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Constructing Pseudo-Involutions in the Riordan Group

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Abstract

Involutions and pseudo-involutions in the Riordan group are interesting because of their numerous applications. In this paper we study involutions using sequence characterizations of the Riordan arrays. For a given B-sequence we find the unique function $f(z)$ such that the array $\big(g(z), f(z)\big)$ is a pseudo-involution. As a combinatorial application, we find the interpretation of each entry in the Bell array $\big(g(z), f(z)\big)$ with a given B-sequence.

1 Introduction

A Riordan array originally introduced by Shapiro et al. [13] is defined in terms of generating functions of its columns. Let $g(z) = g_0 + g_1z + g_2z^2 + \cdots$, $g_0 \neq 0$ and $f(z) = f_1z + f_2z^2 + f_3z^3 + \cdots$, $f_1 \neq 0$ be two formal power series. The Riordan array generated by $g(z)$ and

$f(z)$ is an infinite lower triangular array D whose kth column has the generating function $g(z)\big(f(z)\big)^k$ for all $k \geq 0$. We denote D by $\big(g(z), f(z)\big)$. In other words $D = (d_{n,k})_{n,k\geq 0}$ is Riordan if and only if there exist two generating functions $g(z)$ and $f(z)$ such that $d_{n,k}$ is the coefficient of z^n in the expansion of $g(z)\big(f(z)\big)^k$.

The set $\mathcal{R}$ of all Riordan arrays forms a group under the matrix multiplication operation. In terms of generating functions, the product of two arrays $\big(g(z), f(z)\big)$ and $\big(\alpha(z), \beta(z)\big)$ can be written as

$$\big(g(z), f(z)\big) \cdot \big(\alpha(z), \beta(z)\big) = \big(g(z)\alpha(f(z)), \beta(f(z))\big).$$

The usual identity matrix $(1, z)$ serves as the group identity and for any $\big(g(z), f(z)\big) \in \mathcal{R}$, $\big(\frac{1}{g(\overline{f}(z))}, \overline{f}(z)\big)$ is its inverse, where $\overline{f}$ represents the compositional inverse of f.

Note that the original definitions of Riordan array are based on column construction. Merlini et al. [11] and Sprugnoli [14] characterized Riordan arrays using two sequences called the A-sequence and the Z-sequence which enables us to construct the Riordan array horizontally. That is each entry in the Riordan array can be written as a linear combination of entries in the previous row. More precisely we have

Theorem 1. *An array $D = (d_{n,k})_{n,k\geq 0}$ is Riordan if and only if there exist unique sequences $A = (a_0, a_1, a_2, \ldots)$ and $Z = (z_0, z_1, z_2, \ldots)$ such that*

1. $d_{n+1,k+1} = \sum_{i=0}^{\infty} a_i d_{n,k+i}$ *and*
2. $d_{n+1,0} = \sum_{i=0}^{\infty} z_i d_{n,i}$.

The sequences A and Z in this theorem are called the A-sequence and Z-sequence respectively. See also [8]. See [6] for another characterization of Riordan arrays (in terms of Stieltjes transform matrix). See also [7].

A nontrivial element $D = \big(g(z), f(z)\big) \in \mathcal{R}$ is an involution if and only if $D^2 = I$. Let $M = (1, -z)$, that is M is a diagonal matrix with $(1, -1, 1, -1, \ldots)$ along the diagonal. An element $D = \big(g(z), f(z)\big)$ is a pseudo-involution if and only if DM is an involution or equivalently MD is an involution or $D^{-1} = MDM$. See [1, 2, 3, 4, 5] for more information. In this paper we study pseudo-involutions via B-sequences. It is an important fact that the A-sequence uniquely determines the generating function f. One version is $f(z) = zA(f(z))$ or equivalently $z = \overline{f}(z)A(z)$. This could be called the second fundamental theorem of the Riordan group. A "dot" diagram of this is

$$\begin{array}{|c|c|c|c|c|}\hline a_0 & a_1 & a_2 & a_3 & a_4 \\ \hline \multicolumn{1}{c|}{} & \otimes & \multicolumn{3}{|c}{} \\ \cline{2-2}\end{array}\ \cdots$$

However there are many situations where it is useful to expand to an A-matrix [11]

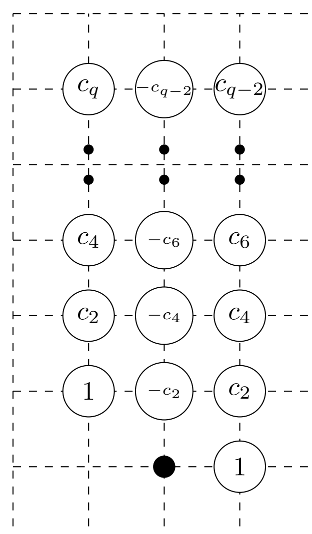
$\cdots$		$\cdots$		
$a_{2,0}$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$\cdots$
$a_{1,0}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$\cdots$
	$\otimes$	$a_{0,2}$	$a_{0,3}$	$\cdots$

A modest example is given by

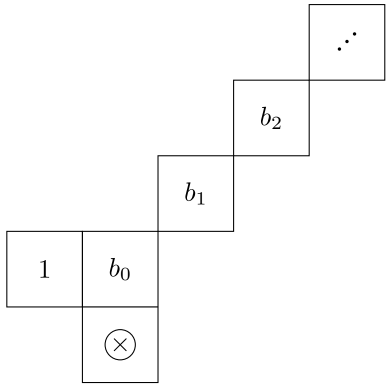
	1	
1		1
	$\otimes$	

This produces the large Schröder numbers.

Recently Merlini and Sprugnoli [10] studied binary words avoiding certain patterns using A-matrices such as



Here is a picture of the A-matrix that produces the B-sequence.



See also [12] for several online software tools for exploring Riordan arrays.

This paper is organized as follows. In Section 2 we present some ways of constructing pseudo-involutions. In Section 3 we discuss those pseudo-involutions via B-sequences, we present a list of 24 examples, and we also present an explicit formula to compute the A-sequence of a pseudo-involution with a given B-sequence. Finally in Section 4 we present combinatorial interpretations for all involutions in the Bell subgroup.

2 (Pseudo)-involutions

Given a formal power series $g(z) = 1+\sum_{n\geq 1} g_n z^n$, we want to find a generating function $f(z)$ such that $\big(g(z), f(z)\big)^2 = I = (1, z)$. For such $f(z)$, we must have $g(z)g(f(z)) = 1$. That is $g(f(z)) = \frac{1}{g(z)}$. We first study a special case in which $g(z)$ can be expressed as $g = 1 + zg^k$, for some $k \in \mathbb{N}$.

Theorem 2. *Let $g(z) = 1+\sum_{n\geq 1} g_n z^n$ be a power series where $g = 1+zg^k$, for some $k \in \mathbb{N}$. Then $\big(g(z), -z(g(z))^{2k-1}\big)^2 = I$.*

Proof. Since $g(z) = 1 + zg^k$, we can write g as

$$\begin{aligned} g(z) &= \frac{1}{1 - zg^{k-1}} \\ &= \big(1 - zg^{k-1}\big)^{-1}. \end{aligned}$$

So $zg^{k-1} = z(1 - zg^{k-1})^{-(k-1)}$. Now set

$$F = F(z) = zg^{k-1}. \tag{1}$$

Then $F = z(1 - F)^{-(k-1)}$. Apply $\overline{F}$ to this equation to get

$$z = \overline{F}(1 - z)^{-(k-1)}.$$

Then

$$\overline{F} = z(1-z)^{k-1}. \tag{2}$$

Starting with Eq. 1, we get

$$F\big(f(z)\big) = f(z)g\big(f(z)\big)^{k-1} = f\cdot\big(g(f)\big)^{k-1} = f\big(\frac{1}{g}\big)^{k-1} = \frac{f}{g^{k-1}}.$$

Now apply $\overline{F}$ to get

$$f(z) = \overline{F}\big(\frac{f}{g^{k-1}}\big) = \frac{f}{g^{k-1}}\big(1-\frac{f}{g^{k-1}}\big)^{k-1}.$$

This implies

$$\begin{aligned} g^{k-1} &= \big(1-\frac{f}{g^{k-1}}\big)^{k-1} \\ \Rightarrow g &= 1-\frac{f}{g^{k-1}} \\ \Rightarrow f &= g^{k-1}-g^{k} \\ \Rightarrow f &= g^{k-1}(1-g). \end{aligned}$$

But $1-g=-zg^k$ so $f=g^{k-1}(-zg^k)=-zg^{2k-1}$. $\square$

The following important examples which are some special cases of this theorem go back at least to 1976 in a paper of Hoggatt and Bicknell in the *Fibonacci Quarterly* [9]. These occur again in Section 3 as examples 1, 10, 18, and 20. Here the generating functions C, T and Q refer to the Catalan, ternary, and quaternary numbers respectively while $P=\frac{1}{1-z}$.

k	$g=1+zg^k$	$f=-zg^{2k-1}$	$(g,-zg^{2k-1})^2=I$
$k=1$	$P=1+zP$	$f=-zP$	$(P,-zP)^2=I$
$k=2$	$C=1+zC^2$	$f=-zC^3$	$(C,-zC^3)^2=I$
$k=3$	$T=1+zT^3$	$f=-zT^5$	$(T,-zT^5)^2=I$
$k=4$	$Q=1+zQ^4$	$f=-zQ^7$	$(Q,-zQ^7)^2=I$

Table 1: Some special cases of Theorem 2

Another interesting relationship among these examples will be presented in Section 3 after presenting the notion of B-sequences.

For $g(z)=1+\sum_{n\geq 1} g_n z^n$ and $f(z)=z+\sum_{n\geq 2} f_n z^n$, we know that if $\big(g(z),f(z)\big)$ is a pseudo-involution then $\big(g(z),-f(z)\big)$ is an involution so $f(-f(-z))=z$. Then we use [15, Exercise 168, p. 134] to specify the even indexed coefficients $(f_2,f_4,\ldots)$ arbitrarily which then determine the odd indexed coefficients $(f_3,f_5,\ldots)$ uniquely. So there are many generating functions $f(z)$ which generate pseudo-involutions. It is easy to show that if $f(-f(-z)) = z$, then $\big(f(z)/z, f(z)\big)$ is a pseudo-involution. Moreover it has been established [2, Thm. 2.3] that if $\big(g(z), f(z)\big)$ is an involution, then $g(z) = \pm\exp(\Phi(z, zf(z)))$ for some antisymmetric function Φ. However we do not know the answer of the following question.

Question 3. *Given $f(z)$ such that $f(-f(-z)) = z$ or $\overline{f}(z) = -f(-z)$, what are the possibilities for $g(z)$ so that $\big(g(z), f(z)\big)$ is an involution, particularly in combinatorial situations?*

Another way to find more involutions and thus pseudo-involutions comes from basic group theory. If α and β are two noncommuting involutions then $\langle\alpha, \beta\rangle$ is a dihedral group and $(\alpha\beta)^n\beta$ is an involution for all integers n, where $\alpha\beta$ plays the role of a rotation. In the Riordan group if M_1 and M_2 are noncommuting involutions then $\langle M_1, M_2\rangle$ is a dihedral group.

Proposition 4. *Let D_1 and D_2 be two pseudo-involutions. Then both $(D_1D_2^{-1})^nD_2$ and $D_1(D_1^{-1}D_2)^n$ are pseudo-involutions for all integers n.*

Proof. Let $H_1 = D_1M$ and $H_2 = D_2M$. Then H_1 and H_2 are involutions and so for all integer n, $(H_1H_2)^nH_2$ is an involution and

$$\begin{aligned}(H_1H_2)^nH_2 &= (D_1MD_2M)^nD_2M\\ &= (D_1D_2^{-1})^nD_2M.\end{aligned}$$

Hence $(D_1D_2^{-1})^nD_2$ is a pseudo-involution. On the other hand if we let $J_1 = MD_1$ and $J_2 = MD_2$, then

$$\begin{aligned}J_1(J_1J_2)^n &= MD_1(MD_1MD_2)^n\\ &= MD_1(D_1^{-1}D_2)^n.\end{aligned}$$

Since $J_1(J_1J_2)^n$ is an involution, $D_1(D_1^{-1}D_2)^n$ is also a pseudo-involution. ∎

The following result was first proved [4] using induction on n but it follows more quickly from the proposition we just proved.

Corollary 5. *Let $D = \big(g(z), f(z)\big)$ be a pseudo involution. Then so is D^n, for all $n \in \mathbb{Z}$.*

Proof. Let $D_1 = D$ and $D_2 = (1, z)$. Then D_1 and D_2 are pseudo-involutions and thus Proposition 4 applies. ∎

3 Relationship with B-sequences

Let $A(z)$ be the generating function of the A sequence of a Riordan array $\big(g(z), f(z)\big)$. We can write $f(z)$ in terms of A as follows

$$f(z) = zA\big(f(z)\big).$$

Replace z by $\overline{f}(z)$ to get $z = \overline{f}(z)A(z)$. So $A(z) = \frac{z}{\overline{f}(z)}$.

In case of pseudo-involutions, we have $\overline{f}(z) = -f(-z)$. Thus we also have

Lemma 6. *Let $A(z)$ be the generating function of the A-sequence of the pseudo-involution Riordan array $\big(g(z), f(z)\big)$. Then*

$$A(z) = \frac{-z}{f(-z)} = \frac{z}{-f(-z)}.$$

The following concept was discussed by Cheon et al. [4]. There the term "Δ-sequence" was used instead of B-sequence.

Definition 7. For a Riordan array $D = (d_{n,k})_{n,k\geq 0}$, a sequence $(b_0, b_1, b_2, \ldots)$ is said to be a *B-sequence* if and only if

$$d_{n+1,k} = d_{n,k-1} + \sum_{j\geq 0} b_j \cdot d_{n-j,k+j}.$$

Note that $d_{n,k} = 0$ for all $n < k$. So the sum on the right side is finite. In other words, we must have $n - j \geq k + j$. That is $2j \leq n - k$.

In the following theorem we link the A-sequence with the B-sequence. It also follows from a notion of A-matrix [11].

Theorem 8. *Let $B(z) = b_0 + b_1 z + b_2 z^2 + \cdots = \sum_{k\geq 0} b_k z^k$ be the B-sequence of a pseudo involution $D = \big(g(z), f(z)\big)$. Then the A-sequence of D is given by*

$$A(z) = 1 + zB\big(-zf(-z)\big) = 1 + zB\left(\frac{z^2}{A(z)}\right).$$

Proof. Since $A(z) = \frac{-z}{f(-z)} = \frac{z}{\overline{f}(z)}$, we have $\overline{f}(z) = -f(-z)$. We also have $f(z) = z + zf(z)B\big(zf(z)\big) = zA\big(f(z)\big)$. So $A\big(f(z)\big) = 1 + f(z)B\big(zf(z)\big)$. Therefore $A\big(f(z)\big) = 1 + f(z)[b_0 + b_1 zf(z) + b_2 z^2 (f(z))^2 + \cdots]$. Replace z by $\overline{f}(z)$. Then

$$\begin{aligned} A(z) &= 1 + z\sum_{k\geq 0} b_k \big(z\overline{f}(z)\big)^k \\ &= 1 + z\sum_{k\geq 0} b_k \big(-zf(-z)\big)^k \\ &= 1 + zB\left(\frac{z^2}{A(z)}\right). \end{aligned}$$

□

Corollary 9. *If $\big(g(z), f(z)\big)$ is in the Bell subgroup, then*

$$A(z) = 1 + \sum_{k\geq 0} b_k z^{2k+1} (g(-z))^k.$$

Proof. We have $f(z) = zg(z)$. So $f(-z) = -zg(-z)$. Therefore

$$A(z) = 1 + z\sum_{k\geq 0} b_k\big(z^2 g(-z)\big)^k = 1 + \sum_{k\geq 0} b_k z^{2k+1}\big(g(-z)\big)^k.$$

□

In Corollary 5, we showed that any power of a pseudo-involution is a pseudo-involution. But we do not know how to easily combine the various B-sequences. The question remains open even in the case of the same pseudo-involution. More precisely,

Question 10. *Let $D = \big(g(z), f(z)\big)$ be a pseudo-involution with the B-sequence $b_0, b_1, b_2, \ldots$. what is the B-sequence for D^2?*

We present a list of some of the examples in the following tables. All these examples are in the Bell subgroup where $f(z) = zg(z)$. One can apply Theorem 11 to construct many examples which are not in the Bell subgroup. Six out of these 24 examples can also be found [4]. We cluster them in families of B-sequences. One can use Theorem 8 to compute the A-sequence in each case. In these examples m, C, T, and r represent the Motzkin, Catalan, ternary, and large Schröder generating functions respectively.

Geometric B-sequences: One can compute the unique $f(z)$ such that $\big(g(z), f(z)\big)$ is a pseudo-involution with the B-sequence $b, bk, bk^2, \ldots$ as follows

$$f(z) = \begin{cases} \frac{z}{1-bz}, & \text{if } \ k = 0; \\ \frac{1-bz+kz^2-\sqrt{(1-bz+kz^2)^2-4kz^2}}{2kz}, & \text{if } k \neq 0. \end{cases}$$

We include some examples in the following table.

	g	B-sequence	Comments
1	$1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$	$1, 0, 0, \ldots$	Pascal
2	$1, 1, 1, 2, 4, 8, 17, 37, 82, 185, \ldots$	$1, 1, 1, \ldots$	RNA (A004148)
3	$1, 2, 4, 10, 28, 82, 248, 770, \ldots$	$2, 2, 2, \ldots$	A187256
4	$1, 4, 16, 68, 304, 1412, 6752, \ldots$	$4, 4, 4, \ldots$	r^2

Table 2: Examples with geometric B-sequences

Linear B-sequences: If the B-sequence is $a, b, 0, 0, 0, \ldots$, then the unique $f(z)$ is

$$f(z) = \frac{1 - az - \sqrt{(1-az)^2 - 4bz^3}}{2bz^2}.$$

Others:

	g	B-sequence	Comments
5	$1, 1, 1, 2, 4, 7, 13, 26, 52, 104, 212, \dots$	$1, 1, 0, \dots$	A023431
6	$1, 1, 1, 3, 7, 13, 29, 71, \dots$	$1, 2, 0, \dots$	A091565
7	$1, 1, 1, 4, 10, 19, 49, 136, 334, 850, \dots$	$1, 3, 0, \dots$	
8	$1, 2, 4, 9, 22, 56, 146, 388, 1048, \dots$	$2, 1, 0, \dots$	A091561
9	$1, 2, 4, 10, 28, 80, 232, 688, 2080, \dots$	$2, 2, 0, \dots$	
10	$1, 3, 9, 28, 90, 297, 1001, 3432, \dots$	$3, 1, 0, \dots$	C^3 (A000245)
11	$1, 1, 1, 5, 13, 25, 73, 221, 565, 1553, \dots$	$1, 4, 0, \dots$	
12	$1, 4, 16, 65, 268, 1120, 4738, 20264, \dots$	$4, 1, 0, \dots$	

Table 3: Examples with linear B-sequences

	g	B-sequence	Comments
13	$1, 1, 1, 2, 4, 9, 21, 50, 122, \dots$	$1, 1, 2, 4, 9, 21, 51, 127, \dots$	
14	$1, 1, 1, 2, 4, 9, 21, 51, 127, \dots$	$1, 1, 2, 5, 14, 42, 132, 429, \dots$	$1 + zm$
15	$1, 1, 1, 2, 4, 10, 28, 85, \dots$	$1, 3, 9, 28, 90, 297, 1001, \dots$	
16	$1, 2, 4, 9, 22, 57, 154, 429, \dots$	$2, 1, 1, 1, 1, 1, \dots$	A105633
17	$1, 4, 16, 68, 304, 1409, \dots$	$4, 4, 1, 0, 0, \dots$	
18	$1, 5, 25, 130, 700, 3876, \dots$	$5, 5, 1, 0, 0, \dots$	Ternary(A102893)
19	$1, 2, 4, 12, 40, 129, 424, \dots$	$2, 4, 1, 0, 0 \dots$	
20	$1, 7, 49, 357, 2695, \dots$	$7, 14, 7, 1, \dots$	Quaternary(A233658)
21	$1, 4, 16, 64, 256, 1024, \dots$	$4, 0, 0, \dots$	(B, zB^2)
22	$1, 0, 0, 2, 0, 0, 5, 0, 0, 14, \dots$	$0, 1, 0, 0, \dots$	$(C(z^3), zC(z^3))$
23	$1, 0, 0, 0, 0, 3, 0, 0, 0, 0, 12, \dots$	$0, 0, 1, 0, 0, \dots$	$(T(z^5), zT(z^5))$
24	$1, 1, 1, 2, 4, 9, 21, 50, 122, \dots$	$1, 1, 2, 4, 8, \dots$	

Table 4: Examples with other B-sequences

For each generating function $g(z)$ and for each B-sequence, there exists a generating function $f(z)$ such that the Riordan array $\big(g(z), f(z)\big)$ is a pseudo-involution. In fact we see from Eq. 8 that for every B-sequence there exists a unique function $f(z)$ such that $(g(z), f(z))$ is a pseudo-involution. Furthermore in the Bell subgroup such an array is unique because $g(z) = \frac{f(z)}{z}$. But the function $g(z)$ is far from being unique. The following result shows that there are infinitely many functions $g(z)$ for given $f(z)$ such that the array $\big(g(z), f(z)\big)$ is pseudo-involution with same B-sequence.

Theorem 11. *Let $(g(z), f(z))$ be a pseudo-involution Riordan array and let $B(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ be the generating function of its B-sequence. Then $\big((g(z))^n, f(z)\big)$ is also a pseudo-involution with the same B-sequence, for all $n \in \mathbb{N}$.*

Proof. Let $\big(g(z), f(z)\big)$ be a pseudo-involution. Then

$$\big((g(z), f(z))(1, -z)\big)^2 = (g(z), -f(z))^2 = (1, z).$$

That is $\big(g(z)g(-f(z)), -f(-f(z))\big) = (1, z)$, so $g(z)g(-f(z)) = 1$ and $-f(-f(z)) = z$. Thus for any n,

$$\begin{aligned}\big(((g(z))^n, f(z))(1, -z)\big)^2 &= \big((g(z))^n, -f(z)\big)^2 \\ &= \big(g(z)g(-f(z))^n, -f(-f(z))\big) \\ &= (1, z).\end{aligned}$$

Also if B_1 and B_2 are the B-sequences of $(g(z), f(z))$ and $((g(z))^n, f(z))$ respectively, then $f(z) = z + zf(z)B_1(zf(z)) = z + zf(z)B_2(zf(z))$. Hence $B_1 = B_2$. $\square$

Now we present a relationship among some of the pseudo-involutions presented in Section 1. The matrices (P, zP), (C, zC), (T, zT), and (Q, zQ) are linked with the g function for one being the A-sequence for the next. For instance using the Catalan numbers as A-sequence one can produce the ternary Bell matrix. See [3] for more detail.

If we look at the B-sequences of the pseudo-involutions $(P, zP), (C, zC^3), (T, zT^5)$ and (Q, zQ^7) (see Examples 1 in Table 2, 10 in Table 3, and 18 and 20 in Table 4) we obtain the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & \cdots \\ 5 & 5 & 1 & 0 & \cdots \\ 7 & 14 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is a Riordan array $\left(\frac{1+z}{(1-z)^2}, \frac{z}{(1-z)^2}\right)$. Each row of this array is the B-sequence of a pseudo-involution in this family.

We also have the following results of independent interest.

Lemma 12.

$$\begin{pmatrix} 1 & & & & \\ -3x & 1 & & & \\ 5x^2 & -5x & 1 & & \\ -7x^3 & 14x^2 & -7x & 1 & \\ 9x^4 & -30x^3 & 27x^2 & -9x & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+x \\ (1+x)^3 \\ (1+x)^5 \\ (1+x)^7 \end{pmatrix} = \begin{pmatrix} 1+x \\ 1+x^3 \\ 1+x^5 \\ 1+x^7 \end{pmatrix}$$

Proof. We apply the Fundamental Theorem of Riordan Arrays which states that if

$$\big(g(z), f(z)\big) \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \end{bmatrix}$$

then $\beta(z) = g(z)\alpha(f(z))$, where $\alpha(z) = \sum \alpha_n z^n$ and $\beta(z) = \sum \beta_n z^n$. We have

$$\begin{aligned}
g(z) &= \frac{1-xz}{(1+xz)^2}, \\
f(z) &= \frac{z}{(1+xz)^2}, \text{ and} \\
\alpha(z) &= \frac{1+x}{1-(1+x)^2 z}.
\end{aligned}$$

Therefore

$$\begin{aligned}
g(z)\alpha(f(z)) &= \frac{1-xz}{(1+xz)^2} \cdot \frac{1+x}{1-(1+x)^2 f(z)} \\
&= \frac{1-xz}{(1+xz)^2} \cdot \frac{1+x}{1-(1+x)^2 \frac{z}{(1+xz)^2}} \\
&= \frac{(1-xz)(1+x)}{(1+xz)^2-(1+x)^2 z} \\
&= \frac{1+x-xz-x^2z}{1+2xz+x^2z^2-z-2xz-x^2z} \\
&= \frac{1-x^2z+x-xz}{1+x^2z^2-z-x^2z} \\
&= \frac{1-x^2z+x-xz}{(1-z)(1-x^2z)} \\
&= \frac{1}{1-z} + \frac{x}{1-x^2z}.
\end{aligned}$$

$\square$

An analogous result for the even powers of $1+x$ is as follows.

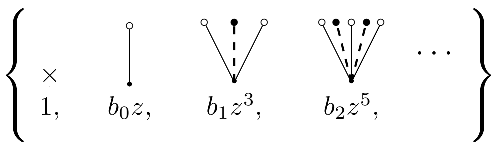
Lemma 13.

$$\begin{pmatrix} 1 & & & & \\ -2x & 1 & & & \\ 2x^2 & -4x & 1 & & \\ -2x^3 & 9x^2 & -6x & 1 & \\ 2x^4 & -16x^3 & 20x^2 & -8x & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ (1+x)^2 \\ (1+x)^4 \\ (1+x)^6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+x^2 \\ 1+x^4 \\ 1+x^6 \end{pmatrix}$$

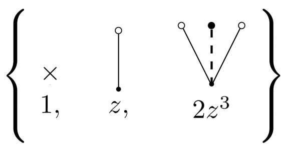
That is $\left(\frac{1-xz}{1+xz}, \frac{z}{(1+xz)^2}\right) \cdot \frac{1}{1-(1+x)^2 z} = \frac{1}{1-z} + \frac{x^2 z}{1-x^2 z}$.

4 Applications

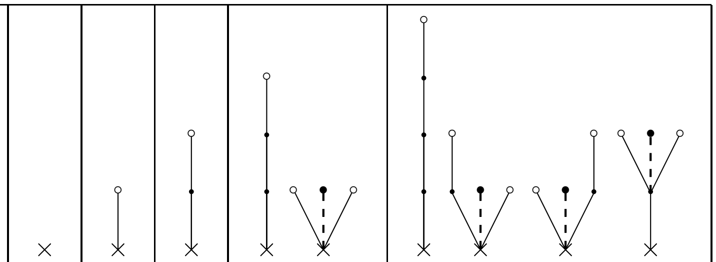
In this section we present an interpretation for each entry in the first column of the Bell array $\left(\frac{f(z)}{z}, f(z)\right)$ with a given B-sequence $b_0, b_1, b_2, \ldots$. For that we define a *PI* (*pseudo-involution*) *tree*. A *PI tree* is built from subtrees which if nontrivial, consist of a root and $2n+1$ edges, $n+1$ of which are active and n of which are sterile with no descendants. The weight b_i is assigned to the building block subtree with $2i+1$ edges. We draw these where the sterile edges are drawn as dotted lines and alternate with the active edges. Some examples are as follows.



For instance if $b_0 = 1$, $b_1 = 2$, and $b_k = 0$, $k \geq 2$, we have the building blocks in which the term $2z^3$ could represent one red and one green block.



This refers to Example 6 in Table 3. In this case we have the following PI trees with edges $n \geq 0$.



Trees					
No. of trees	1	1	1	$1+2=3$	$1+2\cdot 3=7$
No. of edges	0	1	2	3	4

Table 5: PI trees corresponding to the B-sequence $1, 2, 0, 0, \ldots$

For $n = 5$ edges we get 7 trees with root degree 1 and 6 trees with root degree 3. So the total number of such trees with 5 edges is 13.

This sequence also counts Dyck paths where all maximal U and D runs are of length 1 or 3 and each U^3 run has weight 2. We illustrate this with the following table.

In the following theorem we present an interpretation for each entry in the first column of the pseudo-involution Bell array $\left(\frac{f(z)}{z}, f(z)\right)$ with B-sequence $b_0, b_1, b_2, \ldots$. Another interpretation in terms Dyck paths can also be proved in the similar fashion.

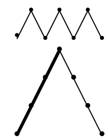
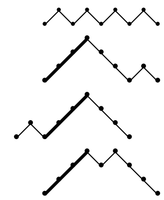
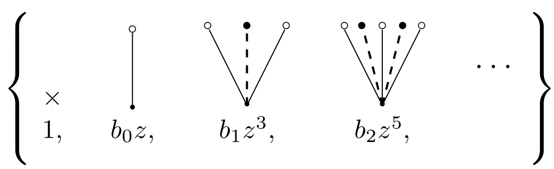
Dyck paths	$\times$				
No. of Dyck paths	1	1	1	$1+2=3$	$1+2\cdot 3=7$
No. of UD's	0	1	2	3	4

Table 6: Dyck paths corresponding to the B-sequence $1, 2, 0, 0, \ldots$

Theorem 14. *Let $\left(\frac{f(z)}{z}, f(z)\right)$ be a pseudo involution Bell array with B-sequence $(b_0, b_1, \ldots)$, where each b_i is a nonnegative integer. Then the function $\frac{f(z)}{z}$ counts the number of PI trees with the following building blocks.*



Proof. Since $\left(\frac{f(z)}{z}, f(z)\right)$ is a pseudo involution with B-sequence $b_0, b_1, b_2, \ldots$, we can write

$$f(z) = z + zf(z)B(zf(z)).$$

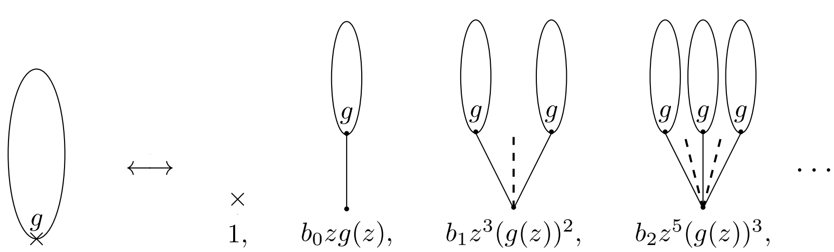
That is

$$\frac{f(z)}{z} = 1 + f(z)B(zf(z)).$$

Let $\frac{f(z)}{z} = g(z)$. Then $g(z) = 1 + zg(z)B(z^2 g(z))$. In expanded form $g(z)$ can be written as

$$\begin{aligned} g(z) &= 1 + zg(z)(b_0 + b_1 z^2 g(z) + b_2 z^4 (g(z))^2 + \cdots) \\ &= 1 + b_0 z g(z) + b_1 z^3 (g(z))^2 + b_2 z^5 (g(z))^3 + \cdots \end{aligned}$$

If the tree is nontrivial the root degree is $2n+1$ with $n+1$ active nodes and the attached weight is b_n. This gives the term $b_n z^{2n+1}\big(g(z)\big)^{n+1}$ and summing yields the generating function $g(z)$.



This shows that the generating function $g(z) = \frac{f(z)}{z}$ counts the PI trees with the given building blocks. $\square$

5 Acknowledgments

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