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On Arithmetic Functions Related to Iterates of the Schemmel Totient Functions

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Abstract

We begin by introducing an interesting class of functions, known as the Schemmel totient functions, that generalizes the Euler totient function. For each Schemmel totient function L_m, we define two new functions, denoted R_m and H_m, that arise from iterating L_m. Roughly speaking, R_m counts the number of iterations of L_m needed to reach either 0 or 1, and H_m takes the value (either 0 or 1) that the iteration trajectory eventually reaches. Our first major result is a proof that, for any positive integer m, the function H_m is completely multiplicative. We then introduce an iterate summatory function, denoted D_m, and define the terms D_m-deficient, D_m-perfect, and D_m-abundant. We proceed to prove several results related to these definitions, culminating in a proof that, for all positive even integers m, there are infinitely many D_m-abundant numbers. Many open problems arise from the introduction of these functions and terms, and we mention a few of them, as well as some numerical results.

1 Introduction

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{P}$ will denote the set of positive integers, the set of nonnegative integers, and the set of prime numbers, respectively. For any function f, we will write $f^{(1)} = f$ and $f^{(k+1)} = f \circ f^{(k)}$ for all $k \in \mathbb{N}$. The letter p will always denote a prime number. For any $n \in \mathbb{N}$, $\upsilon_p(n)$ will denote the unique nonnegative integer k such that $p^k \mid n$

and $p^{k+1} \nmid n$. Finally, in the canonical prime factorization $\prod_{i=1}^{r} p_i^{\alpha_i}$ of a positive integer, it is understood that, for all distinct $i, j \in \{1, 2, \ldots, r\}$, we have $p_i \in \mathbb{P}$, $\alpha_i \in \mathbb{N}$, and $p_i \neq p_j$.

The well-known Euler ϕ function is defined to be the number of positive integers less than or equal to n that are relatively prime to n. For each $m \in \mathbb{N}$, the Schemmel totient function $L_m(n)$ is defined as the number of positive integers $k \leq n$ such that $\gcd(k+s, n) = 1$ for all $s \in \{0, 1, \ldots, m-1\}$ [2]. In particular, $L_1 = \phi$. For no reason other than a desire to avoid cumbersome notation and the possibility of dealing with undefined objects such as $L_2^{(2)}(6)$, we will define $L_m(0)$ to be 0 for all positive integers m.

For any integer $n > 1$, let $p(n)$ be the smallest prime number that divides n. Schemmel [7] showed that, for any positive integer m, L_m is multiplicative. Thus, $L_m(1) = 1$. Furthermore, for $n > 1$,

$$L_m(n) = \begin{cases} 0, & \text{if } p(n) \leq m; \\ n \displaystyle\prod_{p|n} \left(1 - \frac{m}{p}\right), & \text{if } p(n) > m. \end{cases} \tag{1}$$

Letting $\prod_{i=1}^{r} p_i^{\alpha_i}$ be the canonical prime factorization of n, we may rewrite the above formula as

$$L_m(n) = \begin{cases} 0, & \text{if } p(n) \leq m; \\ \displaystyle\prod_{i=1}^{r} p_i^{\alpha_i - 1}(p_i - m), & \text{if } p(n) > m \end{cases} \tag{2}$$

for $n > 1$.

In 1929, S. S. Pillai introduced a function that counts the number of iterations of the Euler ϕ function needed to reach 1 [5]. In the following section, we generalize Pillai's function via the Schemmel totient functions. Then, in the third section, we generalize the concept of perfect totient numbers with the introduction, for each positive integer m, of a function D_m, which sums the first R_m iterates of L_m.

2 The functions R_m and H_m

We record the following propositions, which follow immediately from (2), for later use.

Proposition 1. *For $x, y, m \in \mathbb{N}$, if $x|y$, then $L_m(x)|L_m(y)$.*

Repeatedly applying Proposition 1, we find

Proposition 2. *For $x, y, m, r \in \mathbb{N}$, if $x|y$, then $L_m^{(r)}(x)|L_m^{(r)}(y)$.*

Proposition 3. *For $m, n \in \mathbb{N}$, if m is even and n is odd, then either $L_m(n) = 0$ or $L_m(n)$ is odd.*

In addition, the following theorem is now quite easy to prove.

Theorem 4. *For any prime number p and positive integer x,*

$$L_{p-1}(px) = \begin{cases} L_{p-1}(x), & \text{if } p \nmid x; \\ pL_{p-1}(x), & \text{if } p|x. \end{cases}$$

Proof. If $p \nmid x$, it follows from the multiplicativity of L_{p-1} that $L_{p-1}(px) = L_{p-1}(p)L_{p-1}(x) = L_{p-1}(x)$. If $p|x$, then we have

$$L_{p-1}(px) = L_{p-1}\left(p \cdot p^{\upsilon_p(x)} \cdot \frac{x}{p^{\upsilon_p(x)}}\right) = L_{p-1}\left(p^{\upsilon_p(x)+1}\right) L_{p-1}\left(\frac{x}{p^{\upsilon_p(x)}}\right)$$
$$= p^{\upsilon_p(x)} L_{p-1}\left(\frac{x}{p^{\upsilon_p(x)}}\right) = pL_{p-1}\left(p^{\upsilon_p(x)}\right) L_{p-1}\left(\frac{x}{p^{\upsilon_p(x)}}\right) = pL_{p-1}(x).$$

□

Notice that, for any positive integers m and n with $n > 1$, we have $L_m(n) < n$ and $L_m(n) \in \mathbb{N}_0$. It is easy to see that, by starting with a positive integer n and iterating the function L_m a finite number of times, we must eventually reach either 0 or 1. More precisely, there exists a positive integer k such that $L_m^{(k)}(n) \in \{0, 1\}$. This leads us to the following definitions.

Definition 5. For all $m, n \in \mathbb{N}$, let $R_m(n)$ denote the least positive integer k such that $L_m^{(k)}(n) \in \{0, 1\}$. Furthermore, we define the function H_m by

$$H_m(n) = L_m^{(R_m(n))}(n).$$

Though the functions H_m only take values 0 and 1, they prove to be surprisingly interesting. For example, we can show that, for each positive integer m, H_m is a completely multiplicative function. First, however, we will need some definitions and preliminary results.

Definition 6. For $m \in \mathbb{N}$, we define the following sets:

$$P_m = \{p \in \mathbb{P} : H_m(p) = 1\}$$

$$Q_m = \{q \in \mathbb{P} : H_m(q) = 0\}$$

$$S_m = \{n \in \mathbb{N} : q \nmid n \; \forall \, q \in Q_m\}$$

We define T_m to be the unique set of positive integers defined by the following criteria:

- $1 \in T_m$.
- If p is prime, then $p \in T_m$ if and only if $p - m \in T_m$.
- If x is composite, then $x \in T_m$ if and only if there exist $x_1, x_2 \in T_m$ such that $x_1, x_2 > 1$ and $x_1 x_2 = x$.

Lemma 7. *Let $k, m \in \mathbb{N}$. If all the prime divisors of k are in T_m, then all the positive divisors of k (including k) are in T_m. Conversely, if $k \in T_m$, then every positive divisor of k is an element of T_m.*

Proof. First, suppose that all the prime divisors of k are in T_m, and let d be a positive divisor of k. Then all the prime divisors of d are in T_m. Let $d = \prod_{i=1}^{r} w_i^{\alpha_i}$ be the canonical prime factorization of d. As $w_1 \in T_m$, the third defining criterion of T_m tells us that $w_1^2 \in T_m$. Then, by the same token, $w_1^3 \in T_m$. Eventually, we find that $w_1^{\alpha_1} \in T_m$. As $w_1^{\alpha_1}, w_2 \in T_m$, we have $w_1^{\alpha_1} w_2 \in T_m$. Repeatedly using the third criterion, we can keep multiplying by primes until we find that $d \in T_m$. This completes the first part of the proof. Now we will prove that if $k \in T_m$, then every positive divisor of k is an element of T_m. The proof is trivial if k is prime, so suppose k is composite. We will induct on $\Omega(k)$, the number of prime divisors (counting multiplicities) of k. If $\Omega(k) = 2$, then, by the third defining criterion of T_m, the prime divisors of k must be elements of T_m. Therefore, if $\Omega(k) = 2$, we are done. Now, suppose the result holds whenever $\Omega(k) \leq h$, where $h > 1$ is a positive integer. Consider the case in which $\Omega(k) = h + 1$. By the third defining criterion of T_m, we can write $k = k_1 k_2$, where $1 < k_1, k_2 < k$ and $k_1, k_2 \in T_m$. By the induction hypothesis, all of the positive divisors of k_1 and all of the positive divisors of k_2 are in T_m. Therefore, all of the prime divisors of k are in T_m. By the first part of the proof, we conclude that all of the positive divisors of k are in T_m. $\square$

Theorem 8. *If m is a positive integer, then $S_m = T_m$.*

Proof. Fix $m \in \mathbb{N}$. Let u be a positive integer such that, for all $k \in \{1, 2, \ldots, u-1\}$, either $k \in S_m$ and $k \in T_m$ or $k \notin S_m$ and $k \notin T_m$. We will show that $u \in S_m$ if and only if $u \in T_m$. First, we must show that if $k \in \{1, 2, \ldots, u-1\}$, then $k \in S_m$ if and only if $L_m(k) \in S_m$. Suppose, for the sake of finding a contradiction, that $L_m(k) \in S_m$ and $k \notin S_m$. As $k \notin S_m$, we have that $k > 1$ and $k \notin T_m$. Lemma 7 then guarantees that there exists a prime q such that $q|k$ and $q \notin T_m$. As $q \notin T_m$, the second defining criterion of T_m implies that $q - m \notin T_m$. We know that $q > m$ because, otherwise, $p(k) \leq q \leq m$, implying that $L_m(k) = 0 \notin S_m$. Therefore, $q - m \in \{1, 2, \ldots, u-1\}$ and $q - m \notin T_m$. By the induction hypothesis, $q - m \notin S_m$. Therefore, there exists some $q_0 \in Q_m$ such that $q_0 | q - m$. Because $q|k$, Proposition 1 implies that $L_m(q) | L_m(k)$. Thus, $q_0 | q - m = L_m(q) | L_m(k)$, which implies that $L_m(k) \notin S_m$. This is a contradiction. Now suppose, so that we may again search for a contradiction, that $L_m(k) \notin S_m$ and $k \in S_m$. $L_m(k) \notin S_m$ implies that $k > 1$, and $k \in S_m$ implies (by the induction hypothesis) that $k \in T_m$. By Lemma 7, all positive divisors of k are elements of T_m. Let $k = \prod_{i=1}^{r} p_i^{\alpha_i}$ be the canonical prime factorization of k. Then, by (2),

$$L_m(k) = \begin{cases} 0, & \text{if } p(k) \leq m; \\ \displaystyle\prod_{i=1}^{r} p_i^{\alpha_i - 1}(p_i - m), & \text{if } p(k) > m. \end{cases}$$

If $p(k) \leq m$, then $H_m(p(k)) = 0$, which mean that $p(k) \in Q_m$. As $p(k)|k$, we have contradicted $k \in S_m$. Therefore, $p(k) > m$, so $L_m(k) = \prod_{i=1}^{r} p_i^{\alpha_i - 1}(p_i - m)$. For each

$i \in \{1, 2, \ldots, r\}$, p_i is a positive divisor of k, so $p_i \in T_m$. The second criterion defining T_m then implies that $p_i - m \in T_m$, so all positive divisors (and, specifically, all prime divisors) of $p_i - m$ are in T_m. This implies that all prime divisors of $L_m(k)$ are elements of T_m, so Lemma 7 guarantees that $L_m(k) \in T_m$. However, we have shown that $0 < L_m(k) < k$, so $L_m(k) \in \{1, 2, \ldots, u-1\}$. By the induction hypothesis, we have $L_m(k) \in S_m$, a contradiction. Thus, we have established that if $k \in \{1, 2, \ldots, u-1\}$, then $k \in S_m$ if and only if $L_m(k) \in S_m$.

We are now ready to establish that $u \in S_m$ if and only if $u \in T_m$. Suppose that $u \in S_m$ and $u \notin T_m$. We know that $u > m$ because, otherwise, $L_m(p(u)) = H_m(p(u)) = 0$, implying that $p(u) \in Q_m$ and contradicting $u \in S_m$. If u is prime, then $u \in S_m$ implies that $u \in P_m$. Then $H_m(u) = H_m(u-m) = L_m^{(R_m(u-m))}(u-m) = 1 \in S_m$. As $L_m^{(R_m(u-m))}(u-m) = L_m\left(L_m^{(R_m(u-m)-1)}(u-m)\right) \in S_m$ (we assume here and in the rest of the proof that $R_m(u-m)$ is large enough so that the notation $L_m^{(\cdot)}$ makes sense as we have defined it, but the argument is valid in any case), it follows from the preceding argument that $L_m^{(R_m(u-m)-1)}(u-m) = L_m\left(L_m^{(R_m(u-m)-2)}(u-m)\right) \in S_m$. Continuing this pattern, we eventually find that $L_m(u-m) \in S_m$, so $u-m \in S_m$. By the induction hypothesis, $u-m \in T_m$. However, by the second criterion defining T_m, the primality of u then implies that $u \in T_m$, a contradiction. Thus, u must be composite. We assumed that $u \notin T_m$, so Lemma 7 guarantees the existence of a prime $q \notin T_m$ such that $q|u$. As u is composite, $q \in \{1, 2, \ldots, u-1\}$. The induction hypothesis then implies that $q \notin S_m$, so $q \in Q_m$. However, this contradicts $u \in S_m$, so we have shows that if $u \in S_m$, then $u \in T_m$. Suppose, on the other hand, that $u \notin S_m$ and $u \in T_m$. Again, we begin by assuming u is prime. Then, because $u \in T_m$, we must have $u - m \in T_m$. Therefore, by the induction hypothesis and the fact that $u - m \in \{1, 2, \ldots, u-1\}$, it follows that $u - m \in S_m$. Now, $u \notin S_m$, so we must have $u \in Q_m$. Therefore, $H_m(u) = H_m(L_m(u)) = H_m(u-m) = L_m^{(R_m(u-m))}(u-m) = 0 \notin S_m$. However, as $L_m^{(R_m(u-m))}(u-m) = L_m\left(L_m^{(R_m(u-m)-1)}(u-m)\right) \notin S_m$, it follows that $L_m^{(R_m(u-m)-1)}(u-m) = L_m\left(L_m^{(R_m(u-m)-2)}(u-m)\right) \notin S_m$. Again, we continue this pattern until we eventually find that $L_m(u-m) \notin S_m$, which means that $u - m \notin S_m$. This is a contradiction, and we conclude that u must be composite. From $u \in T_m$ and Lemma 7, we conclude that all of the prime divisors of u are elements of T_m. Furthermore, as u is composite, all of the prime divisors of u are elements of $\{1, 2, \ldots, u-1\}$. Then, by the induction hypothesis, all of the prime divisors of u are in the set S_m. This implies that none of the prime divisors of u are in Q_m, so $u \in S_m$. This is a contradiction, and the induction step of the proof is finally complete. All that is left to check is the base case. However, the base case is trivial because $1 \in S_m$ and $1 \in T_m$. □

We may now use the sets S_m and T_m interchangeably. In addition, part of the above proof gives rise to the following corollary.

Corollary 9. *Let $k, m, n \in \mathbb{N}$. Then $L_m^{(k)}(n) \in S_m$ if and only if $n \in S_m$.*

Proof. The proof follows from the argument in the above proof that $L_m(n) \in S_m$ if and only if $n \in S_m$ whenever $n \in \{1, 2, \ldots, u-1\}$. As we now know that we can make u as large as we need, it follows that $L_m(n) \in S_m$ if and only if $n \in S_m$. Then $L_m^{(2)}(n) \in S_m$ if and only if $L_m(n) \in S_m$, $L_m^{(3)}(n) \in S_m$ if and only if $L_m^{(2)}(n) \in S_m$, and, in general, $L_m^{(r+1)}(n) \in S_m$ if and only if $L_m^{(r)}(n) \in S_m$ $(r \in \mathbb{N})$. The desired result follows immediately. □

Corollary 10. *Let $m, n \in \mathbb{N}$. Then $H_m(n) \in S_m$ if and only if $n \in S_m$.*

Proof. It is clear that $H_m(n) \in S_m$ if and only if $H_m(n) = 1$. Therefore, the proof follows immediately from setting $k = R_m(n)$ in Corollary 9. □

Notice that, for a given positive integer m, Corollary 10, along with Theorem 8 and the defining criteria of T_m, provides a simple way to construct the set of all positive integers x that satisfy $H_m(x) = 1$. Corollary 10 also expedites the proof of the following theorem.

Theorem 11. *The function $n \mapsto H_m(n)$ is completely multiplicative for all $m \in \mathbb{N}$.*

Proof. Choose some $m, x, y \in \mathbb{N}$. First, suppose $H_m(x) = 0$. By Corollary 10, $x \notin S_m$. Therefore, there exists $q \in Q_m$ such that $q|x$. This implies that $q|xy$, so $xy \notin S_m$. Thus, $H_m(xy) = 0$. A similar argument shows that $H_m(xy) = 0$ if $H_m(y) = 0$. Now, suppose that $H_m(x) = H_m(y) = 1$. Then Corollary 10 ensures that $x, y \in S_m$. Therefore, $xy \in S_m$, so $H_m(xy) = 1$. As the function H_m can only take values 0 and 1, the proof is complete. □

In concluding this section, we note that if $m+1$ is composite, then it is impossible for any integer greater than 1 to be in S_m. Therefore, whenever $m+1$ is composite, we have $H_m(1) = 1$ and $H_m(n) = 0$ for all integers $n > 1$.

3 Summing the iterates

A perfect totient number is defined [3] to be a positive integer $n > 1$ that satisfies (using our previous notation)

$$n = \sum_{i=1}^{R_1(n)} \phi^{(i)}(n).$$

In the following definitions, we generalize the concept of perfect totient numbers. We also borrow some other traditional terminology related to perfect numbers.

Definition 12. Let m be a positive integer. We define the arithmetic function D_m by $D_m(1) = 0$ and

$$D_m(n) = \sum_{i=1}^{R_m(n)} L_m^{(i)}(n)$$

for all integers $n > 1$. If $D_m(n) < n$, we say that n is D_m-*deficient*. If $D_m(n) = n$, we say that n is D_m-*perfect*. If $D_m(n) > n$, we say that n is D_m-*abundant*. Finally, in the case when $D_m(n) = 0$, we say that n is D_m-*stagnant*.

We now present a series of theorems related to these definitions.

Theorem 13. *If $m > 1$ is odd, then all positive integers are D_m-deficient.*

Proof. Let $m > 1$ be an odd integer, and let n be any positive integer. If $n = 1$ or $p(n) \le m$, then n is D_m-stagnant. *A fortiori*, n is D_m-deficient. If $p(n) > m$, then $p(n) - m$ is even and $p(n) - m | L_m(n)$ (by Proposition 1). Thus, $2|L_m(n)$, which implies that $L_m^{(2)}(n) = 0$. Hence, $D_m(n) = L_m(n) < n$.

□

Theorem 14. *All positive even integers are D_m-deficient for all positive integers m.*

Proof. The proof is trivial for $m > 1$ because, in that case, any positive even integer is clearly D_m-stagnant. For $m = 1$, we use the fact that all totient numbers greater than 1 are even. Therefore,

$$D_1(n) = \phi(n) + \phi^{(2)}(n) + \cdots + \phi^{(R_1(n))}(n) \le \frac{1}{2}n + \frac{1}{4}n + \cdots + \frac{1}{2^{R_1(n)}}n < n.$$

□

Theorem 14 is nothing revolutionary, but we include it because it fits nicely with the next theorems.

For the next two theorems, which are not quite as trivial as the previous two, we require the following lemma.

Lemma 15. *If $k > 1$ is an odd integer, then at least one element of the set $\{k, L_2(k), L_2^{(2)}(k)\}$ is divisible by 3.*

Proof. Let $k > 1$ be an odd integer with prime factor p, and suppose $3 \nmid k$ and $3 \nmid L_2(k)$. We know that $p \not\equiv 2 \pmod 3$ because $p - 2 | L_2(k)$, so $p \equiv 1 \pmod 3$. As $p - 2 \equiv 2 \pmod 3$, $p - 2$ must have some prime factor p' such that $p' \equiv 2 \pmod 3$. But then, using Proposition 2, $3 | p' - 2 = L_2(p') | L_2(p - 2) = L_2^{(2)}(p) | L_2^{(2)}(k)$. □

Theorem 16. *For any integer $m > 1$, all positive multiples of 3 are D_m-deficient.*

Proof. If $m \ge 3$, then any positive multiple of 3 is clearly D_m-stagnant. Therefore, we only need to check the case $m = 2$. Write $K = \{n \in \mathbb{N} : 3|n, D_2(n) \ge n\}$. Suppose $K \ne \emptyset$ and let n_0 be the smallest element of K. If $n_0 = 3^\alpha$ for some $\alpha \in \mathbb{N}$, then $D_2(n_0) = 3^{\alpha-1} + 3^{\alpha-2} + \cdots + 3 + 1 = \dfrac{n_0 - 1}{2} < n_0$. Therefore, n_0 must have some prime divisor $p \ne 3$. From Theorem 13, $p \ne 2$. Also, by Proposition 2, $L_2(p) | L_2(n_0)$ and $L_2^{(2)}(p) | L_2^{(2)}(n_0)$. By Lemma 15, at least one of $L_2(p)$ and $L_2^{(2)}(p)$ must be divisible by 3. Suppose $3 | L_2(p)$ so that $3 | L_2(n_0)$. By the choice of n_0 as the smallest element of K, we have $D_2(L_2(n_0)) < L_2(n_0)$. This implies that $D_2(n_0) = L_2(n_0) + D_2(L_2(n_0)) < 2L_2(n_0) < \dfrac{2}{3}n_0$ because $3 | n_0$. From this contradiction, we conclude that $3|L_2^{(2)}(p)$, so $3|L_2^{(2)}(n_0)$. Again, by the choice of n_0, we have $D_2\left(L_2^{(2)}(n_0)\right) < L_2^{(2)}(n_0)$. However, this implies that $D_2(n_0) = L_2(n_0) + L_2^{(2)}(n_0) + D_2\left(L_2^{(2)}(n_0)\right) < L_2(n_0) + 2L_2^{(2)}(n_0) < 3L_2(n_0) < n_0$, which is a contradiction. It follows that K is empty. $\square$

Theorem 17. *If $m > 1$ is a positive integer and $m \neq 4$, then all positive multiples of 5 are D_m-deficient.*

Proof. Let m be a positive integer other than 1 or 4, and let n be a multiple of 5. If $m \geq 5$, then n is D_m-stagnant. If $m = 3$, then n is D_m-deficient by Theorem 13. We therefore only need to check the case $m = 2$. Write $n = 5^\alpha k$, where $\alpha, k \in \mathbb{N}$. We may assume that $2 \nmid k$ and $3 \nmid k$ because, otherwise, the desired result follows immediately from either Theorem 14 or Theorem 16. We now consider two cases.

Case 1: $\alpha \geq 2$. Write $L_2(k) = 3^{\alpha_1}5^{\alpha_2}t$, where t is a positive integer not divisible by 2, 3, or 5 and $\alpha_1, \alpha_2 \in \mathbb{N}_0$ (we use Proposition 3 to conclude that t is odd). Then $L_2(n) = L_2(5^\alpha)L_2(k) = 3^{\alpha_1+1}5^{\alpha+\alpha_2-1}t$ and $L_2^{(2)}(n) = L_2(3^{\alpha_1+1})L_2(5^{\alpha+\alpha_2-1})L_2(t) = 3^{\alpha_1+1}5^{\alpha+\alpha_2-2}L_2(t)$. As $3|L_2(n)$, we can use Theorem 16 to write $D_2(n) = L_2(n) + L_2^{(2)}(n) + D_2\left(L_2^{(2)}(n)\right) < L_2(n) + 2L_2^{(2)}(n) = 3^{\alpha_1+1}5^{\alpha+\alpha_2-1}t + 2\left(3^{\alpha_1+1}5^{\alpha+\alpha_2-2}L_2(t)\right) \leq 7\left(3^{\alpha_1+1}5^{\alpha+\alpha_2-2}t\right) \leq \frac{21}{25}5^\alpha k = \frac{21}{25}n$. This completes the proof of the case when $\alpha \geq 2$.

Case 2: $\alpha = 1$. In this case, $n = 5k$, so $L_2(n) = 3L_2(k)$. We may assume that $k > 1$ because the case $n = 5$ is trivial. First, suppose that $3|L_2(k)$. In this case, $L_2^{(2)}(k) \leq \frac{1}{3}L_2(k)$, and, by Theorem 4, $L_2^{(2)}(n) = 3L_2^{(2)}(k)$. Then, using Theorem 16, we have $D_2(n) = L_2(n) + L_2^{(2)}(n) + D_2\left(L_2^{(2)}(n)\right) = 3L_2(k) + 3L_2^{(2)}(k) + D_2\left(3L_2^{(2)}(k)\right) < 3L_2(k) + 6L_2^{(2)}(k) \leq 5L_2(k) \leq n - 10$. Now suppose that $3 \nmid L_2(k)$. By Lemma 15 and our assumption that $3 \nmid k$, we have $3|L_2^{(2)}(k)$. Using Theorem 4 and Theorem 16 again, we have $D_2(n) = L_2(n) + L_2^{(2)}(n) + D_2\left(L_2^{(2)}(n)\right) = 3L_2(k) + L_2^{(2)}(k) + D_2\left(L_2^{(2)}(k)\right) < 3L_2(k) + 2L_2^{(2)}(k) \leq 3(k-2) + 2(k-4) = n - 14$. This completes the proof of all cases. $\square$

The last few theorems have dealt with D_m-deficient numbers, so it is natural to ask questions about D_m-abundant numbers. We might wish to know the positive integers m for which D_m-abundant numbers even exist. How many D_m-abundant numbers exist for a given m? How large can we make $D_m(n) - n$? Theorem 13 deals with these questions for the cases when m is odd and greater than 1. Also, a great deal of literature [3, 4] already exists concerning the case $m = 1$. In the following theorem, we answer all of the preceding questions for the cases when m is a positive even integer.

Theorem 18. *Let m be a positive even integer. For any positive A and δ, there exist infinitely many primes p_0 such that $L_m(p_0^\alpha) + L_m^{(2)}(p_0^\alpha) > p_0^\alpha + Ap_0^{\alpha-\delta}$ for all positive integers α.*

Proof. Fix m, A, and δ to be positive real numbers, where m is an even integer. Let $p_1, p_2, \ldots, p_r$ be all the primes that divide m, and let $q_1, q_2, \ldots, q_t$ be all the primes that are less than m and do not divide m. For each $j \in \{1, 2, \ldots, t\}$, define σ_j by

$$\sigma_j = \begin{cases} 1, & \text{if } m \not\equiv 1 \pmod{q_j}; \\ -1, & \text{if } m \equiv 1 \pmod{q_j}. \end{cases}$$

Write $M = \prod_{p \leq m} p$. By the Chinese remainder theorem, there exists a unique solution modulo M to the system of congruences defined by

$$\begin{cases} x \equiv 1 \pmod{p_i} & \text{if } i \in \{1, 2, \ldots, r\}; \\ x \equiv \sigma_j \pmod{q_j} & \text{if } j \in \{1, 2, \ldots, t\}. \end{cases} \tag{3}$$

It is easy to see that if x_0 is a solution to (3), then x_0 and $x_0 - m$ are each relatively prime to every prime less than or equal to m. By Dirichlet's theorem concerning the infinitude of primes in arithmetic progressions, there must be infinitely many primes that satisfy the system (3). Let p_0 be one such prime, and write

$$\beta = \prod_{p \mid p_0 - m} \left(1 - \frac{m}{p}\right).$$

As p_0 is relatively prime to all primes less than or equal to m, we have

$$\prod_{m < p \leq p_0} \left(1 - \frac{m}{p}\right) \leq \beta \leq 1.$$

It is well-known [6], that, as $p_0 \to \infty$,

$$\prod_{m < p \leq p_0} \left(1 - \frac{m}{p}\right) \sim \frac{c_m}{(\log p_0)^m}$$

for some constant c_m that depends only on m. We find that

$$\beta p_0^2 \geq p_0^2 \prod_{m < p \leq p_0} \left(1 - \frac{m}{p}\right) \sim \frac{c_m p_0^2}{(\log p_0)^m}.$$

Therefore, we may choose p_0 to be large enough so that $\beta p_0^2 > A p_0^{2-\delta} + 3mp_0$. With this choice of p_0, we may write $\beta(p_0 - m)^2 > \beta p_0^2 - 2\beta m p_0 \geq \beta p_0^2 - 2mp_0 > A p_0^{2-\delta} + mp_0$. But $\beta(p_0 - m) = L_m(p_0 - m)$ because $p(p_0 - m) > m$. Thus,

$$(p_0 - m) L_m(p_0 - m) > A p_0^{2-\delta} + mp_0. \tag{4}$$

Let α be an integer, and, for now, assume $\alpha \geq 2$. Rearranging and multiplying the inequality (4) by $p_0^{\alpha-2}$, we have $-mp_0^{\alpha-1} + p_0^{\alpha-1}L_m(p_0-m) > mp_0^{\alpha-2}L_m(p_0-m) + Ap_0^{\alpha-\delta}$. After further algebraic manipulation, we find $p_0^{\alpha-1}(p_0-m) + p_0^{\alpha-2}(p_0-m)L_m(p_0-m) > p_0^{\alpha} + Ap_0^{\alpha-\delta}$. Noticing that the left-hand side of the preceding inequality is simply $L_m(p_0^{\alpha}) + L_m^{(2)}(p_0^{\alpha})$, we have $L_m(p_0^{\alpha}) + L_m^{(2)}(p_0^{\alpha}) > p_0^{\alpha} + Ap_0^{\alpha-\delta}$. This is the desired result for $\alpha \geq 2$. To show that the result holds when $\alpha = 1$, it suffices to show that $L_m(p_0) + L_m^{(2)}(p_0) > \dfrac{L_m(p_0^2) + L_m^{(2)}(p_0^2)}{p_0}$. This reduces to $p_0 - m + L_m(p_0-m) > p_0 - m + \dfrac{p_0-m}{p_0}L_m(p_0-m)$, which is obviously true. □

Corollary 19. *For any positive even integer m, there exist infinitely many D_m-abundant numbers.*

We conclude this section with a remark about D_m-perfect numbers. Using *Mathematica*, one may check that for $m \in \{2, 4, 6\}$, the only D_m-perfect number less than $100,000$ is $37,147$, which is D_2-perfect. Unfortunately, this data is too scarce to make any reasonable conjecture about the nature or distribution of D_m-perfect numbers for positive even integers m.

4 Numerical analysis and concluding remarks

In 1943, H. Shapiro investigated a function C, which counts the number of iterations of the ϕ function needed to reach 2 [8]. Shapiro showed that the function C is additive, and he established bounds for its values. In this paper, we have not gone into much detail exploring the functions R_m because they prove, in general, to be either completely uninteresting or very difficult to handle. For example, for any integer $n > 1$,

$$R_3(n) = \begin{cases} 1, & \text{if } n \not\equiv 1, 5 \pmod 6; \\ 2, & \text{if } n \equiv 1, 5 \pmod 6. \end{cases}$$

On the other hand, the function R_4 does not seem to obey any nice pattern or exhibit any sort of nice additive behavior. There seems to be some hope in analyzing the function R_2, so we make the following conjecture.

Conjecture 20. If $x > 3$ is an odd integer, then

$$R_2(x) \geq \frac{\log\left(\frac{49}{15}x\right)}{\log 7}.$$

We note that it is not difficult to prove, using Lemma 15 and a bit of case work, that $R_2(x) \leq 3\dfrac{\log(x+2)}{\log 3} - 3$ for all integers $x > 1$ (with equality only at $x = 7$). However, as Figure 1 shows, this is a very weak upper bound (at least for relatively small

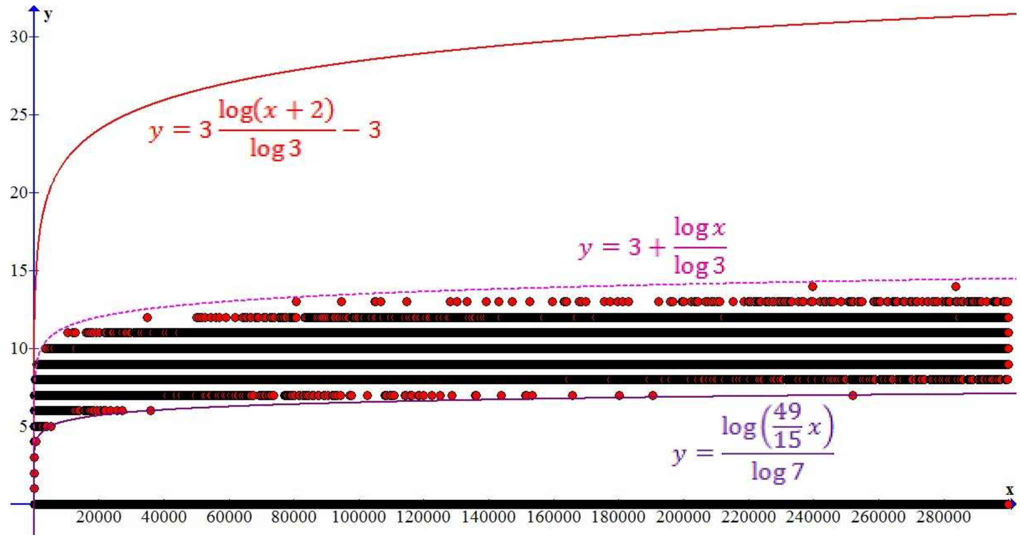

Figure 1: A plot of the first 300,000 values of the function R_2, as well as some important curves. Note that the black streaks in the figure are, in actuality, several overlapping dots.

x). It is tempting to think, based on the figure, that $R_2(x) \leq 3 + \dfrac{\log x}{\log 3}$ for all positive integers x. However, setting $x = 480,314,203$ yields a counterexample because $3 + \dfrac{\log 480,314,203}{\log 3} \approx 21.196 < 22 = R_2(480,314,203)$.

The author has found that investigating bounds of the function R_2 naturally leads to a question about the infinitude of twin primes, which hints at the potential difficulty of the problem. Indeed, Harrington and Jones [1] have arrived at the same conclusion while studying the function $C_2(x) := R_2(x) - 1$, and they conjecture that the values of $C_2(x) + C_2(y) - C_2(xy)$ can be arbitrarily large. To avoid the unpredictability of the values of the function C_2, Harrington and Jones have restricted the domain of C_2 to the set D of positive integers k with the property that none of the numbers in the set $\{k, L_2(k), L_2^{(2)}(k), \ldots\}$ has a prime factor that is congruent to 1 modulo 3. With this restriction of the domain of C_2, these two authors have established results analogous to those that Shapiro gave for the function C mentioned earlier. In fact, we speculate that methods analogous to those that Harrington and Jones have used could easily generalize to allow for analogous results concerning functions $C_m(x) := R_m(x) - 1$ if one is willing to use a sufficiently restricted domain of C_m.

We next remark that, in Theorem 17, the requirement that $m \neq 4$ is essential. For example, write $p_1 = 306,167$, $p_2 = 4 + p_1^2$, $p_3 = 4 + p_2^2$, $p_4 = 4 + p_3^2$, and $p_5 = 4 + p_4^2$. Then the number $5p_5$ is a D_4-abundant multiple of 5.

Lastly, we have not spent much effort analyzing the "sizes" of the functions D_m or searching for D_m-perfect numbers. We might inquire about the average order or possible upper and lower bounds for D_m for a general positive even integer m. In addition, it is natural to ask if there even are any D_m-perfect numbers other than $37, 147$ for even positive integers m.

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