Journal of Integer Sequences, Vol. 16 (2013),
Article 13.6.5

On an Open Problem of Tóth

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Abstract

In a recent paper, Tóth mentioned that it is an open problem to give the asymptotic formula for $\sum_{n\le x} P^k(n)$, where $P(n)$ is the well-known gcd-sum function and $k \ge 2$ is a fixed integer. In this paper, we use the analytic properties of the Dirichlet divisor function to obtain the asymptotic formula for it.

1 Introduction

In 1933, Pillai [8] introduced the gcd-sum function

$$P(n) = \sum_{k=1}^{n} \gcd(k, n). \tag{1}$$

By grouping the terms according to the values of $\gcd(k,n)$ we have

$$P(n) = \sum_{d|n} d\phi(n/d) = n \sum_{d|n} \frac{\phi(d)}{d}, \tag{2}$$

where ϕ is Euler's function. Many authors have studied the properties of $P(n)$, see [1, 2, 3, 4, 5, 8, 9, 10]; it is Sloane's sequence A018804. Chidambaraswamy and Sitaramachandrarao [5] showed that, given an arbitrary $\epsilon > 0$,

$$\sum_{n\le x} P(n) = e_1 x^2 \log x + e_2 x^2 + O(x^{1+\theta+\epsilon}),$$

where θ is the constant appearing in the error term of the Dirichlet divisor problem, e_1, e_2 are certain constants.

It follows from (2) that the arithmetic mean of $\gcd(1,n), \ldots, \gcd(n,n)$ is given by

$$A(n) = \frac{P(n)}{n} = \sum_{d|n} \frac{\phi(d)}{d}. \tag{3}$$

Tóth [9] showed that

$$\sum_{n\le x} A^2(n) = x(C_1 \log^3 x + C_2 \log^2 x + C_3 \log x + C_4) + O(x^{1/2+\epsilon}), \tag{4}$$

where C_1, C_2, C_3, C_4 are computable constants. Furthermore, he listed some open problems concerning the gcd-sum function, one of which is to derive the asymptotic formula for $\sum_{n\le x} P^k(n)$, where $k \ge 2$ is a fixed integer.

In this paper, we use the analytic method to get the asymptotic formula for $\sum_{n\le x} A^k(n)$.

Theorem 1. *Let $k \ge 2$ be a fixed integer. Then*

$$\sum_{n\le x} A^k(n) = x Q_{2^k-1}(\log x) + O(x^{\beta_k+\epsilon}), \tag{5}$$

where $Q_{2^k-1}(t)$ is a polynomial of degree 2^k-1 in t and

$$\beta_2 = \frac{1}{2}, \quad \beta_3 = \frac{5}{8}, \quad \beta_4 = \frac{7}{9}, \quad \beta_5 = \frac{31}{36}, \quad \beta_6 = \frac{207}{224}, \quad \beta_k = 1 - 2^{-\frac{2}{3}k}/50, \quad k \ge 7.$$

Corollary 2. *Let $k \ge 2$ be a fixed integer. Then*

$$\sum_{n\le x} P^k(n) = x^2 Q'_{2^k-1}(\log x) + O(x^{1+\beta_k+\epsilon}), \tag{6}$$

where $Q'_{2^k-1}(t)$ is a polynomial of degree 2^k-1 in t.

Theorem 3. *Let $k \geq 2$ be a fixed integer and*

$$E_k(x) = \sum_{n \leq x} A^k(n) - xQ_{2^k-1}(\log x).$$

Then for $k = 3, 4, 5$, we have

$$\int_1^U E_k(x)dx \ll U^{1+\delta_k+\epsilon},$$

where

$$\delta_3 = 1/2, \quad \delta_4 = 0.6030739, \quad \delta_5 = 0.773114.$$

2 Preliminary Lemmas

Lemma 4. *Let s be a complex number with $Re(s) > 1$. Then*

$$\sum_{n=1}^{\infty} \frac{A^k(n)}{n^s} = \zeta^{2^k}(s) G_k(s),$$

where $G_k(s) = \sum\limits_{n=1}^{\infty} \frac{g(n)}{n^s}$ is a Dirichlet series which is absolutely convergent for $Re(s) > 1/2$.

Proof. Recall that $A(n)$ is multiplicative function. Then it follows from the Euler product representation that for $\mathrm{Re}(s) > 1$,

$$F(s) := \sum_{n=1}^{\infty} \frac{A^k(n)}{n^s} = \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{A^k(p^\alpha)}{p^{\alpha s}}\right). \tag{7}$$

where

$$A(p^\alpha) = 1 + \frac{\phi(p)}{p} + \cdots + \frac{\phi(p^\alpha)}{p^\alpha} = 1 + \alpha - \frac{\alpha}{p}, \quad \alpha \geq 1.$$

Thus we have

$$\begin{aligned}
1 + \sum_{\alpha=1}^{\infty} \frac{A^k(p^\alpha)}{p^{\alpha s}} &= 1 + \sum_{\alpha=1}^{\infty} \frac{(1 + \alpha - \frac{\alpha}{p})^k}{p^{\alpha s}} \\
&= 1 + \frac{2^k}{p^s} - \frac{k \cdot 2^{k-1}}{p^{s+1}} + \ldots + (-1)^k \frac{1}{p^{s+k}} \\
&\quad + \frac{3^k}{p^{2s}} - \frac{2k \cdot 3^{k-1}}{p^{2s+1}} + \ldots + (-1)^k \frac{2^k}{p^{2s+k}} \\
&\quad + \frac{4^k}{p^{3s}} - \frac{3k \cdot 4^{k-1}}{p^{3s+1}} + \ldots + (-1)^k \frac{3^k}{p^{3s+k}} \\
&\quad + \ldots =: x_k(s).
\end{aligned}$$

We substitute the above formula to the formula (7) to get

$$\sum_{n=1}^{\infty}\frac{A^k(n)}{n^s}=\prod_p x_k(s)=\zeta^{2^k}(s)\cdot\prod_p(1-\frac{1}{p^s})^{2^k}\cdot x_k(s)=:\zeta^{2^k}(s)G_k(s),$$

where

$$\begin{aligned}G_k(s)\ &=\prod_p(1-\frac{1}{p^s})^{2^k}\cdot x_k(s)\\ &=\prod_p\Bigg(1-\frac{k\cdot 2^{k-1}}{p^{s+1}}+\ldots+(-1)^k\frac{1}{p^{s+k}}\\ &+\frac{3^k-4^k+\binom{2^k}{2}}{p^{2s}}+\frac{k\cdot 2^{2k-1}-2k\cdot 3^{k-1}}{p^{2s+1}}+\ldots\\ &+\frac{4^k-6^k+\binom{2^k}{2}2^k-\binom{2^k}{3}}{p^{3s}}+\frac{2k\cdot 2^k3^{k-1}-3k\cdot 4^{k-1}-k\cdot 2^{k-1}\cdot\binom{2^k}{2}}{p^{3s+1}}+\ldots\Bigg).\end{aligned}$$

From the above formula, it is easy to see that $G_k(s)$ can be expanded to a Dirichlet series $\sum\limits_{n=1}^{\infty}\frac{g(n)}{n^s}$, which is absolutely convergent for $\mathrm{Re}(s)>1/2$.

□

Lemma 5. *Suppose* $1/2\leq\sigma\leq 1$, *then*

$$\zeta(\sigma+it)\ll(|t|+2)^{\frac{1-\sigma}{3}}\log(|t|+2). \tag{8}$$

Proof. We define the function $\mu(\sigma)$ for each σ as the infimum of number $c\geq 0$ such that $\zeta(\sigma+it)\ll t^c$, or alternatively as

$$\mu(\sigma)=\limsup_{t\to\infty}\frac{\log|\zeta(\sigma+it)|}{\log t}.$$

Then $\mu(\sigma)$ is continuous, nonincreasing and for $\sigma_1\leq\sigma\leq\sigma_2$,

$$\mu(\sigma)\leq\mu(\sigma_1)\frac{\sigma_2-\sigma}{\sigma_2-\sigma_1}+\mu(\sigma_2)\frac{\sigma-\sigma_1}{\sigma_2-\sigma_1}.$$

By the well-known estimates

$$\zeta(1/2+it)\ll t^{1/6},\quad\zeta(1+it)\ll\log t,$$

we can easily get the formula (8). □

Lemma 6. *If* $\zeta(s)=\chi(s)\zeta(1-s)$, *then the estimate*

$$\chi(s)\ll(|t|+2)^{1/2-\sigma}$$

holds uniformly for $0\leq\sigma\leq 1$.

Proof. Using standard properties of the gamma-function one may write the functional equation of $\zeta(s)$ as

$$\zeta(s) = \chi(s)\zeta(1-s), \qquad \chi(s) = (2\pi)^s/(2\Gamma(s)\cos(\pi s/2)).$$

From Stirling's formula

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi|t|}{2}}\left(1+O(|t|^{-1})\right) \qquad (|t| \geq t_0),$$

it follows that

$$\begin{aligned}\chi(s) &= \left(\frac{2\pi}{|t|}\right)^{\sigma+i|t|-\frac{1}{2}} e^{i(|t|+\frac{\pi}{4})}\left(1+O(|t|^{-1})\right)\\ &\ll (|t|+2)^{1/2-\sigma}.\end{aligned}$$

$\square$

3 Proofs of Theorem 1 and Corollary 2

Recall that the generalized divisor function

$$d_k(n) = \sum_{n=n_1\cdots n_k} 1,$$

and its Dirichlet series is

$$\zeta^k(s) = \sum_{n=1}^{\infty}\frac{d_k(n)}{n^s}.$$

From [7, Theorem 13.2 and 13.3] it follows that

$$\sum_{n\leq x} d_{2^k}(n) = xP_{2^k-1}(\log x) + O(x^{\beta_k+\epsilon}), \tag{9}$$

where $P_{2^k-1}(t)$ is a polynomial of degree 2^k-1 in t, and

$$\beta_2 = \frac{1}{2}, \quad \beta_3 = \frac{5}{8}, \quad \beta_4 = \frac{7}{9}, \quad \beta_5 = \frac{31}{36}, \quad \beta_6 = \frac{207}{224}, \quad \beta_k = 1 - 2^{-\frac{2}{3}k}/50, \qquad k \geq 7.$$

Then by Lemma 4, we have that

$$\sum_{n\leq x} A^k(n) = \sum_{m\ell\leq x} d_{2^k}(m)g(\ell) = \sum_{\ell\leq x} g(\ell)\sum_{m\leq x/\ell} d_{2^k}(m),$$

and formula (9) applied to the inner sum gives

$$\sum_{n\leq x} A^k(n) = \sum_{\ell\leq x} g(\ell)\left\{\frac{x}{\ell}P_{2^k-1}\left(\log(\frac{x}{\ell})\right) + O\left((\frac{x}{\ell})^{\beta_k+\epsilon}\right)\right\}$$

$$= xQ_{2^k-1}(\log x) + O(x^{\beta_k+\epsilon}),$$

if we notice from Lemma 4 that the infinite series $\sum_{\ell=1}^{\infty} \frac{g(\ell)}{\ell}$ and $\sum_{\ell=1}^{\infty} \frac{g(\ell)\log^k \ell}{\ell}$ are absolutely convergent, and

$$\sum_{\ell \le x} |g(\ell)| \ll x^{1/2+\epsilon}.$$

From the definitions of $P(n)$ and Abel's summation formula, we can easily get

$$\sum_{n\le x} P^k(n) = x^2 Q'_{2^k-1}(\log x) + O(x^{1+\beta_k+\epsilon}),$$

where $Q'_{2^k-1}(t)$ is a polynomial of degree 2^k-1 in t.

4 Proof of Theorem 3

It suffices to prove that

$$\int_U^{2U} E_k(x)dx \ll U^{1+\delta_k+\epsilon}, \tag{10}$$

where

$$\delta_3 = 1/2, \quad \delta_4 = 0.6030739, \quad \delta_5 = 0.773114.$$

By Perron's formula (see for example, [6, Chapter 5]), we have for $T \le x \le 2T$ that

$$\sum_{n\le x} A^k(n) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \zeta^{2^k}(s)G_k(s)\frac{x^s}{s}ds + O(T^{\varepsilon}).$$

Then we move the integration to the parallel segment with $\mathrm{Re}(s) = 1-\varepsilon$ to get

$$E_k(x) = \frac{1}{2\pi i}\int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \zeta^{2^k}(s)G_k(s)\frac{x^s}{s}ds + O(T^{\varepsilon}).$$

So

$$\begin{aligned}\int_U^{2U} E_k(x)dx &= \frac{1}{2\pi i}\int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \frac{\zeta^{2^k}(s)G_k(s)}{s}\left(\int_U^{2U} x^s dx\right)ds + O(U^{1+\varepsilon}) \\ &= \frac{1}{2\pi i}\int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \frac{\zeta^{2^k}(s)G_k(s)(2^{s+1}-1)U^{s+1}}{s(s+1)}ds + O(U^{1+\varepsilon}). \end{aligned} \tag{11}$$

Moving the integral line in the last integral of (11) to $\sigma = c$, where $\frac{1}{2} < c < 1$, we have

$$\int_U^{2U} E_k(x)dx \ll U^{1+c}\int_{c-iT}^{c+iT} \frac{|\zeta(s)|^{2^k}}{|s(s+1)|}ds \ll U^{1+c}\int_1^T \frac{|\zeta(c+it)|^{2^k}}{T^2}ds, \tag{12}$$

if we notice that $G_k(s)$ is absolutely convergent in $\mathrm{Re}(s) > \frac{1}{2}$.

For the case $k = 3$, it follows from [7, Theorem 8.3] that

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^8 dt \ll T^{\frac{3}{2}}.$$

On taking $c = 1/2 + \varepsilon$ in (12), we have

$$\int_U^{2U} E_3(x)dx \ll U^{3/2}. \tag{13}$$

For the case $k = 4$, from [7, Theorem 8.3] and [7, Theorem 8.4], it follows that

$$\int_1^T \left|\zeta(\frac{1}{2} + it)\right|^{16} dt \ll T^{1+\frac{350}{216}}$$

and

$$\int_1^T |\zeta(\sigma + it)|^{16} dt \ll T^{1+\varepsilon},$$

where σ satisfies

$$\frac{12408}{4537 - 4890\sigma} = 16,$$

which gives $\sigma = 0.7692229$. By [7, Lemma 8.3], we have

$$\int_1^T |\zeta(0.6030739 + it)|^{16} dt \ll T^2.$$

In the formula (12), we take $c = 0.6030739$ to get

$$\int_U^{2U} E_4(x)dx \ll U^{1+0.6030739}. \tag{14}$$

Similarly, we can get $\delta_k, k \geq 5$. For example, $\delta_5 = 0.773114$.

5 Acknowledgments

The authors express their gratitude to the referee for a careful reading of the manuscript and many valuable suggestions, which highly improve the quality of this paper.

Deyu Zhang is supported by National Natural Science Foundation of China (Grant No. 11001154) and Shandong Province Natural Science Foundation (ZR2010AQ009). Wenguang Zhai is supported by National Natural Science Foundation of China (Grant No. 11171344) and Natural Science Foundation of Beijing (No. 1112010).

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2010 *Mathematics Subject Classification*: Primary 11N37.
Keywords: gcd-sum function, Dirichlet divisor problem, asymptotic formula.

(Concerned with sequence A018804.)

Received January 10 2013; revised versions received March 6 2013; June 14 2013. Published in *Journal of Integer Sequences*, July 25 2013.

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