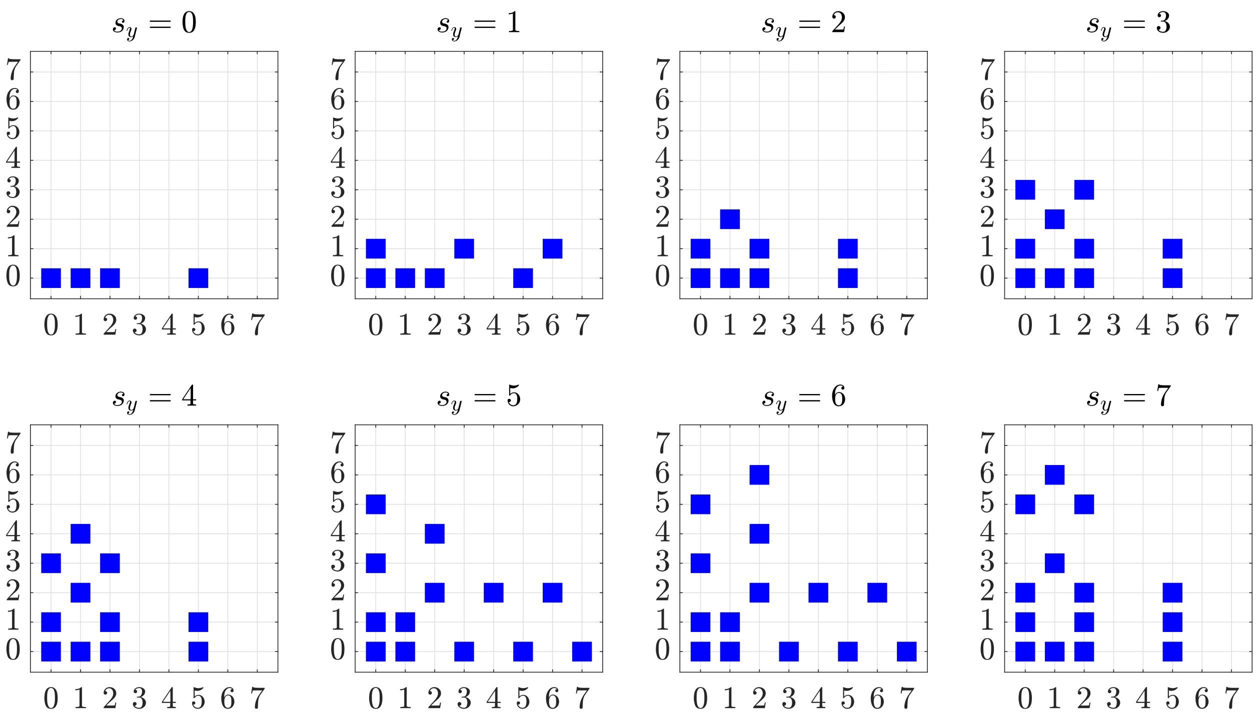


Figure 3: Some minimal bases for $s_x = 7$ and varying s_y.

5.1 Results for squares

Table 1 summarizes the minimal basis sizes for squares up to $s = 13$. For $s \leq 12$ we have generated and counted all minimal bases. Regarding $s = 13$, we deduce that $k(13, 13) = 26$, because Algorithm 1 finds no solutions with 25 elements, but we can construct a basis with 26 elements (see Figure 1d).

We also observe that in small even-sided instances $s = 2, 4, 6, 8, 10, 12$ one of the minimal solutions is the boundary basis. In small odd-sided instances $s = 1, 3, 5, 9, 11$ one of the minimal solutions is the L-shaped basis. Cases $s = 7, 13$ stand out as exceptions where the L-shaped basis is not minimal (recall Figures 1c and 1d).

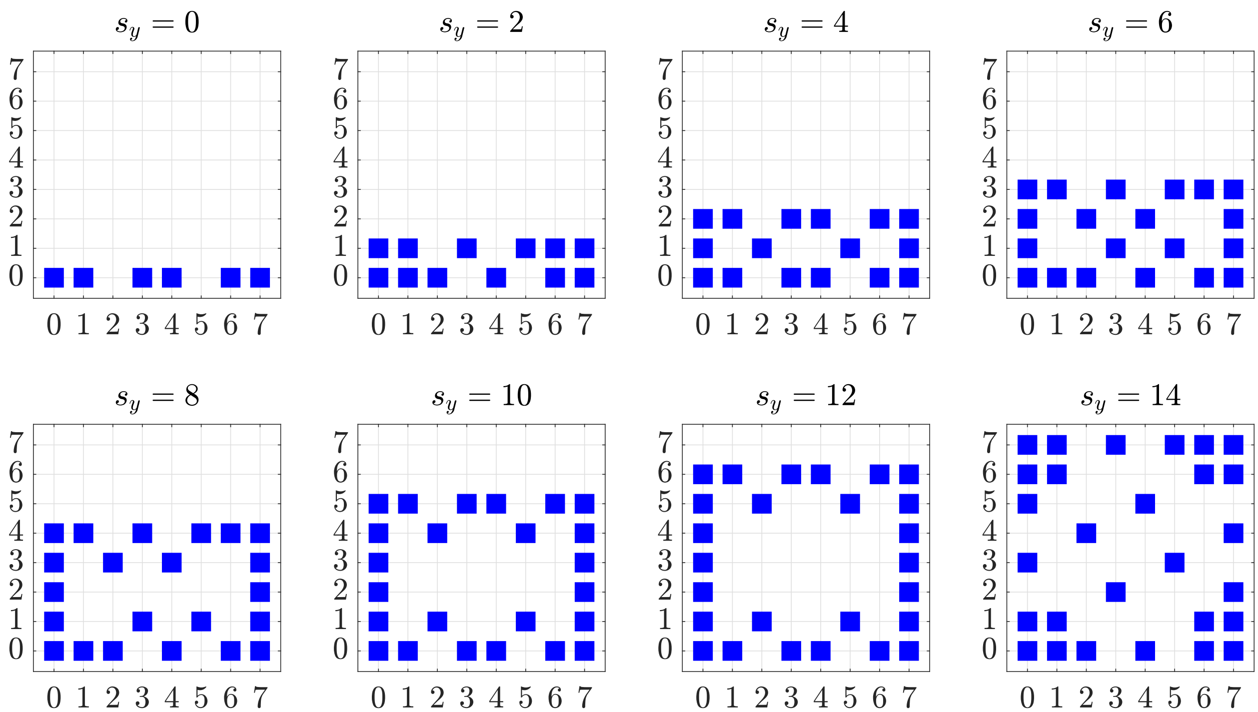


Figure 4: Some minimal restricted bases for $s_x = 14$, varying s_y.

Concerning the restricted case, Table 3 summarizes the results for squares up to $s = 46$. For $s \leq 26$ we generated and counted the minimal bases. For $28 \leq s \leq 46$ we only determined the value of $k^*(s, s)$, but did not generate the bases. For example, since we found that there is no restricted 91-basis for the 46-square, we can deduce that $k^*(46, 46) = 92$ as the boundary basis has this size. In all even-sided squares with $2 \leq s \leq 46$, we have $k^*(s, s) = 2s$, which is attained by the boundary basis.

Although the simple L-shaped and boundary bases provide minimal or almost minimal solutions for small squares, having the full collection of minimal solutions can be useful from an application perspective. In some sensor array applications it is beneficial to avoid placing sensor elements near each other, so as to avoid mutual coupling effects that cause degraded performance [16]. This may lead to a secondary optimization goal or constraint, and one may search the collection of minimal-size bases in order to satisfy this constraint.

5.2 Results for rectangles

The situation with rectangles is quite different from that with squares: if the aspect ratio $\rho = (s_y + 1)/(s_x + 1)$ is not equal to 1, then minimal bases may be much smaller than the L-shaped and boundary bases.

s	k^*	m	m_{u}
0	1	1	1
2	4	1	1
4	8	1	1
6	12	1	1
8	16	9	5
10	20	17	4
12	24	58	16
14	28	163	28
16	32	451	72
18	36	2047	276
20	40	8451	1133
22	44	43807	5575
24	48	213859	27108
26	52	1273607	159744
28	56		
30	60		
32	64		
34	68		
36	72		
38	76		
40	80		
42	84		
44	88		
46	92		

Table 3: Minimal restricted bases for squares.

Minimal bases for rectangles are summarized in Table 2, and Tables 4 and 5 for the restricted case. In order to compare the minimal solutions to the L-shaped and boundary bases, the quantity $\Delta k = k - k_{\mathrm{t}}$ is computed. Here k_{t} is the number of elements in the best applicable trivial solution, which is the boundary basis when s_x and s_y are even, and the L-shaped basis otherwise, except when $s_y = 0$ where the trivial solution is a one-dimensional basis with $\lceil s_x/2 \rceil + 1$ elements at $0, 1, \ldots, \lceil s_x/2 \rceil$.

s_x	s_y	k^*	Δk	$m_{\mathbf{u}}$
0	0	1	0	1
2	0	2	0	1
	2	4	0	1
4	0	3	0	1
	2	6	0	1
	4	8	0	1
6	0	4	0	1
	2	8	0	1
	4	10	0	1
	6	12	0	1
8	0	4	−1	1
	2	8	−2	1
	4	11	−1	1
	6	14	0	3
	8	16	0	5
10	0	5	−1	1
	2	10	−2	2
	4	13	−1	1
	6	16	0	4
	8	18	0	6
	10	20	0	4
12	0	5	−2	1
	2	10	−4	1
	4	14	−2	2
	6	18	0	14
	8	19	−1	1
	10	22	0	14
	12	24	0	16
14	0	6	−2	3
	2	12	−4	7
	4	16	−2	15
	6	20	0	91
	8	22	0	47
	10	24	0	30
	12	26	0	37
14	14	28	0	28
16	0	6	−3	1
	2	12	−6	1
	4	16	−4	1
	6	20	−2	1
	8	22	−2	1
	10	26	0	74
	12	28	0	86
	14	30	0	156
	16	32	0	72
18	0	7	−3	4
	2	14	−6	20
	4	18	−4	12
	6	22	−2	17
	8	25	−1	34
	10	28	0	279
	12	30	0	286
	14	32	0	302
	16	34	0	345
	18	36	0	276
20	0	7	−4	2
	2	14	−8	3
	4	18	−6	1
	6	22	−4	1
	8	25	−3	1
	10	29	−1	1
	12	32	0	1155
	14	34	0	1157
	16	36	0	1202
	18	38	0	1406
	20	40	0	1133
22	0	8	−4	12
	2	16	−8	113
	4	20	−6	14
	6	24	−4	17
22	8	28	−2	381
	10	32	0	8957
	12	34	0	5585
	14	36	0	5601
	16	38	0	5644
	18	40	0	5850
	20	42	0	6705
	22	44	0	5575
24	0	8	−5	4
	2	16	−10	10
	4	20	−8	1
	6	24	−6	1
	8	28	−4	16
	10	32	−2	50
	12	35	−1	4
	14	38	0	27132
	16	40	0	27177
	18	42	0	27381
	20	44	0	28238
	22	46	0	32680
	24	48	0	27108
26	0	8	−6	2
	2	16	−12	2
	4	22	−8	46
	6	26	−6	18
	8	30	−4	302
	10	34	−2	1384
	12	36	−2	4
	14	40	0	159771
	16	42	0	159828
	18	44	0	160019
	20	46	0	160874
	22	48	0	165318
	24	50	0	186849
	26	52	0	159744

Table 4: Minimal restricted bases for rectangles.

s_x	k^*	$m_{\mathbf{u}}$	s_x	k^*	$m_{\mathbf{u}}$	s_x	k^*	$m_{\mathbf{u}}$	s_x	k^*	$m_{\mathbf{u}}$
2	4	1	32	18	1	62	28	125247	92	32	1
4	6	1	34	20	777	64	26	1	94	34	1284
6	8	1	36	20	50	66	28	654	96	34	222
8	8	1	38	20	8	68	28	62	98	34	88
10	10	2	40	20	1	70	28	3	100	34	1
12	10	1	42	22	412	72	28	1	102	36	74170
14	12	7	44	22	20	74	30	2415	104	34	1
16	12	1	46	24	32931	76	30	97	106	36	945
18	14	20	48	24	3126	78	30	6	108	36	242
20	14	3	50	24	369	80	30	1	110	36	104
22	16	113	52	24	37	82	32	18937	112	38	283716
24	16	10	54	24	2	84	32	1561	114	38	42971
26	16	2	56	26	4337	86	32	193	116	36	1
28	18	162	58	26	239	88	32	8	118	38	454
30	18	22	60	26	36	90	32	2	120	38	202

Table 5: Minimal restricted bases for $s_y = 2$.

Tables 2 and 4 show that minimal (unrestricted and restricted) bases use increasingly fewer elements than the trivial solutions as the aspect ratio deviates further from 1. This is also apparent from Figure 5, which shows the efficiency (1) of the minimal bases as a function of aspect ratio, along with the asymptotical efficiencies of the L-shaped basis, and two parametric bases that are introduced in section 6.2 (Definitons 11 and 13). Specifically, the L-shaped basis has efficiency $c \to \rho/(1+\rho)^2$, as $s_x \to \infty$, since it requires $s_x + s_y + 1 = (1+\rho)s_x + \rho$ elements for its sumset to cover the $[0, s_x] \times [0, s_y] = [0, s_x] \times [0, \rho(s_x+1) - 1]$ rectangle. Similarly, the asymptotical efficiency of the dense-sparse and short-bars bases is $c = 1/4$, as shown later in Corollary 15 of section 6.2. The efficiency of the minimal bases in Figure 5 seem to approach $1/4$ as s_x and ρ increase.

A peculiarity is illustrated in Figure 6, which shows two minimal restricted bases for which the number of elements actually decreases as the target width increases. Not only is $k^*(62, 2) = 28 > k^*(64, 2) = 26$, but the number of solutions for the two cases is also drastically different. The former has 125247 unique solutions, whereas the latter has only 1. The solutions for $s_y = 2$ listed in Table 5 reveal that a similar effect also occurs for $s_x = 104$ and 116. The same also applies to $s_y = 0$, since $k^*(s_x, 2) = 2k^*(s_x, 0)$ by Lemma 3.

An overview of currently known minimal restricted bases is shown in Figure 7. The colors of the pixels correspond to the minimal number of elements. At present, bases up to about $k = 50$ are practical to list exhaustively. For clarity of presentation, restricted one-dimensional bases are not plotted for $s_x > 120$.

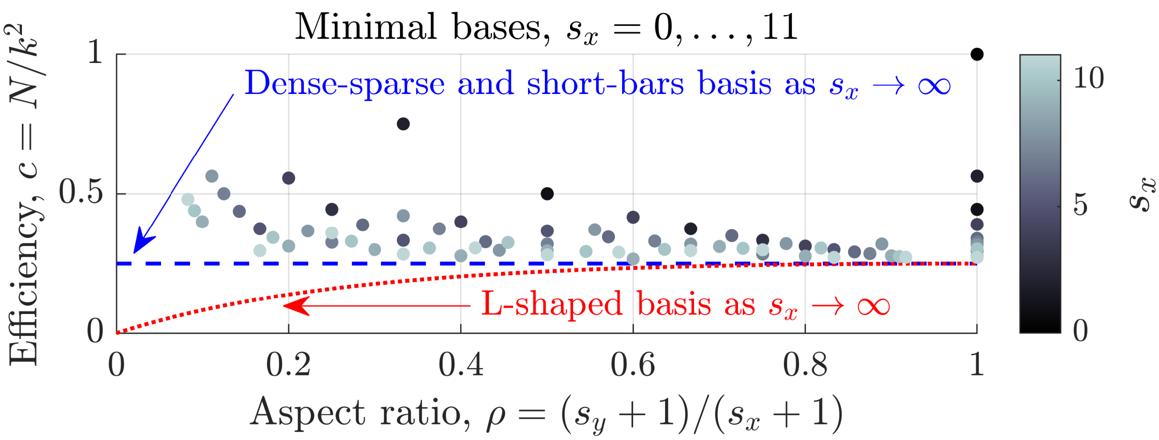


Figure 5: Efficiency of minimal bases, and asymptotical efficiency of L-shaped basis (dotted red line) and dense-sparse/short-bars bases (dashed blue line). The L-shaped basis is suboptimal when $\rho \neq 1$ and $s_x \to \infty$, whereas the dense-sparse and short-bars bases asymptotically achieve $c = 1/4$ for any ρ. The asymptotic efficiency of minimal bases is unknown.

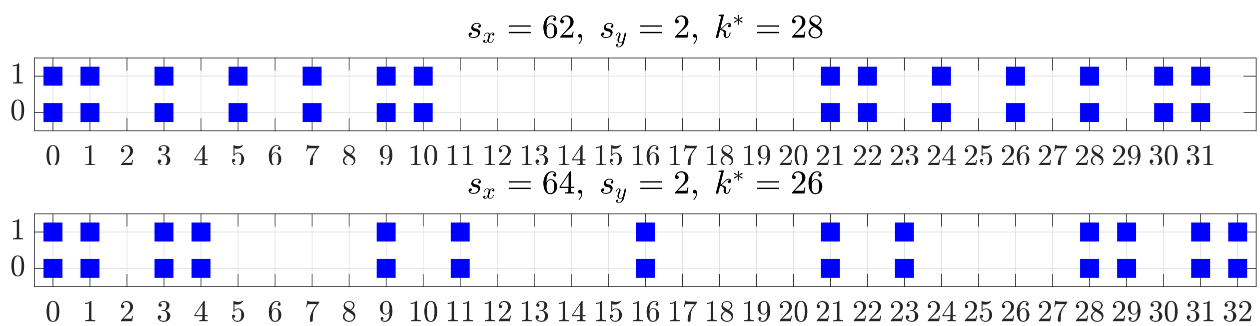


Figure 6: Two restricted bases for $s_y = 2$, for which the minimal number of elements decreases as the rectangle width increases.

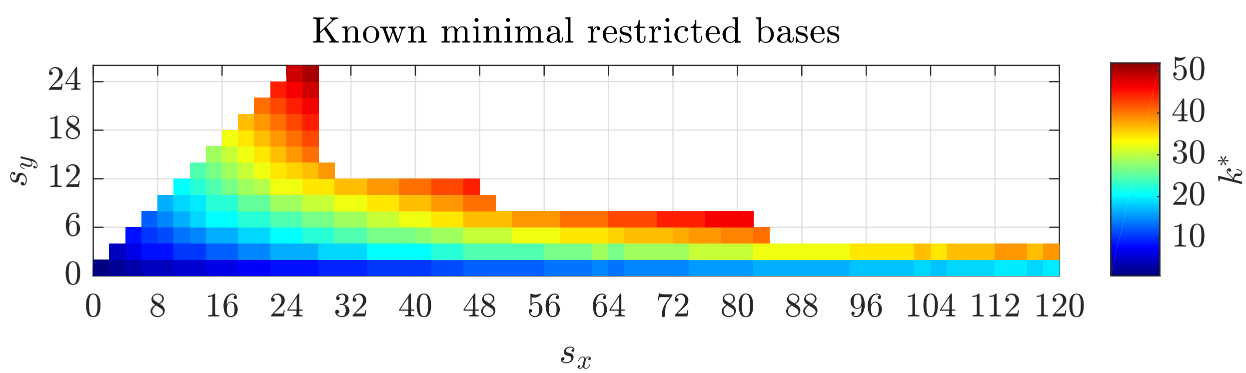


Figure 7: Minimal number of elements in restricted bases.

6 Bounds for large-scale behaviour

For very large rectangles it seems difficult to determine the minimum basis size exactly. Towards understanding the large-scale behaviour, we establish some upper and lower bounds on the efficiency $c = N/k^2$ (recall (1)) of such bases.

6.1 Upper bounds

A crude upper bound on efficiency is obtained by observing that from k elements at most $(k+1)k/2$ different pairwise sums can be formed, considering that $a+b=b+a$ and that sums of the form $a+a$ are allowed. It follows that $N \le (k+1)k/2$, so for any planar basis we have

$$c \le 0.5 + O\big(1/\sqrt{N}\big).$$

In one dimension, upper bounds tighter than 0.5 have been established by analytic and combinatorial methods. For all s_x large enough, by Yu's Theorem 1.1 [25] we have

$$s_x/k(s_x,0)^2 \le 0.45851 = \alpha, \tag{4}$$

and by Yu's Theorem 1.2 [24] we have

$$s_x/k^*(s_x,0)^2 \le 0.41983 = \beta. \tag{5}$$

Combining Yu's theorems with simple counting, we obtain the following bounds with rectangles of small constant height. For brevity, if P is a set of points, we denote $P_y = \{x : (x,y) \in P\}$ and call this the *row* y of P.

Theorem 7. *For all s_x large enough, any basis for $[0,s_x] \times [0,1]$ has efficiency $c < 0.4311$.*

Proof. Assume that s_x is large enough that (4) holds. Without loss of generality let A be admissible, and let its rows A_0, A_1 contain k_0, k_1 elements, respectively. Now $A_0 + A_0$ must cover $R_0 = [0, s_x]$, and $A_0 + A_1$ must cover $R_1 = [0, s_x]$. By applying (4) on row 0, and by counting sums on row 1, we obtain

$$\begin{aligned} s_x &\le \alpha k_0^2, \\ s_x &\le k_0 k_1. \end{aligned}$$

For any k, the minimum of these two bounds is maximized at $k_1 = \alpha k_0$, implying that $k = (1+\alpha)k_0$ and

$$s_x/k^2 \le \frac{\alpha}{(1+\alpha)^2} < 0.215542.$$

Since $N = |R| = 2(s_x+1)$, we have $N/k^2 < 0.4311$ for s_x large enough. $\square$

Theorem 8. *For all s_x large enough, any basis for $[0,s_x] \times [0,2]$ has efficiency $c < 0.4190$.*

Proof. Assume that s_x is large enough that (4) holds. Without loss of generality let A be admissible, and let its rows A_0, A_1, A_2 contain k_0, k_1, k_2 elements, respectively. Now $A_0 + A_0$ must cover $R_0 = [0, s_x]$, and $A_0 + A_1$ must cover $R_1 = [0, s_x]$, and finally $(A_0 + A_2) \cup (A_1 + A_1)$ must cover $R_2 = [0, s_x]$. By applying (4) on row 0, and by counting sums on rows 1 and 2, we obtain

$$\begin{aligned} s_x &\leq \alpha k_0^2, \\ s_x &\leq k_0 k_1, \\ s_x &\leq k_0 k_2 + k_1^2/2 + k_1/2. \end{aligned}$$

For any k, the minimum of these three bounds is maximized at their intersection, and by routine manipulations we obtain

$$s_x/k^2 \leq \frac{\alpha}{(1 + 2\alpha - \alpha^2/2)^2} + o_{s_x}(1) < 0.139663$$

for s_x large enough. Since $N = |R| = 3(s_x + 1)$, we have $c = N/k^2 < 0.4190$ for s_x large enough. □

Any improvements to the one-dimensional bound (4) will imply corresponding improvements to Theorems 7 and 8. One could also apply the same proof technique with larger constant values of s_y, but it then becomes more complicated to maximize the simultaneous upper bounds of s_x. Numerical maximization suggests decreasing upper bounds as s_y increases, for example, around 0.4126 with $s_y = 3$, and around 0.4087 with $s_y = 4$. This begs the question: what happens when s_y goes to infinity?

Turning our attention to the restricted case we obtain the following bounds.

Theorem 9. *For all s_x large enough, any restricted basis for $[0, s_x] \times [0, 2]$ has efficiency $c < 0.3149$.*

Proof. Combine Lemma 3 with the bound (5) and the fact that $|R| = 3(s_x + 1)$. □

Theorem 10. *For all s_x large enough, any restricted basis for $[0, s_x] \times [0, 4]$ has efficiency $c < 0.3585$.*

Proof. Assume that s_x is large enough that (5) holds. Let A be a restricted basis for R, and let k_0, k_1, k_2 be the cardinalities of its rows. By applying (5) on rows 0 and 4 of the target, and by counting sums on rows 1 and 3, we obtain

$$\begin{aligned} s_x &\leq \beta k_0^2, \\ s_x &\leq k_0 k_1, \\ s_x &\leq k_1 k_2, \\ s_x &\leq \beta k_2^2. \end{aligned}$$

The minimum of these four bounds is maximized at their intersection, where $k_0 = k_2$ and $k_1 = \beta k_0$, thus $k = (2+\beta)k_0$. Then we obtain

$$s_x/k^2 \le \frac{\beta}{(2+\beta)^2} < 0.071698.$$

Since $N = |R| = 5(s_x + 1)$, we have $c = N/k^2 < 0.3585$ for s_x large enough. $\square$

6.2 Lower bounds

As with one-dimensional bases, also in planar bases it is relatively easy to obtain an efficiency of approximately 1/4 for large rectangles. For squares this is particularly easy: the L-shaped basis for an s-square has $k = 2s + 1$, so $c = 0.25 + O(1/s)$. The boundary basis has $k = 2s$, so its efficiency has the same asymptotic form.

However, for non-square rectangles, the efficiency of the L-shaped and boundary bases falls below 1/4. Indeed, consider rectangles $[0, s_x] \times [0, s_y]$ with a constant aspect ratio $\rho \neq 1$. The L-shaped basis has $k = s_x + s_y + 1 = (1+\rho)s_x + \rho$, so

$$c \to \rho/(1+\rho)^2 < 1/4 \tag{6}$$

as $s_x \to \infty$. The case with the boundary basis is similar. For example, if the aspect ratio is $\rho = 1/9$, then both the L-shaped and boundary bases have only $c \to 0.09$ in the limit.

The following parametric constructions demonstrate that an asymptotic efficiency of 1/4 can be achieved with rectangles of essentially any constant aspect ratio, as both s_x and s_y go to infinity. The first construction, a *dense-sparse basis*, is the union of a dense part (a filled rectangle) and a sparse part (regularly spaced single elements). The second construction, a *short-bars basis*, consists of short, regularly spaced horizontal and vertical bars. Both constructions are illustrated in Figure 8. We use here the notation

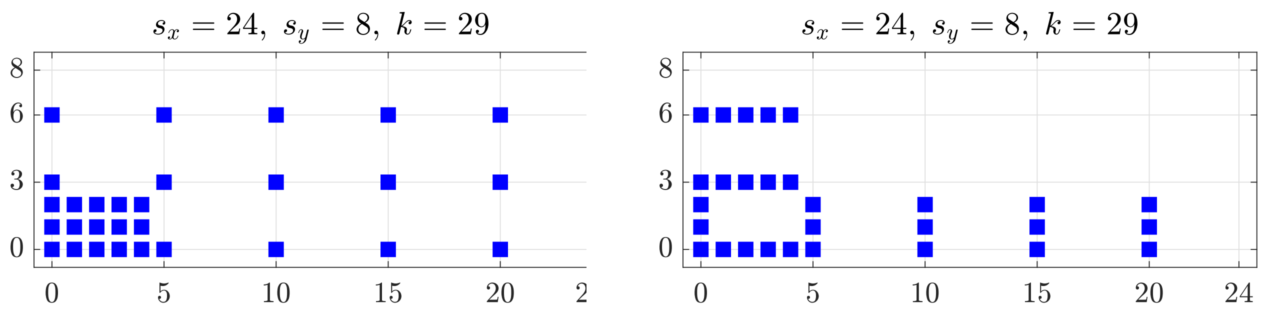
$$[a, (t), b] = \{a,\ a+t,\ a+2t,\ \ldots,\ b\}$$

for a finite arithmetic progression from a to b with step length t, with the provision that $b - a$ is divisible by t.

Definition 11. The *dense-sparse basis* with parameters $t_x, t_y \ge 1$ is the set $A = B \cup C$, where $B = [0, t_x - 1] \times [0, t_y - 1]$ and $C = [0, (t_x), t_x^2 - t_x] \times [0, (t_y), t_y^2 - t_y]$.

Theorem 12. *The dense-sparse basis has* $|A| = 2t_xt_y - 1$ *and* $A + A \supseteq [0, t_x^2 - 1] \times [0, t_y^2 - 1]$.

Proof. Since $|B| = |C| = t_xt_y$ and $B \cap C = \{(0,0)\}$, the claim on $|A|$ follows. For any point $(x, y) \in R$, let $x = b_x + c_x$ with $b_x \in [0, t_x - 1]$ and $c_x \in [0, (t_x), t_x^2 - t_x]$. Similarly let $y = b_y + c_y$ with $b_y \in [0, t_y - 1]$ and $c_y \in [0, (t_y), t_y^2 - t_y]$. Now $(x, y) = (b_x, b_y) + (c_x, c_y)$ with $(b_x, b_y) \in B$ and $(c_x, c_y) \in C$. Thus $(x, y) \in B + C \subseteq A + A$. $\square$



(a) (b)

Figure 8: Two bases for the rectangle $[0, 24] \times [0, 8]$: (a) a dense-sparse basis (Definition 11), (b) a short-bars basis (Definition 13), both with parameters $t_x = 5$, $t_y = 3$. Both bases have only 29 elements, while an L-shaped basis for the same rectangle has 33 elements.

Definition 13. The *short-bars basis* with parameters $t_x, t_y \geq 1$ is the set $A = B \cup C$, where $B = [0, t_x - 1] \times [0, (t_y), t_y^2 - t_y]$ and $C = [0, (t_x), t_x^2 - t_x] \times [0, t_y - 1]$.

Theorem 14. *The short-bars basis has* $|A| = 2t_xt_y - 1$ *and* $A + A \supseteq [0, t_x^2 - 1] \times [0, t_y^2 - 1]$.

Proof. Since $|B| = |C| = t_xt_y$ and $B \cap C = \{(0, 0)\}$, the claim on $|A|$ follows. For any point $(x, y) \in R$, let $x = b_x + c_x$ with $b_x \in [0, t_x - 1]$ and $c_x \in [0, (t_x), t_x^2 - t_x]$. Similarly let $y = b_y + c_y$ with $b_y \in [0, (t_y), t_y^2 - t_y]$ and $c_y \in [0, t_y - 1]$. Now $(x, y) = (b_x, b_y) + (c_x, c_y)$ with $(b_x, b_y) \in B$ and $(c_x, c_y) \in C$. Thus $(x, y) \in B + C \subseteq A + A$. □

Corollary 15. *Let* $\rho = p^2/q^2$ *be a fixed aspect ratio, where* p *and* q *are integers, and let* $h \geq 1$ *be an integer. Then both the dense-sparse basis and the short-bars basis, with parameters* $t_x = qh$ *and* $t_y = ph$*, are bases for the rectangle* $[0, t_x^2 - 1] \times [0, t_y^2 - 1]$*, which has the said aspect ratio. The efficiency of either basis is*

$$c = \frac{t_x^2 t_y^2}{(2t_xt_y - 1)^2} = 0.25 + O(1/h^2).$$

Corollary 16. *For any fixed aspect ratio* $\rho = p^2/q^2 \neq 1$*, with* p *and* q *integers, the L-shaped basis and the boundary bases are asymptotically suboptimal as* $s_x \to \infty$*.*

For arbitrarily wide rectangles of any *constant height* we present a basis construction whose asymptotic efficiency *exceeds* $1/4$. The construction is somewhat analogous to Mrose's one-dimensional basis [18], hence the name. As a mnemonic for our symbols here, note that in I_1, I_2, I_3 the set of x coordinates is an interval; in T it is a t-step arithmetic progression; and in S it is a "sparse" $(t + 1)$-step arithmetic progression.

Definition 17. The *stacked Mrose basis* with parameters $s_y \geq 0$ and $t \geq 1$ is the set $I_1 \cup I_2 \cup I_3 \cup T \cup S$, where

$$\begin{aligned} I_1 &= [0, t] \times Y, \\ T &= [0, (t), at^2 - t] \times \{0\}, \\ S &= [at^2, (t+1), (a+1)t^2 - 1] \times Y, \\ I_2 &= [2at^2, 2at^2 + t] \times Y, \\ I_3 &= [(3a+1)t^2, (3a+1)t^2 + t] \times Y, \end{aligned}$$

and $Y = [0, s_y]$ and $a = 4s_y + 3$.

Theorem 18. *If A is a stacked Mrose basis, then $|A| = (8s_y + 7)t + (3s_y + 1)$ and $A + A \supseteq [0, (16s_y + 14)t^2 - 1] \times [0, s_y]$.*

Proof. Let us first determine the size of the basis. We observe that $|I_1| = |I_2| = |I_3| = (t+1)(s_y+1)$, $|T| = at$, and $|S| = t(s_y+1)$. Because the parts are otherwise disjoint except that $I_1 \cap T = \{(0,0), (t,0)\}$, the claim on $|A|$ follows.

Let us next verify that $A + A$ covers the desired target rectangle. We check seven consecutive subrectangles in turn.

1. $[0, at^2 - 1] \times Y$ is covered by $I_1 + T$.
2. $[at^2, (a+1)t^2 - 1] \times Y$ is covered by $I_1 + S$.
3. $[(a+1)t^2, 2at^2 - 1] \times Y$ is covered by $T + S$.
4. $[2at^2, 3at^2 - 1] \times Y$ is covered by $I_2 + T$.
5. $[3at^2, (3a+1)t^2 - 1] \times Y$ is covered by $I_2 + S$.
6. $[(3a+1)t^2, (4a+1)t^2 - 1] \times Y$ is covered by $I_3 + T$.
7. $[(4a+1)t^2, (4a+2)t^2 - 1] \times Y$ is covered by $I_3 + S$.

Because $I_1, I_2, I_3, T, S \subseteq A$, combining observations (1)–(7) and $4a + 2 = 16s_y + 14$ we have

$$A + A \supseteq [0, (16s_y + 14)t^2 - 1] \times Y$$

as claimed. ∎

Corollary 19. *The stacked Mrose basis has efficiency*

$$c = \frac{N}{k^2} = \frac{(16s_y + 14)t^2 \cdot (s_y + 1)}{((8s_y + 7)t)^2 + O(t)} \xrightarrow[t \to \infty]{} \frac{2s_y + 2}{8s_y + 7}.$$

Example 20. With $s_y = 1$, Definition 17 gives a basis of size $k = 15t + 4$ for the rectangle $[0, 30t^2 - 1] \times [0, 1]$, with efficiency tending to $4/15 > 0.2666$ as $t \to \infty$.

Example 21. With $s_y = 2$, Definition 17 gives a basis of size $k = 23t + 7$ for the rectangle $[0, 46t^2 - 1] \times [0, 2]$, with efficiency tending to $6/23 > 0.2609$ as $t \to \infty$. Figure 9 illustrates this basis in the case of $t = 10$.

Although a stacked Mrose basis can be constructed arbitrarily high, its efficiency tends down to $1/4$ as s_y goes to infinity. We do not know whether $1/4$ can be asymptotically exceeded for rectangles with both dimensions going to infinity (e.g., with a constant aspect ratio).

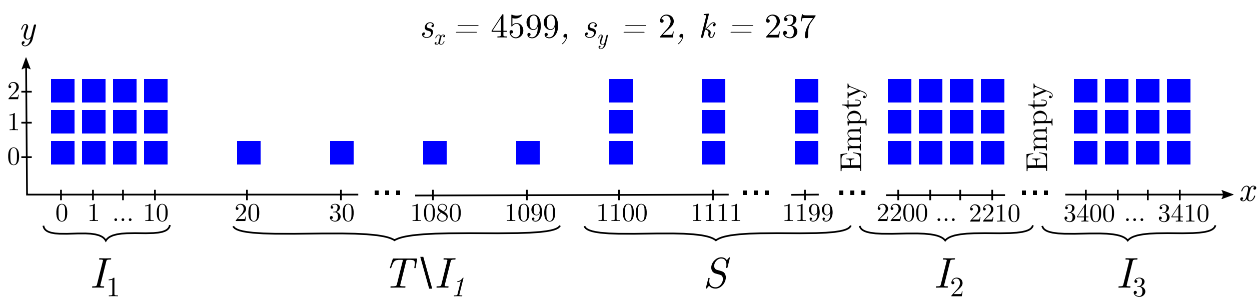


Figure 9: A schematic illustration of the stacked Mrose basis (Definition 17) with parameters $s_y = 2$ and $t = 10$. In this case $a = 11$ and $s_x = 4599$.

7 Conclusion and open questions

Planar additive bases are a natural generalization of the classical one-dimensional additive bases, and it may be slightly surprising that they have not been studied much. In this paper, some initial results have been provided. For small squares and rectangles, we have determined the minimum cardinalities exactly; and for larger instances, we have established some lower and upper bounds, although the bounds are not very tight.

Apart from the obvious desires of extending the finite results and improving the bounds, we would like to pose some open questions. For squares, the "trivial" L-shaped and boundary bases achieve an asymptotic efficiency of $1/4$. For non-square rectangles, the trivial bases fall below $1/4$, but our parametric constructions still attain $1/4$. Can this be improved at all? We pose this question in two forms, one for squares and one for general rectangles.

Question 22. Does any square $[0, s] \times [0, s]$ admit an additive basis of less than $2s$ elements?

Question 23. Is there a constant $c > 1/4$ such that there are arbitrarily high and wide rectangles admitting an additive basis of efficiency at least c?

We would not be too surprised by a positive answer to our questions above. We recall that in one-dimensional additive bases, contrary to Rohrbach's conjecture [22, p. 9], parametric constructions with efficiencies over $1/4$ have been found [5, 18, 11]. Also, for constant-height rectangles, our parametric construction (stacked Mrose basis) exceeds $1/4$. But if both dimensions tend to infinity, the question is open.

For other directions of future research, we note that additive bases are conceptually closely related to *difference bases*, where the object of interest is the difference set $A - A$. One-dimensional difference bases have been studied by, e.g., Leech [14] and Wichmann [23]. Difference bases find applications in sensor arrays for passive sensing, particularly when second-order statistics of the element outputs are processed [8]. Due to the use of data covariance in many applications, such as direction-of-arrival estimation, both one- and two-dimensional difference bases have received attention recently [15, 20, 16]. We also point out that non-rectangular, for example hexagonal, grids have received some attention in array processing using difference bases [7]. Additive bases for non-rectangular targets would be another interesting direction of research.

8 Acknowledgments

We thank the anonymous referee for insightful suggestions to improve the exposition.

References

[1] M. F. Challis, Two new techniques for computing extremal h-bases A_k, *Comput. J.* **36** (1993), 117–126.

[2] S. Eliahou and M. Kervaire, Sumsets in vector spaces over finite fields, *J. Number Theory* **71** (1998), 12–39.

[3] S. Eliahou and M. Kervaire, Minimal sumsets in infinite abelian groups, *J. Algebra* **287** (2005), 449–457.

[4] S. Eliahou, M. Kervaire, and A. Plagne, Optimally small sumsets in finite abelian groups, *J. Number Theory* **101** (2003), 338–3487.

[5] N. Hämmerer and G. Hofmeister, Zu einer Vermutung von Rohrbach, *J. Reine Angew. Math.* **286/287** (1976), 239–247.

[6] S. P. S. Han, The boundary structure of the sumset in $\mathbf{Z}^2$. In D. Chudnovsky, G. Chudnovsky, and M. Nathanson, editors, *Number Theory*, Springer, 2004, pp. 201–218.

[7] R. A. Haubrich, Array design, *Bulletin of the Seismological Society of America* **58** (1968), 977–991.

[8] R. T. Hoctor and S. A. Kassam, The unifying role of the coarray in aperture synthesis for coherent and incoherent imaging, *Proc. IEEE* **78** (1990), 735–752.

[9] J. Kohonen, Early pruning in the restricted postage stamp problem. Preprint, 2015, available at https://arxiv.org/abs/1503.03416.

[10] J. Kohonen, A meet-in-the-middle algorithm for finding extremal restricted additive 2-bases, *J. Integer Sequences* **17** (2014), Article 14.6.8.

[11] J. Kohonen, An improved lower bound for finite additive 2-bases, *J. Number Theory* **174** (2017), 518–524.

[12] J. Kohonen and J. Corander, Addition chains meet postage stamps: Reducing the number of multiplications, *J. Integer Sequences* **17** (2014), Article 14.3.4.

[13] R. J. Kozick and S. A. Kassam, Linear imaging with sensor arrays on convex polygonal boundaries, *IEEE Trans. Systems Man Cybernet.* **21** (1991), 1155–1166.

[14] J. Leech, On the representation of 1, 2, ..., n by differences, *J. Lond. Math. Soc.* **31** (1956), 160–169.

[15] D. A. Linebarger, I. H. Sudborough, and I. G. Tollis, Difference bases and sparse sensor arrays, *IEEE Trans. Inform. Theory* **39** (1993), 716–721.

[16] C. L. Liu and P. P. Vaidyanathan, Hourglass arrays and other novel 2-D sparse arrays with reduced mutual coupling, *IEEE Trans. Signal Process.* **65** (2017), 3369–3383.

[17] W. F. Lunnon, A postage stamp problem, *Comput. J.* **12** (1969), 377–380.

[18] A. Mrose, Untere Schranken für die Reichweiten von Extremalbasen fester Ordnung, *Abh. Math. Semin. Univ. Hambg.* **48** (1979), 118–124.

[19] P. S. Naidu, *Sensor Array Signal Processing*, CRC Press, 2nd edition, 2009.

[20] P. Pal and P. P. Vaidyanathan, Nested arrays: A novel approach to array processing with enhanced degrees of freedom, *IEEE Trans. Signal Process.* **58** (2010), 4167–4181.

[21] J. Riddell and C. Chan, Some extremal 2-bases, *Math. Comp.* **32** (1978), 630–634.

[22] H. Rohrbach, Ein Beitrag zur additiven Zahlentheorie, *Math. Z.* **42** (1937), 1–30.

[23] B. Wichmann, A note on restricted difference bases, *J. Lond. Math. Soc.* **s1-38** (1963), 465–466.

[24] G. Yu, Upper bounds for finite additive 2-bases, *Proc. Amer. Math. Soc.* **137** (2009), 11–18.

[25] G. Yu, A new upper bound for finite additive h-bases, *J. Number Theory* **156** (2015), 95–104.

2010 *Mathematics Subject Classification*: Primary 11B13.
Keywords: additive basis, restricted basis, planar basis, rectangular sumset.

(Concerned with sequences A001212, A008574, A295771, A295774.)

Received August 15 2018; revised version received December 17 2018. Published in *Journal of Integer Sequences*, December 17 2018.

Return to Journal of Integer Sequences home page.