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Convolution Properties of the Generalized Stirling Numbers and the Jacobi-Stirling Numbers of the First Kind

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Abstract

In this paper, we establish several properties of the unified generalized Stirling numbers of the first kind, and the Jacobi-Stirling numbers of the first kind, by means of the convolution principle of sequences. Obtained results include generalized Vandermonde convolution for the unified generalized Stirling numbers of the first kind, triangular recurrence relation for general Stirling-type numbers of the first kind, and linear recurrence formula for the Jacobi-Stirling numbers of the first kind, and so forth, thereby extending and supplementing known knowledge to the existent literature about these Stirling-type numbers.

1 Introduction

We know that [1], if generating functions of two sequences $\langle a(k)\rangle \triangleq (a(0), a(1), a(2), a(3), \ldots)$ and $\langle b(k)\rangle \triangleq (b(0), b(1), b(2), a(3), \ldots)$ are $a(x)$ and $b(x)$ respectively, namely,

$$a(x) = \sum_{k=0}^{\infty} a(k)x^k, \quad b(x) = \sum_{k=0}^{\infty} b(k)x^k,$$

then product function $a(x)b(x)$ is generating function of convolution (sequence) $\langle c(k)\rangle$ of the two sequences $\langle a(k)\rangle$ and $\langle b(k)\rangle$, where each term of the sequence $\langle c(k)\rangle$ is calculated by the following formula:

$$c(k) = \sum_{i=0}^{k} a(i)b(k-i) = \sum_{i=0}^{k} a(k-i)b(i), \quad k = 0, 1, 2, 3, \ldots. \tag{1}$$

For convenience, we occasionally denote the convolution operation by symbol "$*$". For example, equality (1) is also expressed as

$$\langle c(k)\rangle = \langle a(k)\rangle * \langle b(k)\rangle = \langle b(k)\rangle * \langle a(k)\rangle.$$

In this paper, we will call this property of sequences *the convolution principle of sequences.*

The generating functions of several well-known sequences in combinatorics have product form. *Unified generalized Stirling numbers of the first kind* and *Jacobi-Stirling numbers of the first kind* are two examples of such sequences.

The unified generalized Stirling numbers, defined first by Hsu and Shuie [2], are the connection coefficients of linear relations between generalized factorial functions. The generalized factorial functions of a real or complex number x with real increment α, denoted by $(x|\alpha)_n$, are special polynomials in x of degree n, as

$$(x|\alpha)_0 = 1, \quad \text{and} \quad (x|\alpha)_n = x(x-\alpha)\cdots(x-n\alpha+\alpha) = \prod_{i=0}^{n-1}(x-i\alpha), \quad n = 1, 2, \ldots. \tag{2}$$

Thus, the unified generalized Stirling numbers with real parameters α, β, γ, denoted by $S(n, k; \alpha, \beta, \gamma)$, $n, k = 0, 1, 2, \ldots$, are defined as (see [2])

$$S(0, k; \alpha, \beta, \gamma) = \delta_{0,k}, \text{ and } (x|\alpha)_n = \sum_{k=0}^{\infty} S(n, k; \alpha, \beta, \gamma)(x-\gamma|\beta)_k, \quad n = 1, 2, \ldots, \tag{3}$$

or

$$(x+\gamma|\alpha)_n = \sum_{k=0}^{\infty} S(n, k; \alpha, \beta, \gamma)(x|\beta)_k, \quad n = 1, 2, 3, \ldots. \tag{4}$$

We see from (4) that for any α, β, γ, when $k > n$, $S(n, k; \alpha, \beta, \gamma) = 0$; and when $k = n$ $S(n, n; \alpha, \beta, \gamma) = 1$. Therefore, the upper limit ∞ of the summation in the right side of equalities (3) and (4) may be replaced by n.

The most popular special cases of $S(n, k; \alpha, \beta, \gamma)$ are the Kronecker delta $\delta_{n,k}$ ($S(n, k; 0, 0, 0)$), the binomial coefficients $\binom{n}{k}$ ($S(n, k; 0, 0, 1)$), and two kinds of the classical Stirling numbers $s(n, k)$ and $S(n, k)$ ($S(n, k; 1, 0, 0)$ and $S(n, k; 0, 1, 0)$).

Taking $S(n, k; \alpha, \beta, \gamma)$ ($n, k = 0, 1, 2, \ldots$) as entries, we may obtain a ∞-dimensional, lower triangular matrix $\mathbf{S}_{\alpha,\beta,\gamma} = \big(S(n, k; \alpha, \beta, \gamma)\big)_{n,k=0.1.2....}$, named *the Generalized Stirling matrix* with parameters α, β, γ [7]. We also name the sequence

$$\langle S(n, k; \alpha, \beta, \gamma)\rangle \triangleq (S(n, 0; \alpha, \beta, \gamma), S(n, 1; \alpha, \beta, \gamma), S(n, 2; \alpha, \beta, \gamma), S(n, 3; \alpha, \beta, \gamma), \ldots)$$

the n-th row sequence of the unified generalized Stirling numbers $S(n,k;\alpha,\beta,\gamma)$.

We call $S(n,k;\alpha,0,\gamma)$ *the unified generalized Stirling numbers of the first kind*. For $S(n,k;\alpha,0,\gamma)$,

$$S(0,k;\alpha,0,\gamma)=\delta_{0,k}, \text{ and } \prod_{i=0}^{n-1}(x+\gamma-i\alpha)=\sum_{k=0}^{\infty}S(n,k;\alpha,0,\gamma)x^k, \quad n=1,2,3,\ldots, \tag{5}$$

which shows that the (horizontal) generating function, $\prod_{i=0}^{n-1}(x+\gamma-i\alpha)$, of the n-th row sequence $\langle S(n,k;\alpha,0,\gamma)\rangle$ has product form.

For an excellent account of the unified generalized Stirling numbers, see [2].

The Jacobi-Stirling numbers of the first kind, $J(n,k;\zeta)$ ($n,k=0,1,2,\ldots$, and $\zeta>-1$ is a fixed constant parameter), are another special case. In this case, the n-th row sequence $\langle J(n,k;\zeta)\rangle$ also has a (horizontal) generating function of product form, such as $\prod_{i=0}^{n-1}(x-i(i+\zeta))$. Thus,

$$\prod_{i=0}^{n-1}(x-i(i+\zeta))=\sum_{k=0}^{\infty}J(n,k;\zeta)x^k, \quad n=1,2,3,\ldots. \tag{6}$$

(Note: $J(0,k;\zeta)=\delta_{0,k}, k=0,1,2,\ldots$). We see from (6) that for any ζ, when $k>n$, $J(n,k;\zeta)=0$; and when $k=n$, $J(n,n;\zeta)=1$. Therefore, the upper limit ∞ of the summation in the right side of equality (6) may be replaced by n. Particularly, we name $J(n,k;1)$ *the Legendre-Stirling numbers of the first kind*.

For the initial definition, elementary properties (explicit expressions, triangular recurrence relations, similarity between the Jacobi-Stirling and classical Stirling numbers, etc.) and different combinatorial interpretations of special cases of the Jacobi-Stirling numbers, see [3, 4, 5, 6, 8], respectively.

Because the unified generalized Stirling numbers of the first kind, and the Jacobi-Stirling numbers of the first kind, both have generating functions of product form, thus it is reasonable to investigate their several properties by means of the convolution principle of sequences. In the following sections, we will present the obtained results, including generalized Vandermonde convolution for the unified generalized Stirling numbers of the first kind, triangular recurrence relation for general Stirling-type numbers of the first kind, and linear recurrence formulae for the Jacobi-Stirling numbers of the first kind, and so forth.

2 Generalized Vandermonde convolution

For the unified generalized Stirling numbers of the first kind, we may obtain the following theorem by means of the convolution principle of sequences.

Theorem 1. *Let r, t and n be three positive integers, and $n=r+t$. Then*

$$\langle S(n,k;\alpha,0,\gamma)\rangle=\langle S(r,k;\alpha,0,\gamma)\rangle*\langle S(t,k;\alpha,0,\gamma-r\alpha)\rangle. \tag{7}$$

namely for $k = 0, 1, 2, 3, \ldots,$

$$S(n, k; \alpha, 0, \gamma) = \sum_{i=0}^{k} S(r, i; \alpha, 0, \gamma) S(t, k-i; \alpha, 0, \gamma - r\alpha). \tag{8}$$

We name this convolution formula the Generalized Vandermonde convolution.

Proof. We see from (5) that the generalized factorial $\prod_{i=0}^{n-1}(x+\gamma-i\alpha)$ is the generating function of sequence $\langle S(n,k;\alpha,0,\gamma)\rangle$. On the other hand, $\prod_{i=0}^{n-1}(x+\gamma-i\alpha)$ is a product of two factorial functions, $\prod_{i=0}^{r-1}(x+\gamma-i\alpha)$ and $\prod_{i=0}^{t-1}(x+\gamma-r\alpha-i\alpha)$, which are generating functions of sequences $\langle S(r,k;\alpha,0,\gamma)\rangle$ and $\langle S(t,k;\alpha,0,\gamma-r\alpha)\rangle$ respectively. Hence, $\prod_{i=0}^{n-1}(x+\gamma-i\alpha)$ is also the generating function of convolution $\langle S(r,k;\alpha,0,\gamma)\rangle * \langle S(t,k;\alpha,0,\gamma-r\alpha)\rangle$. Thus, $\langle S(n,k;\alpha,0,\gamma)\rangle = \langle S(r,k;\alpha,0,\gamma)\rangle * \langle S(t,k;\alpha,0,\gamma-r\alpha)\rangle$. ☐

Remark 2. We know that $S(n,k;0,0,1) = \binom{n}{k}$, $S(r,k;0,0,1) = \binom{r}{k}$ and $S(t,k;0,0,1) = \binom{t}{k}$, (or $S(n,k;0,0,-1) = (-1)^{n-k}\binom{n}{k}$, $S(r,k;0,0,-1) = (-1)^{r-k}\binom{r}{k}$ and $S(t,k;0,0,-1) = (-1)^{t-k}\binom{t}{k}$). In this special case, we may find that (whether $\gamma = 1$ or $\gamma = -1$) formula (8) lead to the classical Vandermonde convolution [1] (also named Vandermonde's identity or Vandermonde formula) as

$$\binom{n}{k} = \sum_{i=0}^{k}\binom{r}{i}\binom{t}{k-i} = \sum_{i=0}^{k}\binom{r}{k-i}\binom{t}{i}, \tag{9}$$

where $n = r + t$.

Remark 3. The most simple case of formula (7) or (8) is $\alpha = \gamma = 0$. In this case, $S(n,k;0,0,0) = \delta_{n,k}$, $S(r,k;0,0,0) = \delta_{r,k}$, and $S(t,k;0,0,0) = \delta_{t,k}$. Thus, we obtain self-convolution property of the kronecker delta, as

$$\langle \delta_{n,k}\rangle = \langle \delta_{r,k}\rangle * \langle \delta_{t,k}\rangle \tag{10}$$

or

$$\delta_{n,k} = \sum_{i=0}^{k}\delta_{r,i}\delta_{t,k-i} = \sum_{i=0}^{k}\delta_{r,k-i}\delta_{t,i} \tag{11}$$

where $n = r + s$.

Remark 4. The unified generalized Stirling number $S(r,k;\alpha,0,\gamma)$ of the first kind is the (r,k)-th entry of the generalized Stirling matrix $\mathbf{S}_{\alpha,0,\gamma}$, and $S(t,k;\alpha,0,\gamma-r\alpha)$ is the (t,k)-th entry of the generalized Stirling matrix $\mathbf{S}_{\alpha,0,\gamma-r\alpha}$. We know from [7, Theorem 7] that, $S(r,k;\alpha,0,\gamma)$ is the scalar product of the r-th row of the matrix $\mathbf{S}_{\alpha,0,0}$ and the k-th column of the matrix $\mathbf{S}_{0,0,\gamma}$; and $S(t,k;\alpha,0,\gamma-r\alpha)$ is the scalar product of the t-th row of the matrix $\mathbf{S}_{\alpha,0,0}$ and the k-th column of the matrix $\mathbf{S}_{0,0,\gamma-r\alpha}$. Hence, $S(r,k;\alpha,0,\gamma)$ and $S(t,k;\alpha,0,\gamma-r\alpha)$ in (8) may be calculated by using the classical Stirling numbers $s(n,k)$ of the first kind, and the binomial coefficients $\binom{n}{k}$ $(n,k=0,1,2,\ldots)$, as that

$$S(r,k;\alpha,0,\gamma)=\sum_{i=k}^{r}\gamma^{i-k}\alpha^{r-i}s(r,i)\binom{i}{k},$$

and

$$S(t,k;\alpha,0,\gamma-r\alpha)=\sum_{i=k}^{s}(\gamma-r\alpha)^{i-k}\alpha^{t-i}s(t,i)\binom{i}{k}.$$

Remark 5. For the classical Stirling numbers $s(n,k)$ of the first kind, namely $S(n,k;1,0,0)$, we have that

$$\langle S(n,k;1,0,0)\rangle=\langle S(r,k;1,0,0)\rangle*\langle S(t,k;1,0,-r)\rangle,$$

or

$$S(n,k;1,0,0)=\sum_{i=0}^{k}S(r,k-i;1,0,0)S(t,i;1,0,-r).$$

According to Remark 4,

$$S(t,i;1,0,-r)=\sum_{j=i}^{t}(-r)^{j-i}s(t,j)\binom{j}{i}.$$

Finally, we may write the convolution as

$$s(n,k)=\sum_{i=0}^{k}\sum_{j=i}^{t}(-r)^{j-i}\binom{j}{i}s(r,k-i)s(t,j),\quad n=r+t.$$

This is just the Vandermonde convolution for the classical Stirling numbers of the first kind.

3 Triangular recurrence relations of the Stirling-type numbers of the first kind

Remark 6. We see from Equation (7) that when $r=n-1$ and $t=1$,

$$\langle S(n,k;\alpha,0,\gamma)\rangle=\langle S(n-1,k;\alpha,0,\gamma)\rangle*\langle S(1,k;\alpha,0,\gamma-(n-1)\alpha)\rangle,$$

namely,

$$\begin{aligned}S(n,k;\alpha,0,\gamma)&=\sum_{i=0}^{k}S(n-1,k-i;\alpha,0,\gamma)S(1,i;\alpha,0,\gamma-(n-1)\alpha)\\&=S(n-1,k;\alpha,0,\gamma)S(1,0;\alpha,0,\gamma-(n-1)\alpha)\\&\quad+S(n-1,k-1;\alpha,0,\gamma)S(1,1;\alpha,0,\gamma-(n-1)\alpha)\\&=(\gamma-(n-1)\alpha)S(n-1,k;\alpha,0,\gamma)+S(n-1,k-1;\alpha,0,\gamma)\end{aligned}\tag{12}$$

This is the triangular recurrence relation of $S(n,k;\alpha,0,\gamma)$ shown in [2, Theorem 1]. Therefore, the triangular recurrence relation of the unified generalized Stirling numbers of the first kind just is a convolution in essence.

We may generalize this conclusion to a more general case of *the Stirling-type numbers of the first kind*, denoted by $S(n,k;\alpha_i,i=0,1,2,\ldots)$ ($\alpha_0,\alpha_1,\alpha_2,\alpha_3,\ldots$ is a given monotonic non-decreasing or non-increasing sequence). The (horizontal) generating function of row-sequence $\langle S(n,k;\alpha_i,i=0,1,2,\ldots)\rangle$ of the Stirling-type numbers of the first kind is $\prod_{i=0}^{n-1}(x-\alpha_i)$, namely, $S(0,k;\alpha_i,i=0,1,2,3,\ldots)=\delta_{0,k}$, and

$$\prod_{i=0}^{n-1}(x-\alpha_i)=\sum_{k=0}^{n}S(n,k;\alpha_i,i=0,1,2,\ldots)x^k,\quad n=1,2,3,\ldots. \tag{13}$$

For the Stirling-type numbers of the first kind, we may obtain corresponding triangular recurrence relations by means of the convolution principle of sequences.

Theorem 7. *Let $S(n,k;\alpha_i,i=0,1,2,\ldots)$ be the Stirling-type numbers of the first kind defined in (13). Then $S(n,k;\alpha_i,i=0,1,2,\ldots)$ satisfies the following triangular recurrence relation, namely, for $n,k=1,2,3,\ldots$,*

$$S(n,k;\alpha_i,i=0,1,2,\ldots)=-\alpha_{n-1}S(n-1,k;\alpha_i,i=0,1,2,\ldots)+S(n-1,k-1;\alpha_i,i=0,1,2,\ldots) \tag{14}$$

Proof. We see from (13) that the generating function of sequence $\langle S(n,k;\alpha_i,i=0,1,2,\ldots)\rangle$ is $\prod_{i=0}^{n-1}(x-\alpha_i)$. On the other hand, $\prod_{i=0}^{n-2}(x-\alpha_i)$ is the generating function of sequence $\langle S(n-1,k;\alpha_i,i=0,1,2,\ldots)\rangle$, and $(x-\alpha_{n-1})$ is the generating function of sequence $(-\alpha_{n-1},1,0,0,0,\ldots)$. Hence, according to the convolution principle of sequences we have that

$$\begin{aligned}
& S(n,k;\alpha_i,i=0,1,2,\ldots)\\
={}& S(n-1,k;\alpha_i,i=0,1,2,\ldots)\cdot(-\alpha_{n-1})+S(n-1,k-1;\alpha_i,i=0,1,2,\ldots)\cdot 1\\
={}& -\alpha_{n-1}S(n-1,k;\alpha_i,i=0,1,2,\ldots)+S(n-1,k-1;\alpha_i,i=0,1,2,\ldots).
\end{aligned}$$

□

This theorem proves that for the most general Stirling-type numbers of the first kind, exists a triangular recurrence relation, and the triangular recurrence relation is a convolution in essence.

4 Convolution of the Jacobi-Stirling numbers of the first kind

The Jacobi-Stirling numbers of the first kind, $J(n,k;\zeta)$ are a special case of the Stirling-type numbers of the first kind, in which α_i corresponds to $i(i+\zeta)$ ($i=0,1,2,\ldots$).

Because for the Jacobi-Stirling numbers of the first kind, $\alpha_{n-1} = (n-1)(n+\zeta-1)$, according to (14), $J(n,k;\zeta)$ satisfy the following triangular recurrence relation (also see [3, 4, 6]):

$$J(n,k;\zeta) = -(n-1)(n+\zeta-1)J(n-1,k;\zeta) + J(n-1,k-1;\zeta). \tag{15}$$

Furthermore, we may establish several other properties of the Jacobi-Stirling numbers of the first kind by means of the convolution principle of sequences, as shown in the subsections following.

4.1 Convolution of the degenerate Jacobi-Stirling numbers of the first kind

We first investigate $J(n,k;0)$. In this paper, we name $J(n,k;0)$ *the degenerate Jacobi-Stirling numbers of the first kind*. In fact, they are just so-called *central factorial numbers of the first kind with even indices* [8].

In this case, the (horizontal) generating function of the n-th row sequence $\langle J(n,k;0)\rangle$ is $\prod_{i=0}^{n-1}(x-i^2)$, that is,

$$J(0,k;0) = \delta_{0,k}, \quad \text{and} \quad \prod_{i=0}^{n-1}(x-i^2) = \sum_{k=0}^{n} J(n,k;0)x^k, \quad n = 1,2,3,\ldots. \tag{16}$$

For the degenerate Jacobi-Stirling numbers of the first kind, we may obtain the following property by means of the convolution principle of sequences.

Lemma 8. *Let n be a given positive integer, and $\langle J(n,k;0)\rangle$ be the n-th row sequence of the degenerate Jacobi-Stirling numbers of the first kind, whose generating function is shown in (16). Then defining a sequence $\langle \bar{J}(n,k)\rangle$ derived from $\langle J(n,k;0)\rangle$ as*

$$\langle \bar{J}(n,k)\rangle \triangleq (J(n,0;0), 0, J(n,1;0), 0, J(n,2;0), 0, \ldots),$$

we have that

$$\langle \bar{J}(n,k)\rangle = \langle s(n,k)\rangle * \langle (-1)^{n-k}s(n,k)\rangle \tag{17}$$

where $s(n,k)$ are the classical Stirling numbers of the first kind.

Proof. Replacing x in (16) by y^2, we have that

$$\prod_{i=0}^{n-1}(y^2-i^2) = \prod_{i=0}^{n-1}(y-i)\prod_{i=0}^{n-1}(y+i) = \sum_{k=0}^{n} J(n,k;0)y^{2k} = \sum_{k=0}^{2n} \bar{J}(n,k)y^k$$

We know that $\prod_{i=0}^{n-1}(y-i)$ and $\prod_{i=0}^{n-1}(y+i)$ both are the (horizontal) generating functions of two sequences $\langle s(n,k)\rangle$ and $\langle (-1)^{n-k}s(n,k)\rangle$, respectively, Hence according to the convolution principle of sequences, formula (17) holds. $\square$

Theorem 9. *The degenerate Jacobi-Stirling numbers of the first kind, $J(n,k;0)$ may be calculated by the classical Stirling numbers $s(n,k)$ of the first kind, as follows,*

$$J(n,k;0)=\sum_{i=0}^{2k}(-1)^{n-i}s(n,i)s(n,2k-i),\quad n,k=0,1,2,\ldots. \tag{18}$$

Proof. According to (17), we have that

$$\bar{J}(n,k)=\sum_{i=0}^{k}(-1)^{n-i}s(n,i)s(n,k-i).$$

Because $J(n,k;0)=\bar{J}(n,2k)$, thus formula (18) holds. $\square$

Example 10. For example,

$$J(4,2;0)=\sum_{i=0}^{4}(-1)^{4-i}s(4,i)s(4,4-i)=49,$$

and

$$J(5,2;0)=\sum_{i=0}^{4}(-1)^{5-i}s(5,i)s(5,4-i)=-820,$$

and so forth.

Remark 11. Because $\bar{J}(n,2k+1)\equiv 0$ $(k=0,1,2,\ldots)$, from (17) we have the following identity:

$$\sum_{i=0}^{2k+1}(-1)^{n-i}s(n,i)s(n,2k+1-i)=0,\quad (k=0,1,2,\ldots). \tag{19}$$

In fact, this is a trivial identity, for its first $k+1$ terms corresponding to $i=0$, $i=1$, ..., $i=k$ are the contrary numbers of the rest $k+1$ terms corresponding to $i=2k+1$, $i=2k$, ..., $i=k+1$, respectively.

4.2 Linear recurrence formula of the Legendre-Stirling numbers of the first kind

The Jacobi-Stirling numbers of the first kind with $\zeta=1$, $J(n,k;1)$ also are named *the Legendre-Stirling numbers of the first kind* [3]. For $J(n,k;1)$, we may obtain a non-homogeneous linear recurrence relation by means of the convolution principle of sequences.

Theorem 12. *Let n be a given non-negative integer. Then the n-th row sequence, $\langle J(n,k;1)\rangle$, of the Legendre-Stirling numbers of the first kind satisfies the following non-homogeneous linear recurrence formulae:*

$$J(n,0;1)=\delta_{n,0},\quad J(n,1;1)=\sum_{i=0}^{n}(-1)^{n-i}s(n,i)s(n,1), \tag{20}$$

and for $k = 2, 3, \ldots, n$,

$$J(n,k;1) = -\sum_{i=[\frac{k+1}{2}]}^{k-1} \binom{i}{k-i} J(n,i;1) + \sum_{i=0}^{k}\sum_{j=i}^{n} (-1)^{n-j}\binom{j}{i} s(n,j)s(n,k-i), \tag{21}$$

where $[\cdot]$ *is the floor function, and* $s(n,k)$*s are the classical Stirling numbers of the first king.*

Proof. We see from (6) that for the Legendre-Stirling numbers $J(n,k;1)$ of the first kind,

$$\prod_{i=0}^{n-1}(x - i(i+1)) = \sum_{k=0}^{n} J(n,k;1)x^k. \tag{22}$$

Thus, $J(n,0;1) = 0$ for $n = 1, 2, \ldots$. Besides, we know $J(0,0;1) = 1$. Hence, $J(n,0;1) = \delta_{n,0}$. We note that $y(y+1) - i(i+1) = (y-i)(y+(i+1))$. Hence, replacing x in (22) with $y(y+1)$ we may express the left side on (22) as product of two factorial functions in y, $\prod_{i=0}^{n-1}(y-i)$ and $\prod_{i=0}^{n-1}(y+1+i)$, which are the (horizontal) generating functions in y of sequences $\langle s(n,k)\rangle$ and $\langle S(n,k;-1,0,1)\rangle$ respectively. Because according to Remark 4, $S(n,k;-1,0,1) = \sum_{i=k}^{n}(-1)^{n-i}\binom{i}{k}s(n,i)$, by means of the convolution principle of sequences, we may express the left side on (22) as

$$\sum_{k=0}^{n}\{\sum_{j=0}^{k}\sum_{i=j}^{n}(-1)^{n-i}\binom{i}{j}s(n,k-j)s(n,i)\}y^k$$

On the other hand, now we may express the right side on (22) as $\sum_{j=0}^{n} J(n,j;1)y^j(y+1)^j$. In the latter, coefficients of the terms with monomial y^k are respectively $J(n,k;1)\binom{k}{0}$, $J(n,k-1;1)\binom{k-1}{1}$, $J(n,k-2;1)\binom{k-2}{2}$, $\ldots$, $J(n,k-[\frac{k}{2}];1)\binom{k-[\frac{k}{2}]}{[\frac{k}{2}]}$. Hence, noting $k = [\frac{k}{2}] + [\frac{k+1}{2}]$ we also may express the right side as

$$\sum_{k=0}^{n}\{\sum_{i=[\frac{k+1}{2}]}^{k}\binom{i}{k-i}J(n,i;1)\}y^k.$$

Because y is arbitrary, by comparison of coefficients on both sides we obtain that

$$\sum_{j=0}^{k}\sum_{i=j}^{n}(-1)^{n-i}\binom{i}{j}s(n,k-j)s(n,i) = \sum_{i=[\frac{k+1}{2}]}^{k}\binom{i}{k-i}J(n,i;1).$$

Hence, (20) and (21) both hold. □

Example 13. Substituting $0, 720, -1764, 1624, -735, 175, -21, 1$ for the Stirling numbers of the first kind, $s(7,0)$, $s(7,1)$, $s(7,2)$, $\ldots$, $s(7,7)$ respectively , we find the Legendre-Stirling numbers of the first kind, $J(7,0;1)=0$, $J(7,k;1)$ $(k=1,2,\ldots,6)$, and $J(7,7;1)=1$ by using formulae (20) and (21), where $J(7,k;1)$ $(k=1,2,\ldots,6)$ are listed as follows,

$$J(7,1;1)=\sum_{j=1}^{7}(-1)^{7-j}s(7,j)s(7,1)=3628800$$

$$J(7,2;1)=-J(7,1;1)+\sum_{i=0}^{2}\sum_{j=i}^{7}(-1)^{7-j}\binom{j}{i}s(7,j)s(7,2-i)=-3110400,$$

$$J(7,3;1)=-2J(7,2;1)+\sum_{i=0}^{3}\sum_{j=i}^{7}(-1)^{7-j}\binom{j}{i}s(7,j)s(7,3-i)=808848,$$

$$J(7,4;1)=-3J(7,3;1)-J(7,2;1)+\sum_{i=0}^{4}\sum_{j=i}^{7}(-1)^{7-j}\binom{j}{i}s(7,j)s(7,4-i)=-89280,$$

$$J(7,5;1)=-4J(7,4;1)-3J(7,3;1)+\sum_{i=0}^{5}\sum_{j=i}^{7}(-1)^{7-j}\binom{j}{i}s(7,j)s(7,5-i)=4648,$$

$$J(7,6;1)=-5J(7,5;1)-6J(7,4;1)-J(7,3;1)+\sum_{i=0}^{6}\sum_{j=i}^{7}(-1)^{7-j}\binom{j}{i}s(7,j)s(7,6-i)=-112.$$

(see sequence A191936 in [9], and also [3, Table 2]).

4.3 Linear recurrence formula of the Jacobi-Stirling numbers of the first kind

For general cases of the Jacobi-Stirling numbers $J(n,k;\zeta)$ of the first kind, we may obtain a similar linear recurrence relation for its n-th row sequence $\langle J(n,k;\zeta)\rangle$, by means of the convolution principle of sequences.

Theorem 14. *Let n be a given non-negative integer, and ζ (>-1) be a real number. Then the n-th row sequence $\langle J(n,k;\zeta)\rangle$ of the Jacobi-Stirling numbers of the first kind satisfies the following non-homogeneous linear recurrence formulae:*

$$J(n,0;\zeta)=\delta_{n,0},\quad J(n,1;\zeta)=\sum_{i=0}^{n}(-1)^{n-i}\zeta^{i-1}s(n,i)s(n,1), \tag{23}$$

and for $k=2,3,\ldots,n$,

$$\zeta^k J(n,k;\zeta)=-\sum_{i=[\frac{k+1}{2}]}^{k-1}\zeta^{2i-k}\binom{i}{k-i}J(n,i;\zeta)+\sum_{i=0}^{k}\sum_{j=i}^{n}(-1)^{n-j}\zeta^{j-i}\binom{j}{i}s(n,j)s(n,k-i), \tag{24}$$

where $[\cdot]$ is the floor function, and $s(n,k)$s are the classical Stirling numbers of the first kind.

Proof. We see from (6) that $J(n,0;\zeta)=0$ for $n=1,2,$. Besides, we know $J(0,0;\zeta)=1$. Hence, $J(n,0;\zeta)=\delta_{n,0}$. We note that $y(y+\zeta)-i(i+\zeta)=(y-i)(y+(i+\zeta))$. Hence, replacing x in (6) by $y(y+\zeta)$ we may express the left side on (6) as a product of two factorial functions in y, $\prod_{i=0}^{n-1}(y-i)$ and $\prod_{i=0}^{n-1}(y+\zeta+i)$, which are the (horizontal) generating functions in y of sequences, $\langle s(n,k)\rangle$ and $\langle S(n,k;-1,0,\zeta)\rangle$, respectively. Noting that $S(n,k;-1,0,\zeta)=\sum_{i=k}^{n}(-1)^{n-i}\zeta^{i-k}\binom{i}{k}s(n,i)$, by means of the convolution principle of sequences, we may express the left side on (6) as

$$\sum_{k=0}^{n}\{\sum_{j=0}^{k}\sum_{i=j}^{n}(-1)^{n-i}\zeta^{i-j}\binom{i}{j}s(n,k-j)s(n,i)\}y^k.$$

On the other hand, we may express the right side of (6) as

$$\sum_{j=0}^{n}J(n,j;\zeta)x^j=\sum_{j=0}^{n}J(n,j;\zeta)y^j(y+\zeta)^j.$$

In the latter, coefficients of the terms with monomial y^k are respectively $\zeta^k\binom{k}{0}J(n,k;\zeta)$, $\zeta^{k-2}\binom{k-1}{1}J(n,k-1;\zeta)$, $\zeta^{k-4}\binom{k-2}{2}J(n,k-2;\zeta)$, ..., $\zeta^{k-2[\frac{k}{2}]}\binom{k-[\frac{k}{2}]}{[\frac{k}{2}]}J(n,k-[\frac{k}{2}];\zeta)$. Therefore, noting $k=[\frac{k}{2}]+[\frac{k+1}{2}]$, we may rewrite the sum on the right side as

$$\sum_{k=0}^{n}\{\sum_{i=[\frac{k+1}{2}]}^{k}\zeta^{2i-k}\binom{i}{k-i}J(n,i;\zeta)\}y^k.$$

Because y is arbitrary, by comparison of coefficients on both sides we obtain that

$$\sum_{j=0}^{k}\sum_{i=j}^{n}(-1)^{n-i}\zeta^{i-j}\binom{i}{j}s(n,k-j)s(n,i)=\sum_{i=[\frac{k+1}{2}]}^{k}\zeta^{2i-k}\binom{i}{k-i}J(n,i;\zeta).$$

Hence, (23) and (24) both hold. □

Example 15. Substituting $0,24,-50,35,-10,1$ for the row sequence of the classical Stirling numbers of the first kind, $(s(5,0),s(5,1),\ldots,s(5,5))$, we may obtain the row sequence of the Jacobi-Stirling numbers of the first kind, $(J(5,0;\zeta)=0,J(5,1;\zeta),\ldots,J(5,4;\zeta),J(5,5;\zeta)=1)$ according to (23) and (24). $J(5,1;\zeta)$, ..., $J(5,4;\zeta)$ are listed as follows.

$$J(5,1;\zeta)=\sum_{i=0}^{n}(-1)^{5-i}\zeta^{i-1}s(5,i)s(5,1),$$

$$\zeta^2J(5,2;\zeta)=-J(5,1;\zeta)+\sum_{i=0}^{2}\sum_{j=i}^{5}(-1)^{5-j}\zeta^{j-i}\binom{j}{i}s(5,j)s(5,2-i).$$

$$\zeta^3 J(5,3;\zeta) = -2\zeta J(5,2;\zeta) + \sum_{i=0}^{3}\sum_{j=i}^{5}(-1)^{5-j}\zeta^{j-i}\binom{j}{i}s(5,j)s(5,3-i).$$

$$\zeta^4 J(5,4;\zeta) = -3\zeta^2 J(5,3;\zeta) - J(5,2;\zeta) + \sum_{i=0}^{4}\sum_{j=i}^{5}(-1)^{5-j}\zeta^{j-i}\binom{j}{i}s(5,j)s(5,4-i),$$

which then lead to that

$$J(5,1;\zeta) = 576 + 1200\zeta + 840\zeta^2 + 240\zeta^3 + 24\zeta^4,$$

$$J(5,2;\zeta) = -(820 + 1030\zeta + 404\zeta^2 + 50\zeta^3),$$
$$J(5,3;\zeta) = 273 + 200\zeta + 35\zeta^2,$$
$$J(5,4;\zeta) = -(30 + 10\zeta).$$

(see [8, Table 2]).

Remark 16. We may find that the linear recurrence formulae (23) and (24) also verify [8, Theorem 1], that is, $J(n,k;\zeta)$ is a polynomial in ζ of degree $n-k$, the coefficient of the first term with ζ^{n-k} is $s(n,k)$, and the last terms (constant term) is the central factorial numbers $u(n,k)$ of the first kind with even indices, which is identical to $J(n,k;0)$. By the way, we may see that the sum of coefficients of the polynomial is $J(n,k;1)$.

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