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Another Proof of a Conjecture by Hirschhorn and Sellers on Overpartitions

Liuquan Wang
Department of Mathematics
National University of Singapore
Singapore, 119076
Singapore
wangliuquan@nus.edu.sg

Abstract

We provide a new elementary proof for a Ramanujan-type congruence for the overpartition function modulo 40, which was previously conjectured by Hirschhorn and Sellers and later proved by Chen and Xia. We also find some new congruences for the overpartition function modulo 5 and 9.

1 Introduction and Main Results

An overpartition of an integer n is a partition in which the first occurrence of a part may be overlined. The number of overpartitions of n is denoted by $\overline{p}(n)$. For example, $\overline{p}(3)=8$ as there are 8 overpartitions of 3: 3, $\overline{3}$, $2+1$, $\overline{2}+1$, $2+\overline{1}$, $\overline{2}+\overline{1}$, $1+1+1$, $\overline{1}+1+1$. We define $\overline{p}(0)=1$ for convenience. It is well known (see [5], for example) that the generating function of $\overline{p}(n)$ is

$$\sum_{n\geq 0}\overline{p}(n)q^n=\frac{(-q;q)_\infty}{(q;q)_\infty}=\frac{1}{\varphi(-q)}, \tag{1}$$

where $(a;q)_\infty=(1-a)(1-aq)\cdots(1-aq^n)\cdots$ is standard q-series notation, and $\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}$ is one of Ramanujan's theta functions.

Overpartitions were first introduced by MacMahon [14] and have drawn much attention during the past ten years. There are numerous results concerning the arithmetic properties of $\overline{p}(n)$. For more details, we refer the reader to [3, 5, 8, 15].

In 2005, Hirschhorn and Sellers [9] proved numerous Ramanujan-type identities for $\overline{p}(n)$. For example,

$$\sum_{n\geq 0}\overline{p}(4n+3)q^n = 8\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty^6}{(q;q)_\infty^8}, \tag{2}$$

which clearly implies $\overline{p}(4n+3)\equiv 0 \pmod 8$. Meanwhile, they proposed a curious conjecture:

Conjecture 1. For any integer $n\geq 0$, we have

$$\overline{p}(40n+35)\equiv 0 \pmod{40}.$$

In a recent paper, Chen and Xia [3] gave a proof of Conjecture 1 using (p,k)-parametrization of theta functions. Their proof is relatively long and complicated, and in this paper we give a shorter proof.

Let $r_k(n)$ denote the number of representations of n as sum of k squares. We find the following arithmetic relation.

Theorem 2. *For any integer $n\geq 1$, we have*

$$\overline{p}(5n)\equiv (-1)^n r_3(n) \pmod 5.$$

We have two remarkable corollaries.

Corollary 3. *For any integers $n\geq 0$ and $\alpha\geq 0$, we have*

$$\overline{p}(4^\alpha(40n+35))\equiv 0 \pmod 5$$

and

$$\overline{p}(5\cdot 4^{\alpha+1}n)\equiv (-1)^n\overline{p}(5n) \pmod 5.$$

By letting $\alpha=0$ in Corollary 3, we get $\overline{p}(40n+35)\equiv 0 \pmod 5$. Together with (2), Conjecture 1 follows immediately.

Corollary 4. *For any prime $p\equiv -1 \pmod 5$, we have*

$$\overline{p}(5p^3n)\equiv 0 \pmod 5$$

for all n coprime to p.

Corollary 4 was first proved by Treneer (see [17, Proposition 1.4]) in 2006 using the theory of modular forms, which is not elementary.

Furthermore, with Corollary 3 in mind, we are ready to generalize Conjecture 1 to the following result.

Theorem 5. *For any integers $n\geq 0$ and $\alpha\geq 0$, we have*

$$\overline{p}(4^\alpha(40n+35))\equiv 0 \pmod{40}.$$

Some miscellaneous congruences can be deduced from Theorem 2, and we list some of them here.

Theorem 6. *For any integers $\alpha \geq 1$ and $n \geq 0$, we have*

$$\overline{p}(5^{2\alpha+1}(5n+1)) \equiv \overline{p}(5^{2\alpha+1}(5n+4)) \equiv 0 \pmod{5}.$$

Theorem 7. *Let $p \geq 3$ be a prime, and N a positive integer which is coprime to p. Let α be any nonnegative integer.*
(1) If $p \equiv 1 \pmod{5}$, then $\overline{p}(5p^{10\alpha+9}N) \equiv 0 \pmod{5}$.
(2) If $p \equiv 2, 3, 4 \pmod{5}$, then $\overline{p}(5p^{8\alpha+7}N) \equiv 0 \pmod{5}$.

Using properties of $r_3(n)$, we can establish some other congruences as corollaries of Theorem 2. Recently, Chen et al. [2] found some Ramanujan-type congruences mainly based on the theory of modular forms. Their work also contains a proof of Theorem 5. In addition, they proved some congruences such as $\overline{p}(5n) \equiv (-1)^n \overline{p}(4 \cdot 5n) \pmod{5}$.

Finally, we mention some congruences for overpartitions modulo 3. In 2011, based on the generating function of $\overline{p}(3n)$ discovered by Hirschhorn and Sellers [8], Lovejoy and Osburn [13] proved the following result. For any integer $n \geq 1$, we have

$$\overline{p}(3n) \equiv (-1)^n r_5(n) \pmod{3}.$$

Using the same method as in the proof of Theorem 2, we are able to improve this congruence to the following one.

Theorem 8. *For any integer $n \geq 1$, we have*

$$\overline{p}(3n) \equiv (-1)^n r_5(n) \pmod{9}.$$

Similar to Theorem 7, we can deduce the following interesting congruences from Theorem 8.

Theorem 9. *Let $p \geq 3$ be a prime and N a positive integer which is coprime to p.*
(1) If $p \equiv 1 \pmod{3}$, then $\overline{p}(3p^{6\alpha+5}N) \equiv 0 \pmod{3}$ and $\overline{p}(3p^{18\alpha+17}N) \equiv 0 \pmod{9}$.
(2) If $p \equiv 2 \pmod{3}$, then $\overline{p}(3p^{4\alpha+3}N) \equiv 0 \pmod{9}$.

2 Preliminaries

Lemma 10. *(Cf. [16, Lemma 1.2].) Let p be a prime and α a positive integer. Then*

$$(q;q)_\infty^{p^\alpha} \equiv (q^p;q^p)_\infty^{p^{\alpha-1}} \pmod{p^\alpha}.$$

Lemma 11. *(Cf. [1, Theorem 3.3.1, 3.5.4].) For any integer $n \geq 1$, we have*

$$r_4(n) = 8 \sum_{d|n, 4\nmid d} d, \ \ r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3.$$

Lemma 12. *For any prime $p \geq 3$, we have*

$$r_4(pn) \equiv r_4(n) \pmod{p}, \; r_8(pn) \equiv r_8(n) \pmod{p^3}.$$

Proof. By Lemma 11, we have

$$r_4(n) = 8 \sum_{d|n, 4\nmid d} d = 8 \sum_{\substack{d|n \\ 4\nmid d, p\nmid d}} d + 8 \sum_{\substack{d|n \\ 4\nmid d, p|d}} d \equiv 8 \sum_{\substack{d|n \\ 4\nmid d, p\nmid d}} d \pmod{p},$$

and

$$r_4(pn) = 8 \sum_{d|pn, 4\nmid d} d = 8 \sum_{\substack{d|pn \\ 4\nmid d, p\nmid d}} d + 8 \sum_{\substack{d|pn \\ 4\nmid d, p|d}} d = 8 \sum_{\substack{d|n \\ 4\nmid d, p\nmid d}} d + 8p \sum_{\substack{d|n \\ 4\nmid d}} d.$$

Combining them together, we deduce that $r_4(pn) \equiv r_4(n) \pmod{p}$.

Similarly we can prove $r_8(pn) \equiv r_8(n) \pmod{p^3}$. □

Lemma 13. *(Cf. [6, Theorem 1 in Chapter 4].) For any integers $\alpha \geq 0$ and $n \geq 0$, we have $r_3(4^\alpha(8n+7)) = 0$ and $r_3(4^\alpha n) = r_3(n)$.*

Lemma 14. *(Cf. [7].) Let $p \geq 3$ be a prime. For any integers $n \geq 1$ and $\alpha \geq 0$, we have*

$$r_3(p^{2\alpha}n) = \left(\frac{p^{\alpha+1}-1}{p-1} - \left(\frac{-n}{p}\right)\frac{p^\alpha - 1}{p-1}\right) r_3(n) - p\frac{p^\alpha - 1}{p-1} r_3(n/p^2).$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol, and we take $r_3(n/p^2) = 0$ unless $p^2|n$.

Lemma 15. *(Cf. [10, Theorem 3].) Let n be an integer which is neither a square nor twice a square. Then $\overline{p}(n) \equiv 0 \pmod{8}$.*

Lemma 16. *(Cf. [4].) Let $p \geq 3$ be a prime, and n a positive integer such that $p^2 \nmid n$. For any integer $\alpha \geq 0$, we have*

$$r_5(p^{2\alpha}n) = \left(\frac{p^{3\alpha+3}-1}{p^3-1} - p\left(\frac{n}{p}\right)\frac{p^{3\alpha}-1}{p^3-1}\right) r_5(n).$$

3 Proofs of The Theorems

Proof of Theorem 2. Replacing q by $-q$ in (1), we get

$$\sum_{n\geq 0} \overline{p}(n)(-q)^n = \frac{1}{\varphi(q)}.$$

Hence we have

$$\varphi(q)^5 \sum_{n\geq 0} \overline{p}(n)(-q)^n = \varphi(q)^4 = \sum_{n\geq 0} r_4(n) q^n.$$

By Lemma 10, we have $\varphi(q)^5 \equiv \varphi(q^5) \pmod 5$ and thus

$$\varphi(q^5) \sum_{n\geq 0} \overline{p}(n)(-q)^n \equiv \sum_{n\geq 0} r_4(n)q^n \pmod 5.$$

Collecting all the terms of the form q^{5n} on both sides, we get

$$\varphi(q^5) \sum_{n\geq 0} \overline{p}(5n)(-q)^{5n} \equiv \sum_{n\geq 0} r_4(5n)q^{5n} \pmod 5.$$

Replacing q^5 by q and applying Lemma 12 with $p = 5$, we obtain

$$\varphi(q) \sum_{n\geq 0} \overline{p}(5n)(-q)^n \equiv \sum_{n\geq 0} r_4(5n)q^n \equiv \sum_{n\geq 0} r_4(n)q^n = \varphi(q)^4 \pmod 5.$$

Hence we have

$$\sum_{n\geq 0} \overline{p}(5n)(-q)^n \equiv \varphi(q)^3 = \sum_{n\geq 0} r_3(n)q^n \pmod 5.$$

Theorem 1 follows by comparing the coefficients of q^n on both sides. □

Proof of Corollary 3. This corollary follows immediately from Theorem 2 and Lemma 13. □

Proof of Corollary 4. Let $\alpha = 1$ and we replace n by np in Lemma 14. We have

$$r_3(p^3n) = (p+1)r_3(n) - pr_3(n/p).$$

Since n is coprime to p and $p+1 \equiv 0 \pmod 5$, we get $r_3(p^3n) \equiv 0 \pmod 5$. By Theorem 2, we deduce that $\overline{p}(5p^3n) \equiv 0 \pmod 5$. □

Proof of Theorem 5. Since $40n + 35 = 5(8n + 7)$ is an odd number, it cannot be twice a square. If $5(8n + 7) = x^2$ is a square where x is an odd number, then we know $5|x$. Let $x = 5y$ where y is an odd number, we get $8n + 7 = 5y^2$. But $5y^2 \equiv 5 \pmod 8$, which is a contradiction! Hence we know $4^\alpha(40n + 35)$ is neither a square nor twice a square. By Lemma 15, we have $\overline{p}(4^\alpha(40n+35)) \equiv 0 \pmod 8$. Combining this with Corollary 3 completes the proof. □

Proof of Theorem 6. Set $p = 5$ and $n = 5m + r$ where $r \in \{1, 4\}$ in Lemma 14. It is easy to deduce that $r_3(5^{2\alpha}(5m + r)) \equiv 0 \pmod 5$ for any integer $\alpha \geq 1$. By applying Theorem 2, we complete the proof. □

Proof of Theorem 7. (1) Let $n = pN$ and then replace α by $5\alpha + 4$ in Lemma 14. Since

$$\frac{p^{5\alpha+5} - 1}{p - 1} = 1 + p + \cdots + p^{5\alpha+4} \equiv 0 \pmod 5,$$

we have $r_3(p^{10\alpha+9}N) \equiv 0 \pmod 5$. By Theorem 2, we deduce that $\overline{p}(5p^{10\alpha+9}N) \equiv 0 \pmod 5$.

(2) Let $n = pN$ and then replace α by $4\alpha + 3$ in Lemma 14. Since $p^{4\alpha+4} \equiv 1 \pmod 5$, we deduce that $r_3(p^{8\alpha+7}N) \equiv 0 \pmod 5$. By Theorem 2, we deduce that $\overline{p}(5p^{8\alpha+7}N) \equiv 0 \pmod 5$. □

Proof of Theorem 8. We have

$$\varphi(q)^9 \sum_{n\geq 0} \overline{p}(n)(-q)^n = \varphi(q)^8 = \sum_{n\geq 0} r_8(n)q^n.$$

By Lemma 10, we have $\varphi(q)^9 \equiv \varphi(q^3)^3 \pmod 9$ and thus

$$\varphi(q^3)^3 \sum_{n\geq 0} \overline{p}(n)(-q)^n \equiv \sum_{n\geq 0} r_8(n)q^n \pmod 9.$$

Collecting all the terms of the form q^{3n} on both sides, we get

$$\varphi(q^3)^3 \sum_{n\geq 0} \overline{p}(3n)(-q)^{3n} \equiv \sum_{n\geq 0} r_8(3n)q^{3n} \pmod 9.$$

Replacing q^3 by q and applying Lemma 12 with $p = 3$, we obtain

$$\varphi(q)^3 \sum_{n\geq 0} \overline{p}(3n)(-q)^n \equiv \sum_{n\geq 0} r_8(3n)q^n \equiv \sum_{n\geq 0} r_8(n)q^n = \varphi(q)^8 \pmod 9.$$

Hence we have

$$\sum_{n\geq 0} \overline{p}(3n)(-q)^n \equiv \varphi(q)^5 = \sum_{n\geq 0} r_5(n)q^n \pmod 9.$$

Theorem 8 follows by comparing the coefficients of q^n on both sides. □

Proof of Theorem 9. (1) Let $n = pN$ and then replace α by $3\alpha + 2$ in Lemma 16. Since

$$\frac{p^{9\alpha+9} - 1}{p^3 - 1} = 1 + p^3 + \cdots + p^{3(3\alpha+2)} \equiv 0 \pmod 3,$$

we have $r_5(p^{6\alpha+5}N) \equiv 0 \pmod 3$. By Theorem 8, we deduce that $\overline{p}(3p^{6\alpha+5}N) \equiv 0 \pmod 3$.

Similarly, let $n = pN$ and replace α by $9\alpha+8$ in Lemma 16. Since $p \equiv 1 \pmod 3$ implies $p^3 \equiv 1 \pmod 9$, we have

$$\frac{p^{27\alpha+27} - 1}{p^3 - 1} = 1 + p^3 + \cdots + p^{3(9\alpha+8)} \equiv 0 \pmod 9.$$

Hence $r_5(p^{18\alpha+17}N) \equiv 0 \pmod 9$, and we deduce by Theorem 8 that $\overline{p}(3p^{18\alpha+17}N) \equiv 0 \pmod 9$.

(2) Let $n = pN$ and replace α by $2\alpha + 1$ in Lemma 16. Note that $p \equiv 2 \pmod 3$ implies $p^3 \equiv -1 \pmod 9$. Since $p^{6\alpha+6} \equiv 1 \pmod 9$, we have $r_5(p^{4\alpha+3}N) \equiv 0 \pmod 9$. By Theorem 8, we deduce that $\overline{p}(3p^{4\alpha+3}N) \equiv 0 \pmod 9$. □

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