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Valuations of v-adic Power Sums and Zero Distribution for the Goss v-adic Zeta Function for $\mathbb{F}_q[t]$

Dinesh S. Thakur[1]
Department of Mathematics
University of Arizona
Tucson, AZ 85721
USA
thakur@math.arizona.edu

Abstract

We study the valuation at an irreducible polynomial v of the v-adic power sum, for exponent k (or $-k$), of polynomials of a given degree d in $\mathbb{F}_q[t]$, as a sequence in d (or k). Understanding these sequences has immediate consequences, via standard Newton polygon calculations, for the zero distribution of the corresponding v-adic Goss zeta functions. We concentrate on v of degree one and two and give several results and conjectures describing these sequences. As an application, we show, for example, that the naive Riemann hypothesis statement which works in several cases, needs modifications, even for a prime of degree two. In the last section, we give an elementary proof of (and generalize) a product formula of Pink for the leading term of the Goss zeta function.

Dedicated to Jean-Paul Allouche on his 60th birthday

1 Introduction

In a previous paper [9], we investigated valuation sequences at the infinite place of the power sums of polynomials in $\mathbb{F}_q[t]$, with positive and negative exponents. We gave

(i) a simple recipe to find these;

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(ii) a duality between valuations for positive and negative exponents;

(iii) a simple recursion in case q was prime; and

(iv) applications to the zero distribution of the Goss zeta function [4, 5, 8], giving a cleaner approach to the Riemann hypothesis (due to Wan and Sheats [10, 3, 6]) and to the study of the multizeta values in this case.

In this paper, we look at finite places v of $\mathbb{F}_q[t]$ and study the v-adic valuations of power sums with the v-factor removed. We give conjectural formulas for them describing very interesting patterns, for v of degree one and two, and proving full results in some special cases. We then give applications to the zero distribution of the Goss v-adic zeta function, showing that in the degree-two case, the Riemann hypothesis type statement, holding at the infinite degree-one place, needs modification. We also calculate a particular power sum, which is a leading term of the Goss zeta, when q is a prime, and in particular, give an elementary proof of (and generalize) Pink's nice product formula for the same when $q = 2$.

Patterns of these valuation sequences exhibit symmetries remarkably similar to those occurring in several papers of Allouche, Shallit, Mendès France, Lasjaunias, etc., as well as those found by the author for continued fractions for analogs of e, Hurwitz numbers, and some algebraic quantities.

2 Power sums

2.1 Notation

$$\begin{array}{rcl}
\mathbb{Z} & = & \{\text{integers}\} \\
\mathbb{Z}_+ & = & \{\text{positive integers}\} \\
\mathbb{Z}_{\geq 0} & = & \{\text{nonnegative integers}\} \\
q & = & \text{a power of a prime } p,\ q = p^f \\
A & = & \mathbb{F}_q[t] \\
A+ & = & \{\text{monics in } A\} \\
A_d+ & = & \{\text{monics in } A \text{ of degree } d\} \\
K & = & \mathbb{F}_q(t) \\
K_v & = & \text{completion of } K \text{ at the place } v \text{ of } K \\
C_v & = & \text{the completion of an algebraic closure of } K_v \\
[n] & = & t^{q^n} - t \\
d_n & = & \prod_{i=0}^{n-1}(t^{q^n} - t^{q^i}) \\
\ell_n & = & \prod_{i=1}^{n}(t - t^{q^i}) \\
\ell(k) & = & \text{sum of the digits of the base } q \text{ expansion of } k \\
\deg & = & \text{function assigning to } a \in A \text{ its degree in } t,\ \deg(0) = -\infty
\end{array}$$

While in the notation above, we let v be any place of K, we use v for the finite places (i.e., those corresponding to irreducible polynomials of A), and we use ∞ for the usual infinite place of K (i.e., the place corresponding to the valuation coming from the degree in t).

2.2 Power sums

For $k \in \mathbb{Z}$ and $d \in \mathbb{Z}_{\geq 0}$, and v a prime of A, write

$$S_d(k) := \sum_{a \in A_d+} \frac{1}{a^k} \in K, \quad S_{d,v}(k) := \sum_{\substack{a \in A_d+ \\ (v,a)=1}} \frac{1}{a^k} \in K.$$

2.3 Goss Zeta and v-adic zeta

For $k \in \mathbb{Z}$, put

$$\zeta(k) := \sum_{d=0}^{\infty} S_d(k) \in K_\infty, \quad \zeta_v(k) := \sum_{d=0}^{\infty} S_{d,v}(k) \in K_v.$$

More generally, we have two-variable Goss zeta functions defined as follows. Define exponent spaces $S_\infty := C_\infty^* \times \mathbb{Z}_p$ and $S_v := C_v^* \times \lim \mathbb{Z}/(q^{\deg v} - 1)p^j\mathbb{Z}$.

For $s = (x, y) \in S_\infty$, put

$$\zeta(s) = \sum_{d=0}^{\infty} x^d \sum_{a \in A_d+} (a/t^d)^y \in C_\infty.$$

Note that for y an integer, the coefficient of the d-th term in this power series in x is nothing but $S_d(-y)t^{-dy}$, hence $\zeta(k)$ as above is $\zeta(t^{-k}, -k)$.

For $s = (x, y) \in S_v$, put

$$\zeta_v(s) = \sum_{d=0}^{\infty} x^d \sum_{\substack{a \in A_d+ \\ (a,v)=1}} a^y \in C_v.$$

Note that for y an integer, the coefficient of the d-th term in this power series is nothing but $S_{d,v}(-y)$, hence $\zeta_v(k)$ above is $\zeta(1, -k)$.

See [2, 4, 8] and references there for more details on these interpolations, properties.

We begin with definitions and basic results on the valuations of these power sums.

2.4 Valuations

For a finite prime v of A, and the usual place at infinity, put

$$s_d(k) := \mathrm{val}_\infty(S_d(k)) = -\deg(S_d(k)), \quad v_d(k) := \mathrm{val}_v(S_{d,v}(k)).$$

2.5 Valuation sequence at ∞

In this paper, we concentrate on valuations at finite primes, but for comparison only, we mention some results on valuations at infinity, and give some remarks.

Previously [9], we described interesting behavior of $s_d(k)$. For example, when q is a prime, we proved [9, Thm. 1]

$$s_d(k) = s_{d-1}(s_1(k)) + s_1(k) = \sum_{i=1}^{d} s_1^{(i)}(k),$$

where $s_1^{(i)}$ is the i-th iteration of s_1. There are several other formulas, and applications in [9], including Riemann hypothesis type zero-distribution results for the Goss two-variable zeta function for the place at infinity.

2.6 Remarks

1. For non-prime q, $s_d(k)$ is not even determined by $s_1(k)$. Example: for $q = 4$, $s_1(75) = s_1(93) = 96$, whereas $s_2(75) = 348$ and $s_2(93) = 480$. For general q, [9, Thm. 2] gives a sort of 'duality' connection between the values at positive and negative k, linking values at $-k$ and $q^n - k$ under certain conditions. An example dual to the one above is $q = 4$, $s_1(-181) = s_1(-163) = -160$ and $s_2(-181) = -164$, whereas $s_2(-163)$ is infinite.

2. The recursion above when q is a prime and the duality for $s_d(k)$ for general q [9, Thm. 2] also gives a fast way of calculating these valuations at positive k. Note that the case of negative k (even when $d = 1$) is a much easier polynomial calculation than rational calculation at positive k.

3. We do not know whether there is any duality of actual power sums $S_d(k)$ or zeta values $\zeta(k)$ themselves, giving some kind of functional equation. For the two-parameter special family $k = q^n - q^r$, with $r < n$, we have the following nice relation, for $n - r \geq d$,

$$S_d(q^r - q^n)/S_d(q^r) = (q^{n-r}!)^{q^r}/(q^{n-(r+d)}!)^{q^{r+d}},$$

using the Carlitz factorial [8, p. 102] and Carlitz's results (see [8, Sec.5.6]). While all the factorial-gamma values are monomials in $[i]$'s, the ratios of power sums for dual exponents are not these kinds of monomials for general k, so we do not know what, if any, a correct generalization of such phenomena would be.

4. In [9, 2.2.5 (ii)], I showed that if $S_d(k_1) = S_d(k_2)$ holds for $d = 1$, then it holds for all d. While true, it should be pointed out that, in addition to the proof there, in case of positive k, it is vacuously true, because [9, 2.3] shows that $S_d(k)\ell_d^k$ is a polynomial congruent to 1 (mod t), so the equality above for any d and $k_2 > k_1 > 0$ would imply $\ell_d^{k_2-k_1} \equiv 1 \pmod{t}$, whereas it is divisible by t. On the other hand, interestingly, the statement is true, when q is prime, with S_d replaced by s_d by [9, Thm. 1], and not in general as we saw above, for example. (In my editing gaffe, these motivating remarks were inadvertently left off in [9]).

2.7 Valuation sequence at v

We have $v_d(k) \geq 0$ for all $k \in \mathbb{Z}$, with it being infinite when the corresponding power sum is zero. Note $v_d(pk) = pv_d(k)$, so without loss of generality, we can restrict to k prime to p.

Another simple remark is that two primes related by automorphisms $t \to t + \theta$, $(\theta \in \mathbb{F}_q)$ of A (e.g., any two degree-one primes) give the same valuation sequence.

2.8 Non-vanishing of power sums

The power sums $S_{d,v}(k)$ are non-zero for $k > 0$, as can be seen by choosing a monic prime P, unequal to v and of degree d (which can be done unless $q = 2$, v is of degree 2 and $d = 2$, which we can check separately), and noticing that except for $a = P$ term, all other terms in the power sum $S_{d,v}(k)$ have valuation 0 at P.

For $k \leq 0$, these power sums can be zero. In fact, $S_{d,v}(k)$ is zero if $d > \ell(-k)/(q-1) + \deg(v)$, by the Carlitz result [8, Cor. 5.6.2] that $S_d(k) = 0$ if $d > \ell(-k)/(q-1)$ (with the converse of the latter holding also for q prime). We have not investigated the exact conditions corresponding to the non-vanishing. For a similar necessary and sufficient condition for vanishing of $S_d(k)$, for general q, due to Carlitz, Sheats and Böckle, see [9, A5] and [1].

3 Some evaluations and simple bounds for $k > 0$

We recall [8, Sec. 5.6] the formulas for power sums and valuations of $[n]$ and ℓ_n.

$$S_d(r) = \frac{1}{\ell_d^r}, \quad 0 < r \leq q, \quad S_d(q+1) = \frac{[1] - [d]^q}{[1]\ell_d^{q+1}}.$$

Note that $[n]$ is the product of all monic irreducible polynomials of degree dividing n. When $q = 2$ and $v = t^2 + t + 1$, we have

$$[d] = v^{2^{d-1}} + v^{2^{d-2}} + \cdots + v, \quad \text{or } [d] = v^{2^{d-1}} + \cdots + v^2 + v + 1$$

according to whether d is even or odd.

Hence, the valuation of $[n]$ (resp., ℓ_n) at a degree one prime is 1 (resp., n), whereas at the degree 2 prime v, when $q = 2$, the valuation is 1 or 0 according as n is even or odd (resp., $\lfloor n/2 \rfloor$).

Claim 1. *If v is a prime of degree 1, then $v_d(1) = q^d - (d+1)$.*

Proof. Without loss of generality, using automorphisms $t \to t + \theta$ $(\theta \in \mathbb{F}_q)$ of $\mathbb{F}_q[t]$ which preserve sign, we may assume that $v = t$. Then

$$S_{d,v}(1) = \frac{1}{\ell_d} - \frac{1}{t\ell_{d-1}} = \frac{t - (t - t^{q^d})}{t\ell_d}.$$

□

Claim 2. *If $q = 2$, v is a prime of degree 1, then $v_d(3) = 2^{d+1} - 3d - 1$, for $d \geq 1$.*

Proof. Since $S_d(3) = (1+[d]^q/[1])/\ell_d^3 = [d+1]/([1]\ell_d^3)$, we have $S_{d,v}(3) = S_d(3) - S_{d-1}(3)/t^3 = (t^{2^{d+1}} + t^{2^{d+2}-3})/([1]\ell_d^3)$. ☐

Claim 3. *If $q = 2$, v is the prime of degree 2, namely $v = t^2 + t + 1$, then $v_d(1) = 2^{d-1} - \lfloor (d+1)/2 \rfloor$, for $d \geq 1$.*

Proof. Proceeding as before, we have

$$S_{d,v}(1) = \frac{v + [d][d-1]}{v\ell_d}$$

so the valuation of the denominator is $1 + \lfloor d/2 \rfloor$. Let us write $M = v^{2^{d-2}} + \cdots + v^2 + v$ temporarily, so that $M^2 + M = v^{2^{d-1}} + v$. The numerator is $v + (v^{2^{d-1}} + M)(M+1) = v^{2^{d-1}}M$ or $v + (v^{2^{d-1}} + M + 1)M = v^{2^{d-1}} + v^{2^{d-1}}M$ according as whether d is even or odd, so its valuation is $2^{d-1} + 1$ or 2^{d-1}, respectively. ☐

Claim 4. *If $q = 2$, $v = t^2 + t + 1$, then $v_d(3) = 2^d - 3\lfloor d/2 \rfloor + (-1)^{d-1}$, for $d > 1$. We have $v_1(3) = 1$, $v_0(3) = 0$.*

Proof. We have

$$S_{d,v}(3) = \frac{[d+1]v^3 + [d-1]^4[d]^3}{v^3[1]\ell_d^3}.$$

The valuation of the denominator is $3 + 3\lfloor d/2 \rfloor$. The valuation of the numerator is $2^d + 2$ for d even, and $2^d + 4$ for d odd, by a straight calculation (we omit the details) as in the proof above. ☐

More generally, we have the following *conjectural recipe* guessed from a small numerical data computed:

If $q = 2$, $v = t^2 + t + 1$, $d > 1$,

$$v_d(2^n - 1) = 2^{n+d-2} - (2^n - 1)\lfloor d/2 \rfloor + (-1)^{d-1}, \text{ if } n \text{ is even,}$$

and

$$v_d(2^n - 1) = 2^{n+d-2} - (2^n - 1)\lfloor d/2 \rfloor - \lfloor (d+1)/2 \rfloor + \lfloor d/2 \rfloor, \text{ if } n \text{ is odd.}$$

3.1 Trivial lower bound

Claim 5. *If $k > 0$, and $d > m \deg(v)$, then $v_d(k) \geq m$.*

Proof. The terms in the sum $S_{d,v}(k)$ can be grouped by orbits $1/(n+\theta v^m)^k$, as θ runs through elements of $\mathbb{F}_q$. The terms of each orbit add to zero (mod v^m). ☐

3.2 Trivial upper bound

Claim 6. *For $k > 0$, $d \geq \deg(v)$, we have $v_d(k) \leq dk(q^d - q^{d-\deg(v)} - 1)/\deg(v)$.*

Proof. By definition $S_{d,v}(k)$ is the sum of $q^d - q^{\deg(v)}$ terms of the form $1/n^k$ with n monic and prime to v, so with the common denominator the product of n^k's, the numerator is sum of products of $q^d - q^{d-\deg(v)} - 1$ terms of degree dk. ☐

3.3 Congruences and periodicity

In contrast to $(\mathbb{Z}/p^n)^*$ which is cyclic for odd prime p, the analog $(A/v^n)^*$ is far from cyclic in general, when $n > 1$. If v has degree D, then $(A/v^n)^*$ has order $(q^D - 1)q^{D(n-1)}$, but it has exponent

$$e_n = (q^D - 1)p^{\lceil \log_p(n) \rceil}.$$

So $S_{d,v}(k) \equiv S_{d,v}(k + me_n) \pmod{v^n}$. In particular,

$$\text{if } v_d(k) < n, \text{ then } v_d(k) = v_d(k + me_n), \ m \in \mathbb{Z}.$$

(Hence for any fixed d, $v_d(k)$ can be small even for large k.)

We will see applications resulting in nice patterns below. Also note that as m can be negative, we can replace calculation with rational functions to get $v_d(k)$ at positive k, by easier calculation with polynomials to get it from negative k.

We know [9, Thm. 2] says that $s_d(q^n - k) - s_d(-k) = dq^n$, if $s_d(-k) \neq 0$ and $q^n > k > 0$. Using the congruence idea at the infinite prime together with a switching trick, let us now give another proof of this equality claimed, but using a slightly stronger hypothesis that $s_d(-k) \neq 0$ and $q^n > k\ell(k)/(q-1)$.

Let us temporarily denote by w the prime $1/t$ of $\mathbb{F}_q[1/t]$. We have

$$S_d(-k) = t^{dk} \sum (1 + \frac{f_{d-1}}{t} + \cdots + \frac{f_0}{t^d})^k, \quad S_d(q^n - k) = t^{dk - dq^n} \sum (1 + \cdots)^{k - q^n}.$$

Now the first sum is (term-by-term) congruent modulo w^{q^n} to the second sum. Since the first sum is non-zero, we get a trivial upper bound dk on its valuation at w. On the other hand, we know by [8, 5.6.2] that $d \leq \ell(k)/(q-1)$, because $s_d(-k) \neq 0$. Hence the claim follows as before.

4 Recipes and inter-relations for degree 1 primes, $k \in \mathbb{Z}$

Let us write $a \oplus b \oplus c + \cdots$ to denote the sum $a + b + c + \cdots$, where this sum has no carry over base p. In Theorem 7 below, by the "greedy algorithm" we mean first choosing among valid decompositions with $m_d \leq \cdots \leq m_1$, the least m_d, then among these, the least m_{d-1}, and so forth.

Theorem 7. *(i) Let k be negative and $m = -k$. Then either $s_d(k) = -dm + \min(m_1 + 2m_2 + \cdots + dm_d)$, where $m = m_0 \oplus \cdots \oplus m_d$, $m_i \geq 0$, and for $i \geq 1$, $(q-1)$ divides $m_i > 0$, or $s_d(k)$ is infinite, if there is no such decomposition. When the decompositions exist, the minimum is uniquely given by greedy algorithm.*

(ii) Let k be positive. Then $s_d(k) = dk + \min(m_1 + \cdots + dm_d)$, with $(k-1) + m = (k-1) \oplus m$ and $m = m_1 \oplus \cdots \oplus m_d$, with m_i positive and divisible by $q-1$. The minimum is uniquely given by the greedy algorithm.

(iii) Let k be negative and $m = -k$. Let v be a prime of A of degree one. Then either $v_d(k) = \min(m_1 + \cdots + dm_d)$, where $m = m_0 \oplus \cdots \oplus m_d$, where $(q-1)$ divides $m_i > 0$

for $0 < i < d$, and $q-1$ divides $m_0 \geq 0$; or $v_d(k)$ is infinite, if there is no such decomposition. When the decompositions exist, the minimum is uniquely given by the greedy algorithm. (We see immediately that m_d is the least non-negative residue of m (mod $q-1$) for the minimal.)

(iv) *Let k be positive. Let v be a prime of A of degree one. Then $v_d(k) = \min(m_1 + \cdots + dm_d)$, with $(k-1) + m = (k-1) \oplus m$, $m = m_1 \oplus \cdots m_d$, and $(q-1)$ divides $m_j > 0$ for $j < d$ and $(q-1)$ divides $k + m_d$. The unique minimum is given by the greedy algorithm.*

Proof. Part (i) follows from results of Carlitz, Diaz-Vargas, Sheats as explained and referenced in [8, 5.8] or [9, Sec. 4]. Part (ii) is proved in [9, Sec. 4]. The parts (iii) and (iv) follow by exactly the same arguments, once we note a crucial difference as follows. First consider negative $k = -m$, then we have

$$S_d(k) = \sum_{f_i \in \mathbb{F}_q} (t^d + f_{d-1}t^{d-1} + \cdots + f_0)^m = \sum \binom{m}{m_0, \cdots, m_d} (t^d)^{m_d} \cdots (f_0)^{m_0},$$

whereas for $v = t$, $S_{d,v}(k)$ is given by exactly the same sum, except that the condition $f_0 \neq 0$ is added and rather than looking at highest power of t present in the answer, we now look at the lowest power. The parity and positivity conditions come from the well-known fact that $\sum f^n$, (where f runs through elements of $\mathbb{F}_q$ and n is a positive integer) is -1 or 0, according as whether n is divisible by $q-1$ or not. The carry over conditions come from Lucas' theorem that the binomial coefficient above is non-zero exactly when there is no carry. Hence our assertions follow easily if the minimum is unique. This is the hard part, proved in Sheats [6] in general (and by Diaz-Vargas [3] for much easier case of prime q). So (iii) follows similarly by reduction to the Sheats minimization theorem and then (iv) is deduced exactly the same way as in the proof of [9, Theorem 1, Sec. 4]. □

Corollary 8. *Let v be a prime of degree one.*

(i) *Let k be positive. Then $v_{d+1}(k) = s_d(k) - dk$ if $q-1$ divides k.*

(ii) *Let k is negative. Then $v_{d+1}(k) = s_d(k) - dk$ if $q-1$ divides k and $s_d(k)$ is finite.*

(iii) *If $q-1$ divides k, then $v_d(k)$ is also divisible by $q-1$.*

(iv) *If q is prime and k is divisible by $q-1$, then $v_{d+1}(k) = v_d(v_2(k) + k) + dv_2(k)$.*

Proof. Part (i) (resp., part (ii)) follows from comparing (ii) and (iv) (resp., (i) and (iii)) of Theorem 7. Part (iii) follows from (i) and (ii) and the fact [9, Thm. 6, Thm. 14] that $s_d(k) \equiv dk \pmod{q-1}$. Part (iv) follows from (i), (ii) and the fact [9, Thm. 1] that for q prime, we have $s_d(k) = s_{d-1}(s_1(k)) + s_1(k)$. □

Another direct way to prove Corollary 8 is as follows. For k divisible by $q-1$, we have

$$t^{-dk}S_d(k) = \sum (1 + f_1/t + \cdots + f_0/t^d)^{-k} = \sum_{i=0}^{d} S'_i(k),$$

where we temporarily write S'_d for the power sum $S_{d,v}$ for $A = \mathbb{F}_q[1/t]$ for its degree-one finite prime $1/t$. Replacing t by $1/t$, and telescoping, we see that $v_{d+1}(k)$ is $s_d(k) - k$ (as it is greater than $s_{d+1}(k) - (d+1)k$, by the last paragraph in the proof of [9, Theorem 4], combined with $s_1(k) > k$).

4.1 Remarks

1. By part (iii) of Corollary 8, the recursion in (iv) can be continued, so that $v_2(k)$ determines $v_d(k)$, for $d \geq 2$, for a prime v of degree one.

2. All parts no longer hold if we drop the divisibility condition.

3. The second proof mentioned above is achieved by developing the ideas of Wan [10] and Goss [5, Prop. 9] further.

5 $v_d(k)$ when q is a prime, v is of degree 1, and $k < 0$

Theorem 9. *Let $q = p$ be a prime, v a prime of A of degree one and $-m = k < 0$. Write $m = \sum_{i=1}^{\ell} p^{e_i}$, with e_i monotonically increasing and with not more than $p-1$ of the consecutive values being the same (i.e., consider the base p-digit expansion sequentially one digit at a time). Also, let r be the least non-negative residue of m (mod $q-1$). Then $v_d(k)$ is infinite if $\ell < (p-1)(d-1)$, and otherwise*

$$v_d(k) = dr + \sum_{j=1}^{d-1} j \sum_{s=1}^{p-1} p^{e_{d-1-j(p-1)+s}}.$$

Proof. Note that $p^i \equiv 1 \pmod{q-1}$ when q is prime. Hence $p-1$ powers together give divisibility by $q-1$. Hence the recipe in (iii) of the Theorem 7 simplifies and has for the minimum the choice $m_d = r$, $m_{d-1}, \cdots, m_1$ obtained by picking $p-1$ digits from the base p expansion of m starting from the lowest digits (and dumping the rest of the expansion, if any, into m_0). $\square$

Note that it looks even simpler for $q = 2$, so that $r = 0$ and inner sums are singletons, and also that in this case $v_1(k) = 0$, $v_2(k)$ is, in fact, the valuation of k at 2.

6 $v_d(k)$ when $q=2$, v is of degree 1, and $k>0$

First note that $v_d(2k) = 2v_d(k)$, so we will focus on the case where k is odd.

Next, observe that $v_1(k) = 0$ for all k and $v_2(k) = 1$ for all odd k.

From [9, Sec. A.2 (1)] and duality part (i) of Corollary 8, we see that

$$w_n := v_d(2^n - 1) = 2^{d+n-1} - 2^n d + (d-1).$$

We define sequence f_n (it is $v_{n-3}(5) - v_{n-3}(1)$) by

$$f_0 = 0 \text{ and } f_n = 2f_{n-1} + 4n.$$

Here is the recipe: For a given d, we describe the sequence of $v_d(k)$ with k odd, so that the n-th entry will correspond to $k = 2n-1$ and thus w_n is 2^{n-1}-th entry. Let X_n be the vector of the first $2^{n-1}-1$ entries, and write $X_{n+1} = X_n, w_n, X_n'$, in two halves. In other words, the whole sequence is of the form $X_1, w_1, X_1', w_2, X_2', w_3, X_3', \cdots$ or equivalently,

$$X_n, w_n, X_n', w_{n+1}, \cdots,$$

Theorem 10. *The second half X_n' is obtained from the first half X_n by adding $2^{n-2}f_m$ to the entries with index (i.e., $(k+1)/2$) having the base-2 expansion*

$$\sum_{i=w}^{n-2} b_i 2^i, \;\; b_w \neq 0, \textit{with exactly } d-3-m \textit{ of the } b_i\textit{'s zero},$$

where $1 \leq m \leq d-3$.

Proof. We replace d by $d+1$ for convenience. We then need to prove $v_{d+1}(k+2^n) - v_{d+1}(k) = 2^{n-2}f_m$. By using the duality (i) of Theorem 9, we are reduced to proving $s_d(k+2^n) - s_d(k) = d2^n + 2^{n-2}f_m$, for $2^n - 1 > k$ odd and m as in Theorem 10, but with d replaced by $d+1$. By [9, 3.3], we have

$$s_d(k) - dk = d \cdot 2^{e_0} + \cdots + 1 \cdot 2^{e_{d-1}},$$

where we write the base 2 expansion of k as

$$k = \cdots 0_{e_{t+1}} 0_{e_t} 1 \cdots 1 0_{e_{t-1}} 1 \cdots 1 \cdots 0_{e_2} 1 \cdots 1 0_{e_1} 1 \cdots 1_{e_0} 0 \cdots 0.$$

Since, for us, k is odd, less than 2^n, (with $n = e_t + r$, $r \geq 0$) $k + 2^n$ has expansion of the form

$$\cdots 0_{e_{t+r}} 1 0_{e_{t+r-1}} \cdots 0_{e_t} 1 \cdots 1 0_{e_{t-1}} 1 \cdots 1 \cdots 0_{e_2} 1 \cdots 1 0_{e_1} 1 \cdots 1_{e_0}.$$

Observe that $(d+1) - 3 - m = t + r - 2$, by counting the relevant zeros in the expansion of $(k+1)/2$. If $m \leq 0$, the relevant e_i's are the same for k and for $k+2^n$, and hence the left side of the first formula is $d2^n$, which agrees with the right side as $f_m = 0$ then. Now we proceed by an induction on m. Now the e_i's which are different are $e_{d-m}, \cdots, e_{d-1}$ which are $n, \cdots, n+(m-1)$ for k whereas $n+1, \cdots, n+m$ for $k+2^n$ resulting in the difference $X_m := (2^{n+m} + 2 \cdot 2^{n+m-1} + \cdots + m2^{n+1}) - (2^{n+m-1} + \cdots + m2^n)$. Hence $X_{m+1} - 2X_m = (m+1) \cdot 2^n = 4 \cdot 2^{n-2}(m+1)$, matching the recursion for $2^{n-2}f_m$'s. ∎

Remark 11. Using duality, we have converted the recursion in d for valuations at infinity into proving nice fast 'doubling' pattern for a fixed d.

6.1 Examples

For $d \leq 3$, $X'_n = X_n$, so there is block repetition after new entries w_n. The case $d = 0$ (resp., $d = 1$) corresponds to vectors with all entries zero (resp, one), as mentioned above. On the other hand, for $d = 3$, we get

$$\overline{4}, \overline{6}, 4, \overline{10}, 4, 6, 4, \overline{18}, 4, 6, \cdots, 4, \overline{34}, 4, 6, 4, \cdots,$$

where the over-lined entries are the w_n's.

For $d = 4$, the quantity X'_n is obtained by keeping its first half the same and adding $2^n = 2^{n-2} \cdot 4$ to the half-way entry, half of the next half-way, etc., entries (i.e., $\sum_w^{n-2} 2^i$-th entries).

More generally, X_n and X'_n share their first $X_{n-(d-3)}$ part. This also follows another way from 3.3.

7 When $q = 3$, v is of degree 1, and $k > 0$

We list valuation sequence $v_d(k)$ for k not divisible by 3.

For $d = 1$, $v_d(k)$ is zero for even k and 1 for odd k. So the sequence is periodic of period $1, 0, 0, 1$ of length 4, when only k not divisible by 3 are used.

7.1 Conjectural recipe for $d = 2$

The valuation sequence is of the form $X_1, a_1, X_2, a_2, X_3, a_3, \cdots$ where

(i) $X_1 = X_{3n} = X_{3n+1} = [6, 4, 2, 14, 12, 10, 2, 8, 6, 4, 2]$,

(ii) X_{3n+2} is the same as X_1 except $a_{3n+1} - a_{3n+2}$ is added to fourth, fifth and sixth terms of X_1 to get the corresponding entries,

(iii) The a_n's (which correspond to $k = 17 + 18(n-1)$) look like $3^r + 3^s + 2$, with $r \geq s \geq 2$ (and $a_{3m+2} = 20$, so that $r = s = 2$ and for n of the form $3m+1$, we have $s = 2$). More precisely, we describe the full sequence $[a_1, a_2, \cdots]$ as follows: $a_{3^n} = 3^{n+3} + 3^{n+2} + 2$, $a_{2 \cdot 3^n} = 3^{n+2} + 3^{n+2} + 2$, the block between $a_{2 \cdot 3^n + 1}$ to $a_{3^{n+1}-1}$ is exactly the block between a_1 to $a_{3^n - 1}$, whereas you get the block between a_{3^n+1} to $a_{2 \cdot 3^n - 1}$ by taking the block between a_1 to a_{3^n-1} and replacing the entries $3^{n+2} + 3^k + 2$ by the new entries $3^{n+3} + 3^k + 2$.

So the sequence is X_1 followed by 38 followed by 6, 4, 2, 32, 30, 28, 2, 8, 6, 4, 2, followed by 20, X_1, 110, X_1, 92, followed by 6, 4, 2, 86, 84, 82, 2, 8, 6, 4, 2, followed by 20, X_1, 56, X_1, 38, ...

8 When $q = 4$, v is of degree 1, and $k > 0$

Here is the *conjectural recipe* for $d = 1$:

We describe the sequence $v_1(k)$ with k odd, so that the n-th entry will correspond to $k = 2n - 1$ and thus w_n is 2^{n-1}-th entry:

The whole sequence is limit of vectors X_n (of increasing sizes with initial portion being X_{n-1}) with

(i) X_1 being $[2, 0, 1, 8, 0, 1]$ and

(ii) X_n consisting of X_{n-1} followed by X'_{n-1}, where the entries of X'_{n-1} are the same as that of X_{n-1}, except one entry is changed as follows: If n is odd, change the $k = 2^n - 1$-th entry (which is 2^n) to 2^{n+2} and if n is even, change $k = 2^{n+1} - 1$-th entry (which is 2^{n+1}) to $2^n + 1$.

So the sequence is X_1 followed by 2, 0, 1, 5, 0, 1, 2, 0, 1, 32, 0, 1, 2, 0, 1, 5, 0, 1, X_1, ...

9 When $q = 2$, v is of degree 2, and $k > 0$

For a given d, we describe the sequence of $v_d(k)$ with k odd, so that the n-th entry will correspond to $k = 2n - 1$.

By definition, it is easy to see that $v_1(k) = 1$ or 0, respectively, and $v_2(k) = 0$ or 1, respectively, according to whether k is divisible by 3 or not. Let us check this for $d = 1$: the numerator of $1/t^k + 1/(t+1)^k$ is congruent to $t^k + (t+1)^k \equiv t^k + (t^2)^k \equiv t^k(1 + t^k)$ modulo v, but $t^{3m} \equiv 1$ and $(t^{3m} + 1)/(v(t+1)) \equiv t^{3m-3} + t^{3m-6} + \cdots + 1 \equiv m \not\equiv 0 \pmod{v}$ (as m is odd).

Now we fix $d \geq 3$, so it will be dropped from the notation sometimes.

We write the sequence in the form $X_1, w_1, X'_1, w_2, X'_2, w_3, X'_3, \cdots$ where $X_{n+1} = X_n, w_n, X'_n$. In other words, For every n, we write the sequence in the form

$$X_n, w_n, X'_n, w_{n+1}, \cdots,$$

where X_n and X'_n are vectors of entries of length $3 \cdot 2^{n-1} - 1$, with X_n containing entries (odd k's) from $v_d(1)$ to $v_d(3 \cdot 2^n - 3)$ and $w_n = v_d(3 \cdot 2^n - 1)$. We can further subdivide X_n in 'thirds'. More precisely, $X_n = (A_n, B_n, C_n)$ with A_n consisting of first 2^{n-1} entries, namely for (odd) $k = 1$ to $k = 2^n - 1$, i.e. those with at most n (base 2) digits, with B_n consisting of entries with k from $2^n + 1$ to $2^{n+1} - 1$, i.e., with $n + 1$ digits and with C_n consisting of entries with k from $2^{n+1} + 1$ to $3 \cdot 2^n - 3$, i.e. with those entries in X_n with k of $n + 2$ digits. We write $X'_n = (A'_n, B'_n, C'_n)$ in the obvious fashion.

Then as we have already proved in Section 3, we have

(i) **Initial value** $X_1 = [2^{d-1} - \lfloor (d+1)/2 \rfloor, 2^d + (-1)^{d-1} - 3\lfloor d/2 \rfloor]$,

Here is the *conjectural recipe* for the rest:

(ii) **The sequence** $w_n = w_{n,d}$ For d odd, put

$$w_1 = 2^d - (5d - 1)/2, \quad w_{n+1} = 2w_n - (d-1)/2,$$

so that $w_{n,1} = 0$, put $w_{n,2} = 1$, and for $d > 2$ even, put

$$w_1 = 2^d + 6 - 5d/2, \quad w_{2n} = 2w_{2n-1} - d/2 + 5, \quad w_{2n+1} = 2w_{2n} - 4 - d/2.$$

We now give conjectural description of the value of

$$t_k := t_{k,n,d} := v_d(k + 3 \cdot 2^n) - v_d(k)$$

depending on whether k belongs to A_n, B_n, C_n, respectively:

(iii) **Description of A_n to A'_n transition**: $t_k = v_{d-i}(2^{n+2}-1) - v_{d-i}(2^n-1)$ when k has $n-i$ ones, with $0 \le i \le d-3$, otherwise $t_k = 0$.

In other words, add $v_{d-i}(2^{n+2}-1) - v_{d-i}(2^n-1)$ (which is $3 \cdot 2^n(2^{d-i-2} - \lfloor (d-i)/2 \rfloor)$ by conjecture above) to the entry in A_n with k having $n-i$ ones, with $0 \le i \le d-3$, to get the corresponding entry in A_n. All other entries remain unchanged.

(iv) **Description of B_n to B'_n transition**: Let us temporarily write $f(n,d) := v_d(5 \cdot 2^n - 1) - v_d(2^{n+1}-1)$.

Then $t_{k,n,d} = -f(n, d-i-1)$ and 0, respectively, if the base 2 expansion of k has $n-i$ number of ones (there is exactly one term with $i=-1$, otherwise $i \ge 0$), with $i < d-4$ and $i > d-4$, respectively. The special case $i = d-4$ has $t_k < 0$ and is described later.

(v) **Description of C_n to C'_n transition**: Put $g(n,i) := w_{n+1,i} - w_{n,i}$, when $i \ge 0$ and 0 otherwise, and $r(n,i) := (-1)^{n+i-1} \cdot 3 \cdot 2^{n-i}$, when $1 \le i \le n-1$. Note $g(n,i) = 0$ for $i \le 3$.

Let k have $n-m$ number of ones in its base-2 expansion, so that $0 \le m \le n-2$. If $m \ge d-4$, then $t_k = 0$. So fix $m > d-4$. If d and m have the same parity, then $t_k = g(n, d-(m+1))$. If d and m have opposite parity, list such k's (there are $\binom{n-1}{n-m-2} = \binom{n-1}{m+1}$ of these, as out of the $n+2$ digits, first two are $1, 0$ and the last is 1) in increasing order and add to corresponding entries the amounts

$$g(n, d-(m+1)) + r(n, i_1) + r(n, i_2) + \cdots + r(n, i_{m+1}),$$

with $1 \le i_1 < \cdots < i_{m+1} \le n-1$, in lexicographic order, where smaller i (i.e., larger absolute value of $r(n,i)$) comes first.

9.1 Remarks on consequences

1. In the A_n to A'_n and C_n to C'_n transition to $t_k \ge 0$, while in B_n to B'_n transition, $t_k \le 0$.

2. The first $2^{n-d+2} - 1$ entries of X_n are unchanged (i.e., with $t_k = 0$) in X'_n. More generally, those entries in A_n (resp., B_n, C_n) with k having at most $n+2-d$ (resp., $n+3-d$, $n+4-d$) ones in their base-2 expansion remain unchanged in the corresponding places of X'_n. (Some other entries also remain unchanged, so this is not an if and only if condition.)

   The first consequence can be proved from the hypothesis that the maximum of $v_d(k)$ with odd $k \le 2^n-1$ occurs at $k = 2^n-1$ and the conjectural formula of $v_d(2^n-1)$ as well as the congruence noted above. (Another way to see this is to note $1/a^k + 1/a^{k+3\cdot 2^n} = (a^{3\cdot 2^n}+1)/a^{k+3\cdot 2^n}$, for a prime to v, so that $a^3 \equiv 1 \pmod{v}$ implies the sum has valuation at least 2^n.). In more detail, if $k \le 2^{n-d+2} - 1$, then by these hypotheses we have $v_d(k) \le v_d(2^{n-d+2}-1) < 2^n$, so that $v_d(k) = v_d(k + m \cdot 3 \cdot 2^n)$.

3. In (iii) the bound $d-3$ could have been changed to d, since the addition amount for $d-2 \le i \le d$ is zero.

4. In (iv) we also conjecture that $f(n,d)$ is $(3d-6)2^n$ for d even and $(3d-15) \cdot 2^n$ for $d > 3$ odd, while for $d = 3$, it is 2^n for n odd and $2^n - 1$ for n even. Note $f(n,5) = 0$, so that by (iv) more entries are unchanged than listed in the part 2 of these Remarks.

Special case (iv), $i = d - 4$: Since k is odd with $n+1$ digits, we have $3 \leq d \leq n+2$ and there are $\binom{n-1}{d-3}$ such entries in B_n. Let a_n denote $2^n - 1$, if n is even and 2^n, if n is odd.

For $d = 3$ (resp., $d = n+2$), we have a unique such k and we have $t_k = -a_n$ (resp., $t_k = -2^n$).

For $d = 4$, the $n-1$ differences $-t_k - a_n$ are $(-1)^{i-1}(2^{n-i}-1)$ with increasing $1 \leq i \leq n-1$, if n is even and they are $2^{n-1}+1, -2^{n-2}, \cdots, 2^2-1, -2$ if n is odd.

For $d = n+1$, the differences $-t_k - a_n$ are $0, 2^3, -(2^2-1), 2^5, \cdots, 2^{n-1}, -(2^{n-2}-1)$ if n is even and $2^2+1, -2, \cdots, 2^{n-1}+1, -2^{n-2}$ if n is odd.

For $d = 5$, $-t_k - a_n$ are given as follows. Write these differences in $d = 4$ case described above as $c_{n-1}, \cdots, c_1$, then in $d = 5$ case, the first $n-2$ differences are (for k's in the special case written in increasing order) $0, c_{n-1} + c_{n-3}, c_{n-1} + c_{n-2} + c_{n-3} + c_1, c_{n-1} + c_{n-5}, c_{n-1} + c_{n-4} + c_{n-5} + c_1, \cdots$, (thus ending with $c_{n-1} + c_4 + c_3 + c_1, c_{n-1} + c_1$ when n is even and with $c_{n-1} + c_2, c_{n-1} + c_3 + c_2 + c_1$ when n is odd. These are followed by $n-3$ copies of $-a_{n-2}$, followed by the differences at $(n-2, d)$ level (there are $\binom{n-3}{2}$ of them.)

For $d \geq 3$, for k written in increasing order, for the $\binom{n-1}{d-3}$ k's that we consider at the (n, d)-level, the differences $t_{k,n,d} - a_n$ are given by repeating these differences ($\binom{n-3}{d-5}$ of them) at the $(n-2, d-2)$ level, followed by a portion denoted by Q, as we have not been able to guess it yet. It consists of $\binom{n-3}{d-4}$ entries), followed by $-a_{n-2}$ repeated $\binom{n-3}{d-4}$ times, followed by the differences at $(n-2, d)$ level ($\binom{n-3}{d-3}$ of them.

The portion Q is $2^{n-1} + (-1)^{n-1}$ for $d = 4$. For $d = 5$, it is sum of first and third differences at $(n, 4)$ level, followed by $3 \cdot 2^{n-3}+$ top entries ($n-4$ of them) from $(n-2, 5)$ level. The first term of the portion Q in the $d = 6$ case is $2^{n-1} + (-1)^{n-1}$. For $d = n+1$, the portion Q is $2^{n-1}+1$ and 2^{n-1}, according to whether n is odd or even, respectively. More generally, the first (resp., last) entry of the portion Q is $2^{n-1} + (-1)^{n-1}$ (resp., 2^{n-1}) according to whether d is even (resp., d is odd and n is even).

Special cases of low d: We consider the $v_d(k)$ sequence for k odd.

($d = 1$) The pattern is 0, 1, 0, 0, 1, 0, ..., periodic with period 3.

($d = 2$) The pattern is 1, 0, 1, 1, 0, 1, ... , periodic with period 3.

($d = 3$) $w_n = 1$ for $d = 3$, X'_n is obtained from X_n by adding $3 \cdot 2^n$ to the 2^{n-1}-th entry (i.e., $k = 2^n - 1$) and subtracting from 2^n-th entry (i.e., $k = 2^{n+1} - 1$) either 2^n or $2^n - 1$, respectively, as n is odd or even.

So the pattern is 2, 6, 1, 8, 4, 1, 2, 18, 1, 5, 4, 1, 2, 6, 1, 32, 4, 1, 2, 10, $\ldots$ with k being 5, 11, 1, 9 modulo 12 giving the entries 1, 1, 2, 4, respectively.

10 Zero distribution of Goss zeta functions

For a given y, $\zeta(x, y)$ is a power series in x with coefficients in K_∞. Wan [10] noticed and proved using a Newton polygon calculation, using estimates of $s_d(-y)$, that the zeros of $\zeta(x, y)$ are simple and always lie in K_∞, when q is a prime. This was later generalized by Sheats to any q. See [8, Sec. 5.8] and [4, 5] for the references for this development and discussion of higher genus case.

Noting that K_∞ and C_∞ are, respectively, the analogues of the real and complex number fields, this restrictive behavior reminds one of the Riemann hypothesis situation [4, 5, 8].

This looks even more remarkable, if one notices further that algebraic degree of C over R is just 2, whereas in our case, it is infinity.

We now turn to the v-adic case and the zero distribution for the power series $\zeta_v(x,y)$ in x with coefficients in K_v, for a given y. We can ask whether the zeros of $\zeta_v(x,y)$ are in K_v and whether they are simple. Our results can be used to calculate Newton polygons and the zero distribution. For this we can approach the p-adic integers y through the sequence of positive k's, or through negative k, or indeed through any dense subsequence (See [8, 5.8] and [9]). We leave this for a future paper and note here only two simple applications requiring only a few things we have proved.

(I) When degree of v is one, and $q=2$, Wan [10] (see also [5, Prop. 9]) already showed that the zeros are simple and in K_v. For general q, one has the same results for $y \in (q-1)S_v$. (Note that when $q=2$, non-zero is the same as monic, otherwise we restrict to 'even' k to kill the signs.) This can be also derived immediately from Corollary 8, which in fact provides much more precise information to calculate the Newton polygon.

We remark that, when $q=2$, the one unit part of $S_d(k)$ when you substitute $1/t$ for t is $S_{\leq d}(k)$ with t factor removed. So $s_d(k) - k \geq \min v_i(k)$ $(i \leq d)$ with equality if there are no clashes. (But there can be clashes, even for $d=1$, k odd.)

To do the general q case, without restriction of 'evenness', for degree one primes, we need to use information provided by Theorems 7 and 9. We leave this for a future paper.

(II) We now show that when $q=2$, $v=t^2+t+1$, the zeros need not be in K_v.

For the first example, let $k=5$. In fact, we saw that $v_d(5) = 0, 0, 1, 1, 12, 20, \ldots$. We do not need the full pattern. We can say that the first two slopes are 0 and 1/2 because degree of v being 2, we have easy estimate $v_d(k) \geq m$, if $d > 2m$, by the trivial lower bound described above. (In fact, since the valuations are not zero infinitely often, the slope would be at most 1/2 in any case.)

Here is a second example: $q=2$, $v=t^2+t+1$, and $k=3$. Now $v_d(3)$ for $d=0,1,2,\cdots$ is $0,1,0,6,9,27,\ldots$, so that the first two slopes are 0 and 9/2. To see this, trivial lower bounds above are not sufficient, but in this case, we know $v_d(3)$, by Section 3, so a straightforward calculation justifies this.

11 Leading term formulas

It follows [8, Cor. 5.6.2] from Carlitz' work that for $k>0$, $S_d(-k)=0$ if and only if $d > \ell(k)/(q-1)$, for q prime. (See [1] and [9, A.5] for the general situation). Hence, for $k>0$, $S_{\lfloor \ell(k)/(q-1) \rfloor}(-k)$ is the leading term of the Goss zeta series, at least when q is prime, and also for general q, when $\ell(k)$ is the minimum of $\ell(p^i k)$.

Theorem 12. *Let q be any prime power. Let $k>0$ and $\ell(k) = (q-1)d + r$, with $0 \leq r < (q-1)$, so that $d = \lfloor \ell(k)/(q-1) \rfloor$. Write the base q-expansion $k = \sum_1^{d(q-1)+r} q^{k_i}$. Then*

$$S_d(-k) = (-1)^d \sum t^{\sum_{i=1}^{d-1} i \sum_1^{q-1} q^{k_j} + d \sum_1^r q^{k_m}},$$

where the sum is over all assignments to i's of groups of $q-1$ of the powers q^{k_j}'s corresponding to indices in partitions of $d(q-1)+r$ indices into d groups of $q-1$ each and one group of r powers.

Proof. Let us first see the simplest $q = 2$ case. Then $r = 0$ and we have

$$\begin{aligned} S_d(-k) &= \sum_{f_0,\ldots,f_{d-1}\in\mathbb{F}_q} (t^d + f_{d-1}t^{d-1} + \cdots + f_0)^k \\ &= \sum \prod_{i=1}^{d(q-1)} (t^{dq^{k_i}} + f_{d-1}t^{(d-1)q^{k_i}} + \cdots + f_0) \end{aligned}$$

Keep the sum and consider the terms obtained by expanding the product. Any term not containing all f_i's will vanish after summing over that missing f_i (compare proof of [8, Thm. 5.1.2]). So terms that matter are of the form $f_0 \cdots f_{d-1}t^{\sum_1^{d-1} iq^{k_{j_i}}}$, where there is really only one non-zero term corresponding to $f_i = 1$, so that $S_d(-k)$ is exactly the sum of these t powers, over j_i's which are permutations of 1 to d.

Now consider the general q case. Then, as before, we have exactly the same displayed expression, and as before, when we expand the product, all f_i's need to be there and each f_i with minimal power $q-1$ to get the non-zero sum, so only terms that matter have coefficient $(f_0 \cdots f_{d-1})^{q-1}$ (as the relation between d and $\ell(k)$ shows) so that we get the expression as claimed. $\square$

11.1 Example

Let $q = 3, k = 38 = 27 + 9 + 1 + 1$, so that $d = 2$ and our our formula gives $S_d(-k) = t^{27+9} + 2 \cdot t^{27+1} + 2 \cdot t^{9+1} + t^2$.

Corollary 13. *With the notation as in Theorem 12, when $q = 2$, we have a product formula*

$$S_d(-k) = \prod_{d\geq n>m} (t^{2^{k_n}} + t^{2^{k_m}}).$$

More generally, for any q, but for the special family $k = (q-1)\sum_1^d q^{k_i} > 0$ (with k_i distinct) we have the leading term

$$S_d(-k) = (-1)^d \prod_{d\geq n>m} (t^{q^{k_n}} - t^{q^{k_m}})^{q-1}.$$

Proof. Put $T_n = t^{q^{k_n}}$. The product formula follows immediately, when $q = 2$, by simple counting of monomials in $\prod(T_n + T_m)$. For general q, one has to only note, in addition, that $\binom{q-1}{i} = (-1)^i$, so that $(T_n - T_m)^{q-1} = \sum T_n^a T_m^b$, where the sum is over a, b with $a + b = q - 1$. $\square$

11.2 Remarks

1. The product formula in the $q = 2$ case was obtained earlier by Pink using a cohomological formula for the leading power sum. See [1, 7.1] for this, as well as the proof of the Corollary using the Vandermonde determinantal formula combined with cohomological machinery.

2. When $q > 2$, we do not have a product formula involving only monomials in $[n]$'s, in the general case, for the leading term, even if q is a prime. For example, when $q = 3$, $k = 13$, $S_1(-13) = -(t^3 - t)(t^3 - t + 1)(t^3 - t - 1)$. On the other hand, for many families of q, k, we can prove the product expression (for the leading term $S_d(-k)$ as above)

$$c \prod (t^{q^j} - t^{q^i})^{r_{i,j}},$$

where $c \in \mathbb{F}_q$ expressed in terms of multinomial coefficient, product being over $i < j$ such that $k_i + k_j = q - 1 + r_{i,j}$, with $r_{i,j} > 0$, where $k = \sum k_j q^j$ is the base q expansion of k. We leave it to a future paper to investigate the exact scope of when it works, and the cohomological explanation of the prime factors which enter in terms of the p-ranks of the Jacobians (components) of the corresponding cyclotomic extensions.

3. When $q = 2$, the product has $d(d-1)/2$ terms of two terms. When expanded, it has $2^{(d-1)d/2}$ terms, and the sum has $d!$ terms (some can cancel), whereas if we just use the definition there are 2^d terms to be added each consisting of $(d+1)^k$ (or rather $(d+1)^{\ell(k)}$ using $p-$th powers) terms.

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