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A Generalization of the Catalan Numbers

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Abstract

In this paper, we generalize the Catalan number C_n to the (m,n)th Catalan number $C(m,n)$ using a combinatorial description, as follows: the number of paths in $\mathbb{R}^m$ from the origin to the point $\big(\underbrace{n,\ldots,n}_{m-1},(m-1)n\big)$ with m kinds of moves such that the path never rises above the hyperplane $x_m = x_1 + \cdots + x_{m-1}$.

1 Introduction

Catalan numbers (A000108) are a very prominent sequence of numbers that arises in a wide varity of combinatorial situations [1, 2]. Stanley [10] gave a list of 66 different combinatorial descriptions of Catalan numbers and he added to the list some more [11]. Some of the specific instances are

- The number of movements in xy-plane from $(0,0)$ to (n,n) with two kinds of moves
$$R:(x,y)\to(x+1,y),\quad U:(x,y)\to(x,y+1),$$
such that the path never rises above the line $y=x$.
- Triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.

- Binary parenthesizations of a string of $n+1$ letters.
- Binary trees with n vertices.

The solution to these problems is the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

and the sequence $C_0, C_1, C_2, \ldots, C_n, \ldots$ is called the Catalan sequence.

There have been many attempts to generalize the Catalan numbers. Probably the most important generalization consists of the k-ary numbers or k-Catalan numbers, defined by

$$C_n^k = \frac{1}{kn+1}\binom{kn+1}{n} = \frac{1}{(k-1)n+1}\binom{kn}{n} = \frac{1}{n}\binom{kn}{n-1},$$

where $k, n \in \mathbb{N}$. Clearly, $C_n^2 = C_n$. The k-good paths (below the line $y = kx$) from $(0,-1)$ to $\big(n,(k-1)n-1\big)$, staircase tilings and k-ary trees are structures known to be enumerated by k-ary numbers [5, 6, 8, 10]. Moreover, Kim [7, Thm. 2] showed that C_n^k is the number of partitions of $n(k-1)+2$ polygon by $(k+1)$-gon where all vertices of all $(k+1)$-gon lie on the vertices of $n(k-1)+2$ polygon. Gould [3] developed a generalization that has both the Catalan numbers and the k-Catalan numbers as special cases, defined as

$$A_n(a,b) = \frac{a}{a+bn}\binom{a+bn}{n},$$

and showed the following convolution formula for these sequences:

$$\sum_{k=0}^{n} A_k(a,b)A_{n-k}(c,b) = A_n(a+c,b).$$

These numbers are also known as the Rothe numbers [9] and Rothe-Hagen coefficients of the first type [4]. Clearly, $A_n(1,2) = C_n$ and $A_n(1,k) = C_n^k$.

We know that one of the interpretations of the Catalan numbers is the movements in $\mathbb{R}^2$ with two kinds of moves such that the path never rises above the line $y = x$. In this paper, a new generalization of the Catalan numbers using this interpretation is introduced. Consider m kinds of moves in $\mathbb{R}^m$ such that they are one unit parallel to the positive axes. We show that the number of paths from the origin to the point $\big(\underbrace{n,\ldots,n}_{m-1},(m-1)n\big)$ using these moves such that the path never rises above the hyperplane $x_m = x_1 + \cdots + x_{m-1}$ is

$$\frac{1}{n(m-1)+1}\binom{2n(m-1)}{\underbrace{n,\ldots,n}_{m-1},\,n(m-1)}.$$

We call this number the (m,n)th Catalan number $C(m,n)$. Clearly, $C(2,n)$ is the ordinary nth Catalan number C_n.

2 Generalization

In this section, we prove the our main theorem. We show that the generalized Catalan numbers $C(m,n)$ are given by

$$\frac{1}{n(m-1)+1}\binom{2n(m-1)}{\underbrace{n,\ldots,n}_{m-1},n(m-1)}.$$

Theorem 1. *Let $\mathbb{R}^m$ be the m-dimensional vector space. Consider*

$$\begin{aligned} R_1 &: (x_1,x_2,\ldots,x_m) \longrightarrow (x_1+1,x_2,\ldots,x_m),\\ R_2 &: (x_1,x_2,\ldots,x_m) \longrightarrow (x_1,x_2+1,\ldots,x_m),\\ &\vdots\\ R_m &: (x_1,x_2,\ldots,x_m) \longrightarrow (x_1,x_2,\ldots,x_m+1), \end{aligned}$$

be m kinds of moves in $\mathbb{R}^m$ (i.e., R_i denotes the move one unit parallel to the x_i-axis in the positive direction). Then the number of paths from $\boldsymbol{0} = (\underbrace{0,\ldots,0}_{m})$ to the point $N := (\underbrace{n,\ldots,n}_{m-1},(m-1)n)$ using the moves $R_1,R_2,\ldots,R_m$ such that the path never rises above the hyperplane $x_m = x_1+\cdots+x_{m-1}$ is

$$\frac{1}{n(m-1)+1}\binom{2n(m-1)}{\underbrace{n,\ldots,n}_{m-1},n(m-1)}.$$

Proof. We call a path from $\mathbf{0}$ to N of n R_1's, n R_2's, …, n R_{m-1}'s, and $(m-1)n$ R_m's acceptable if the path never rises above the hyperplane $x_m = x_1+\cdots+x_{m-1}$ and unacceptable otherwise. Let A_n^m and U_n^m denote the number of acceptable and unacceptable paths, respectively. It is easy to see that each path from $\mathbf{0}$ to N corresponds to an arrangement of n R_1's, n R_2's, …, n R_{m-1}'s, and $(m-1)n$ R_m's. Then

$$A_n^m + U_n^m = \frac{\big(2n(m-1)\big)!}{n!^{m-1}\big(n(m-1)\big)!}.$$

Now, consider an unacceptable path and its arrangement $r_1,r_2,\ldots,r_{2n(m-1)}$, where $r_i \in \{R_1,R_2,\ldots,R_m\}$ indicates the ith step of the path. Since the path rises above the hyperplane, there is a first t such that the number of R_m's in $r_1,\ldots,r_t$ exceeds the sum of the numbers $R_1,R_2,\ldots,R_{m-1}$. Moreover, $r_t = R_m$. We only change $r_{t+1},\ldots,r_{2n(m-1)}$ the part of the path after the crossing in the arrangement as follows: Mark all the positions of the R_m's in that part of the path and fill those positions with the sequence (in order) consisting of all but the last of the non-R_m's. Then replace those non-R_m's that have been used in the replacement with R_m's. Here is an example: let $m=3$, $n=2$ and the path be given by $R_1R_3R_2R_3R_3R_1R_2R_3$. Then $t=5$, and the part of the path to be modified is $R_1R_2R_3$. There is just one position of the R_m's, so we replace R_3 with R_1, and then fill the R_1R_2 with R_3R_3 to obtain the modified sequence $R_1R_3R_2R_3R_3R_3R_3R_1$. The resulting arrangement $r_1^{'}, r_2^{'}, \ldots, r_{2n(m-1)}^{'}$ is an arrangement of $(m-1)n+1$ R_m's, n R_1's, …, n R_{i-1}'s, n R_{i+1}'s, …, n R_{m-1}'s, and $n-1$ R_i's for a $1 \le i \le m-1$. It is not difficult to see that this process is reversible:

$$\begin{array}{c}
r_1, r_2, \ldots, r_{2n(m-1)} \\
\Updownarrow \\
\underbrace{*, *, \ldots, *, R_m}_{\substack{rR_m\text{'s},\\ r-1R_1,\ldots,R_{m-1}\text{'s.}}} \quad | \quad \underbrace{*, *, \ldots, *}_{\substack{(m-1)n-rR_m\text{'s},\\ (m-1)n-r+1R_1,\ldots,R_{m-1}\text{'s.}}} \\
\Updownarrow \\
\underbrace{*, *, \ldots, *, R_m}_{\substack{rR_m\text{'s},\\ r-1R_1,\ldots,R_{m-1}\text{'s.}}} \quad | \quad \underbrace{*, *, \ldots, *}_{\substack{(m-1)n-r+1R_m\text{'s},\\ (m-1)n-rR_1,\ldots,R_{m-1}\text{'s.}}} \\
\Updownarrow \\
r_1^{'}, r_2^{'}, \ldots, r_{2n(m-1)}^{'}
\end{array}$$

Hence, there are as many unacceptable arrangements as there are arrangements of $(m-1)n+1$ R_m's, n R_1's, …, n R_{i-1}'s, n R_{i+1}'s, …, n R_{m-1}'s, and $n-1$ R_i's for a $1 \le i \le m-1$. Then

$$U_n^m = (m-1)\frac{\big(2n(m-1)\big)!}{n!^{m-2}(n-1)!\big(n(m-1)+1\big)!}.$$

So,

$$\begin{aligned}
A_n^m =& \frac{\big(2n(m-1)\big)!}{n!^{m-1}\big(n(m-1)\big)!} - (m-1)\frac{\big(2n(m-1)\big)!}{n!^{m-2}(n-1)!\big(n(m-1)+1\big)!} \\
=& \frac{\big(2n(m-1)\big)!}{n!^{m-2}(n-1)!\big(n(m-1)\big)!}\Big(\frac{1}{n} - \frac{m-1}{n(m-1)+1}\Big) \\
=& \frac{\big(2n(m-1)\big)!}{n!^{m-1}\big(n(m-1)+1\big)!} \\
=& \frac{1}{n(m-1)+1}\binom{2n(m-1)}{\underbrace{n, \ldots, n}_{m-1}, n(m-1)}.
\end{aligned}$$

□

We denote A_n^m in the above proof by $C(m,n)$. The first few generalized Catalan numbers are evaluated to be

$n\backslash m$	3	4	5
0	1	1	1
1	4	30	336
2	84	11880	3603600
3	2640	8168160	76881235200
4	100100	7207615800	2229760743210000
5	4232592	7336632122820	77015151194691790080
6	192203088	8193001579963200	2978057806800232994982144
7	9178678080	9763825599821779200	124625746332992720112321024000
8	455053212900	12209602888667136003480	5529032167369807343550830945418000

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