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A Multidimensional Continued Fraction Generalization of Stern's Diatomic Sequence

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Abstract

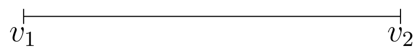
Continued fractions are linked to Stern's diatomic sequence $0, 1, 1, 2, 1, 3, 2, 3, 1, 4, \ldots$ (given by the recursion relations $\alpha_{2n} = \alpha_n$ and $\alpha_{2n+1} = \alpha_n + \alpha_{n+1}$, where $\alpha_0 = 0$ and $\alpha_1 = 1$), which has long been known. Using a particular multidimensional continued fraction algorithm (the Farey algorithm), we generalize the diatomic sequence to a sequence of numbers that quite naturally can be termed Stern's triatomic sequence (or a two-dimensional Pascal sequence with memory). As both continued fractions and the diatomic sequence can be thought of as coming from a systematic subdivision of the unit interval, this new triatomic sequence arises by a systematic subdivision of a triangle. We discuss some of the algebraic properties of the triatomic sequence.

1 The classical Stern diatomic sequence

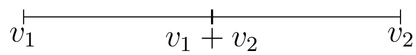
The goal of this paper is to take a particular generalization of continued fractions (the Farey algorithm) and find the corresponding generalization of Stern's diatomic sequence, which we will call Stern's triatomic sequence. In this first section, though, we will quickly review the basics of Stern's diatomic sequence (sequence A002487 in Sloane's *Encyclopedia*). Hence experts should skip to the next section. In Section 2 we give a formal presentation of Stern's triatomic sequence. Section 3 shows how Stern's triatomic sequence can be derived from a systematic subdivision of a triangle. Section 4 shows how Stern's triatomic sequence arises from the multidimensional continued fraction stemming from the Farey map, in analog to the link between Stern's diatomic sequence and classical continued fractions. Sections 5 and 6 discuss some of the properties for this sequence. Many of these properties are analogs of corresponding properties for Stern's diatomic sequence. We will conclude with open questions.

A recent *Monthly* article by Northshield [34] gives a good overview of Stern's diatomic sequence and, more importantly for this paper, its relationship to continued fractions. In much earlier work, Lehmer [29] lists a number of properties of this sequence. In part, our eventual goal is showing how these properties usually generalize for our new Stern's triatomic sequences.

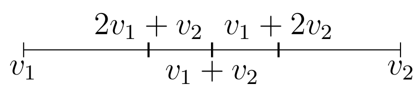
Consider an interval, with endpoints weighted by numbers v_1 and v_2:



Then subdivide:



We can continue, getting:



The traditional Stern diatomic sequence is doing this for the case when $v_1 = v_2 = 1$. If instead we start with vectors

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and do the standard identification of

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \frac{x}{y},$$

we get the Farey decomposition of the unit interval, which in turn can be used to recover the continued fraction expansion of any real number on the unit interval. The case of letting $v_1 = v_2 = 1$ is equivalent to keeping track of the denominators for the Farey decomposition. (We mention this as motivation for why we later link our new Stern's triatomic sequence to a particular multidimensional continued fraction algorithm.)

Thus Stern's diatomic sequence is given by the recursion formulas

$$\alpha_0 = 0, \alpha_1 = 1$$

and

$$\begin{aligned} \alpha_{2n} &= \alpha_n \\ \alpha_{2n+1} &= \alpha_n + \alpha_{n+1} \end{aligned}$$

The first few terms are

$$0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 4, \ldots.$$

It is standard to write this as

$$\begin{array}{ccccccccc}
\alpha_1 & & & & & & & & \alpha_2 \\
\alpha_2 & & & & \alpha_3 & & & & \alpha_4 \\
\alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 & & \alpha_8 \\
\alpha_8 & \alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16}
\end{array}$$

For $\alpha_1 = 1$, we get

$$\begin{array}{ccccccccccccccccc}
1 & & & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & 2 & & & & & & & & 1 \\
1 & & & & 3 & & & & 2 & & & & 3 & & & & 1 \\
1 & & 4 & & 3 & & 5 & & 2 & & 5 & & 3 & & 4 & & 1 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1
\end{array}$$

Thus to create the next level, we simply interlace the original level with the sum of adjacent entries. This is why this sequence is called by Knauf "Pascal with memory" by Knauf [23]. (It was Knauf's work on linking this sequence with statistical mechanics, which is also in [22, 24, 25, 26, 27], that led the author to look at these sequences. Other work linking statistical mechanics with Stern diatomic sequences, though these works do not use the rhetoric of Stern diatomic sequences, include [5, 10, 12, 13, 14, 16, 20, 22, 31, 32, 33, 36, 37, 38]).

A matrix approach for Stern's diatomic sequence $\alpha_1, \alpha_2, \ldots$ is as follows. Set

$$b_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $N > 2$ be a positive integer. Then there is a tuple $(i_k, \ldots, i_1)_S$ with k a positive integer and each $i_j \in \{0, 1\}$ such that

$$N = 2^k + 1 + i_k 2^{k-1} + \cdots + i_1.$$

(The subscript S stands for 'Stern'.) For example $3 = 2^1 + 1 + 0 \cdot 1 = (0)_S, 4 = 2 + 1 + 1 \cdot 1 = (1)_S, 5 = 2^2 + 1 + +0 \cdot 2^1 + 0 \cdot 1 = (0,0)_S,\ 6 = 2^2 + 1 + 0 \cdot 2 + 1 \cdot 1 = (0,1)_S$, $7 = 2^2 + 1 + 1 \cdot 2 + 0 \cdot 1 = (1,0)_S$, $8 = (1,1)_S$ and $16 = 2^3 + 1 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 1 = (1,1,1)_S$. Then we have

$$\alpha_N = \begin{pmatrix} 0 & 1 \end{pmatrix} M b_{i_k} \cdots b_{i_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is just systemizing the subdivision of the unit interval via Farey fractions. As described by the author [14], these matrices can be used to find the continued fraction expansion for any real number, which is in part why one would expect any natural generalization of Stern's diatomic sequence to be linked with a generalization of continued fractions.

Finally, Allouche and Shallit [2, 3] have shown that Stern's diatomic sequence is a 2-regular sequence.

2 Stern's triatomic sequence

We now start with explaining the new sequence of this paper. Start with an initial triple (which we will also think of as a 1×3 row matrix)

$$\triangle = (v_1, v_2, v_3) = (1, 1, 1).$$

We are using the notation $\triangle$ to suggest the vertices of a triangle. For any index

$$I = (i_1, i_2, \ldots, i_n),$$

with each i_k being zero, one or two, we will associate a triple which we denote by $\triangle(I)$.

Suppose we know the triple

$$\triangle(I) = (v_1(I), v_2(I), v_3(I)).$$

Consider the three matrices:

$$A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then we set

$$\begin{array}{rcccl} \triangle(I,0) & = & \triangle(I)A_0 & = & (v_1(I), v_2(I), v_3(I))A_0 \\ \triangle(I,1) & = & \triangle(I)A_1 & = & (v_1(I), v_2(I), v_3(I))A_1 \\ \triangle(I,2) & = & \triangle(I)A_2 & = & (v_1(I), v_2(I), v_3(I))A_2. \end{array}$$

Thus we have

$$\begin{array}{rclrclrcl} v_1(I,0) & = & v_1(I), & v_2(I,0) & = & v_2(I), & v_3(I,0) & = & v_1(I) + v_2(I) + v_3(I) \\ v_1(I,1) & = & v_2(I), & v_2(I,1) & = & v_3(I), & v_3(I,1) & = & v_1(I) + v_2(I) + v_3(I) \\ v_1(I,2) & = & v_3(I), & v_2(I,2) & = & v_1(I), & v_3(I,2) & = & v_1(I) + v_2(I) + v_3(I). \end{array}$$

Thus for any $I = (i_1, i_2, \ldots, i_n)$, we have

$$\begin{array}{rcl} \triangle(I) & = & (v_1(I), v_2(I), v_3(I)) \\ & = & (1,1,1)A_{i_1} \cdots A_{i_n} \\ & = & (v_1, v_2, v_3))A_{i_1} \cdots A_{i_n} \\ & = & \triangle A_{i_1} \cdots A_{i_n} \end{array}$$

For $I = (i_1, \dots i_n)$ and $J = (j_1, \dots, j_m)$, we say that $I < J$ if $n < m$ or, when $n = m$, there is an integer k such that $1 \leq k \leq n$ and $i_l = j_l$, for all integers $1 \leq l < k$, and $i_k < j_k$. Thus we have the ordering

$$(0) < (1) < (2) < (00) < (01) < (02) < (10) < (11) < \cdots < (000) < (001) < \cdots$$

and hence an ordering of triples

$$\triangle, \triangle(0), \triangle(1), \triangle(2), \triangle(00), \triangle(01), \triangle(02), \triangle(10), \dots, \triangle(000), \triangle(001), \dots.$$

For each triple $\triangle(I)$, we have the ordered list of three numbers $v_1(I), v_2(I), v_3(I)$. Stern's triatomic sequence is simply putting in these three numbers for each triple $\triangle(I)$:

Definition 1. Stern's triatomic sequence is the sequence formed from all of the triples $\triangle(I)$ using the above ordering on the indices I and is hence $v_1, v_2, v_3,\ v_1(0), v_2(0), v_3(0)$, $v_1(1), v_2(1),\ v_3(1), v_1(2), v_2(2), v_3(2),\ v_1(00), v_2(01), v_3(02), \dots$.

Thus the first terms are $1, 1, 1$,$1, 1, 3$, $1, 1, 3$,$1, 1, 3$, $1, 1, 5$, $1, 3, 5$, $3, 1, 5$, $1, 1, 5$, $1, 3, 5$, $3, 1, 5$, $1, 1, 5$, $1, 3, 5$, $3, 1, 5$.

Definition 2. If $I = (i_1, \dots, i_n)$, then the three numbers $v_1(I), v_2(I), v_3(I)$ are said to be in level n.

Thus the level 0 elements are v_1, v_2, v_3, or simply $1, 1, 1$. The level 1 elements of the triatomic sequence are $v_1(0), v_2(0), v_3(0), v_1(1), v_2(1), v_3(1), v_1(2), v_2(2), v_3(2)$ and hence the nine numbers

$$1, 1, 3, 1, 1, 3, 1, 1, 3,$$

while the level 2 elements come from the nine triples

$$\triangle(00), \triangle(01), \triangle(02), \triangle(10), \triangle(11), \triangle(12), \triangle(20), \triangle(21), \triangle(22)$$

and hence are the 27 numbers

$$1, 1, 5, 1, 3, 5, 3, 1, 5, 1, 1, 5, 1, 3, 5, 3, 1, 5, 1, 1, 5, 1, 3, 5, 3, 1, 5.$$

The naturalness of "levels" will become more apparent in the next section when we describe the triatomic sequence in terms of a partitioning of a triangle. Further, in section 5, we will see that many of the combinatorial properties of the triatomic sequence deals with which numbers are in different levels.

We can write out Stern's triatomic sequence via a recursive formula, which will take a bit to define. Start with three numbers a_1, a_2, a_3, which for us are the three numbers $1, 1, 1$. For each positive integer $N \geq 4$, there is a unique tuple $I = (i_1, i_2, \dots, i_n)$, with each i_j a zero, one or two, and a k chosen to be as large as possible from $k \in \{1, 2, 3\}$ such that

$$\begin{aligned} N &= 3(1 + 3 + 3^2 + \dots + 3^{n-1}) + i_1 3^n + i_2 3^{K-1} + \cdots + i_n 3 + k \\ &= \frac{3(3^n - 1)}{2} + i_1 3^n + i_2 3^{n-1} + \cdots + i_n 3 + k. \end{aligned}$$

Define the function τ from tuples $\times\{1,2,3\}$ to positive integers greater than three by setting

$$\tau(I:k)=\tau(j_n,\ldots,j_1;k)=3(1+3+3^2+\ldots+3^{n-1})+j_n3^n+j_{n-1}3^{n-1}+\cdots+j_13+k.$$

Then we have for example that

$$\tau(0;2)=3+0\cdot 3+2=5$$

while

$$\tau(2,1;3)=3(1+3)+2\cdot 3^2+1\cdot 3+3=36.$$

We extend the map τ to when we have the "empty" tuple, namely we define

$$\tau(\emptyset;1)=1,\tau(\emptyset;2)=2,\tau(\emptyset;3)=3.$$

We write our eventual sequence $a_1,a_2,a_3,\ldots$ as

$$a_{\tau(\emptyset;1)},a_{\tau(\emptyset;2)},a_{\tau(\emptyset;3)}$$

$$a_{\tau(0;1)},a_{\tau(0;2)},a_{\tau(0;3)},a_{(1;1)},a_{\tau(1;2)},a_{(\tau 1;3)},a_{\tau(2;1)},a_{\tau(2;2)},a_{\tau(2;3)}$$

$$\vdots$$

We define the following three functions

$$\tau_1:\mathbb{N}\to\mathbb{N},\tau_2:\mathbb{N}\to\mathbb{N},\tau_3:\mathbb{N}\to\mathbb{N}$$

by setting

$$\begin{array}{rcccl}\tau_1(N) & = & \tau_1(\tau(j_n,\ldots,j_1;k)) & = & \tau(j_n,\ldots,j_2;k)\\ \tau_2(N) & = & \tau_2(\tau((j_n,\ldots,j_1;k)) & = & \tau(j_n,\ldots,j_2,;2)\\ \tau_3(N) & = & \tau_3(\tau(j_n,\ldots,j_1;k)) & = & \tau(j_n,\ldots,j_2;3),\end{array}$$

for

$$N=\tau(j_n,\ldots,j_1;k).$$

Definition 3. Given three numbers a_1,a_2 and a_3, the corresponding Stern triatomic sequence is

$$a_N=\begin{cases}a_{\tau_1(N)}, & \text{if}\quad j_1=0,l=1;\\ a_{\tau_2(N)}, & \text{if}\quad j_1=0,l=2;\\ a_{\tau_1(N)}+a_{\tau_2(N)}+a_{\tau_3(N)}, & \text{if}\quad j_1=0,l=3;\\ a_{\tau_2(N)}, & \text{if}\quad j_1=1,l=1;\\ a_{\tau_3(N)}, & \text{if}\quad j_1=1,l=2;\\ a_{\tau_1(N)}+a_{\tau_2(N)}+a_{\tau_3(N)}, & \text{if}\quad j_1=1,l=3;\\ a_{\tau_3(N)}, & \text{if}\quad j_1=2,l=1;\\ a_{\tau_1(N)}, & \text{if}\quad j_1=2,l=2;\\ a_{\tau_1(N)}+a_{\tau_2(N)}+a_{\tau_3(N)}, & \text{if}\quad j_1=2,l=3.\end{cases}$$

Of course, we want to make sure that this definition agrees with our earlier one. Luckily the following proposition is a straightforward calculation.

Proposition 4. *We have for $N = \tau(I; k)$ that*

$$a_N = v_I(k).$$

The proof is just an unraveling of the definitions.

We also have a description for the series $\sum a_N x^N$. Start with three column vectors v_1, v_2, v_3 in $\mathbb{R}^3$ and put them into the matrix

$$V = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}.$$

Set

$$P(x) = A_0 + xA_1 + x^2 A_2.$$

Let e_i be the standard three by one column vector with entries all zero save in the ith entry, which is one.

Theorem 5. *When*

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

then $\sum a_N x^N$ is

$$e_3^T \cdot V \cdot \left[I + \sum_{k=1}^{\infty} x^{\frac{3(3^k-1)}{2}} P\left(x^{3^k}\right) P\left(x^{3^{k-1}}\right) \cdots P\left(x^3\right) \right] \left(xe_1 + x^2 e_2 + x^3 e_3\right).$$

(For this theorem, we are only interested in the bottom row of the matrix V, which is $\triangle = (1, 1, 1)$. The matrix V, though, will later be important when we link Stern's triatomic sequence to the Farey multidimensional continued fraction.)

Proof. The proof is again an unraveling of the notation, as follows.

Multiplying any three-by-three matrix by e_3^T on the left picks out the bottom row of the matrix. For any tuple $I = (i_1, \ldots, i_n)$, with each i_k a 0, 1 or 2, we have

$$(v_1(I), v_2(I), v_3(I)) = (1, 1, 1) A_{i_1} \cdots A_{i_n} = e_3^T \cdot V A_{i_1} \cdots A_{i_n}.$$

In particular,

$$e_3^T \cdot V = (v_1, v_2, v_3) = \triangle.$$

Then

$$\begin{aligned} e_3^T \cdot V \cdot I \cdot \left(xe_1 + x^2 e_2 + x^3 e_3\right) &= v_1 x + v_2 x^2 + v_3 x^3 \\ &= a_1 x + a_2 x^2 + a_3 x^3, \end{aligned}$$

which are the desired first three terms in the series. Similarly,

$$e_3^T \cdot V \cdot P(x^3) \cdot \left(xe_1 + x^2 e_2 + x^3 e_3\right)$$

is

$$e_3^T \cdot V \cdot (A_0 + x^3 A_1 + x^6 A_2) \cdot \left(xe_1 + x^2 e_2 + x^3 e_3\right),$$

which in turn equals

$$(\triangle(0) + x^3 \triangle(1) + x^6 \triangle(2)) \left(xe_1 + x^2 e_2 + x^3 e_3\right).$$

Thus

$$\begin{aligned} e_3^T \cdot V \cdot P(x^3) \cdot \left(xe_1 + x^2 e_2 + x^3 e_3\right) &= v_1(0)x + v_2(0)x^2 + v_3(0)x^3 \\ &\quad + v_1(1)x^4 + v_2(1)x^5 + v_3(1)x^6 \\ &\quad + v_1(2)x^7 + v_2(2)x^8 + v_3(2)x^9 \\ &= a_4 x + a_5 x^2 + a_6 x^3 \\ &\quad + a_7 x^4 + a_8(1)x^5 + a_9(1)x^6 \\ &\quad + a_{10}(2)x^7 + a_{11}(2)x^8 + a_{12}(2)x^9. \end{aligned}$$

After multiplying through by the factor of x^3, we get the next nine terms in the series.

For the next 27 terms, we start with

$$\begin{aligned} P(x^9)P(x^3) &= (A_0 + A_1 x^9 + A_2 x^{18})(A_0 + A_1 x^3 + A_2 x^6) \\ &= A_0 A_0 + A_0 A_1 x^3 + A_0 A_2 x^6 \\ &\quad + A_1 A_0 x^9 + A_1 A_1 x^{12} + A_1 A_2 x^{15} \\ &\quad + A_2 A_0 x^{18} + A_2 A_1 x^{21} + A_2 A_2 x^{24}. \end{aligned}$$

Then

$$\begin{aligned} e_3^T \cdot V P(x^9) P(x^3) &= \triangle(00) + \triangle(01)x^3 + \triangle(02)x^6 \\ &\quad + \triangle(10)x^9 + \triangle(11)x^{12} + \triangle(12)x^{15} \\ &\quad + \triangle(20)x^{18} + \triangle(21)x^{21} + \triangle(22)x^{24} \end{aligned}$$

and hence

$$\begin{aligned} e_3^T \cdot V P(x^9) P(x^3) \left(xe_1 + x^2 e_2 + x^3 e_3\right) &= v_1(00)x + v_2(00)x^2 + v_3(00)x^3 \\ &\quad + v_1(01)x^4 + v_2(01)x^5 + v_3(01)x^6 \\ &\quad + v_1(02)x^7 + v_2(02)x^8 + v_3(02)x^9 \\ &\quad + v_1(10)x^{10} + v_2(10)x^{11} + v_3(10)x^{12} \\ &\quad + v_1(11)x^{13} + v_2(11)x^{14} + v_3(11)x^{15} \\ &\quad + v_1(12)x^{16} + v_2(12)x^{17} + v_3(12)x^{18} \\ &\quad + v_1(20)x^{19} + v_2(20)x^{20} + v_3(20)x^{21} \\ &\quad + v_1(21)x^{22} + v_2(21)x^{23} + v_3(21)x^{24} \\ &\quad + v_1(22)x^{25} + v_2(22)x^{26} + v_3(22)x^{27} \end{aligned}$$

which is

$$a_{13}x + a_{14}x^2 + a_{15}x^3 + \cdots + a_{37}x^{25} + a_{38}x^{26} + a_{39}x^{27}$$

Using that for $k = 2$

$$\frac{3(3^k - 1)}{2} = 12,$$

we multiply the above sum by the factor of x^{12}, which does indeed give us the next 27 terms.

The next 81 terms are similarly computed. To see how this would work, note that $P(x^{27})P(x^9)P(x^3)$ equals

$$(A_0 + A_1x^{27} + A_2x^{54})(A_0 + A_1x^9 + A_2x^{18})(A_0 + A_1x^3 + A_2x^6),$$

which gives us

$$\begin{aligned} P(x^{27})P(x^9)P(x^3) &= A_0A_0A_0 + A_0A_0A_1x^3 + A_0A_0A_2x^6 \\ &\quad +A_0A_1A_0x^9 + A_0A_1A_1x^{12} + A_0A_2A_2x^{15} \\ &\quad +A_0A_2A_0x^{18} + A_0A_2A_1x^{21} + A_0A_2A_2x^{24} \\ &\quad +A_1A_0A_0x^{27} + A_1A_0A_1x^{30} + A_1A_0A_2x^{33} \\ &\quad +A_1A_1A_0x^{36} + A_1A_1A_1x^{39} + A_1A_1A_2x^{42} \\ &\quad +A_1A_2A_0x^{45} + A_1A_2A_1x^{48} + A_1A_2A_2x^{51} \\ &\quad +A_2A_0A_0x^{54} + A_2A_0A_1x^{57} + A_2A_0A_2x^{60} \\ &\quad +A_2A_1A_0x^{63} + A_2A_1A_1x^{66} + A_2A_1A_2x^{69} \\ &\quad +A_2A_2A_0x^{72} + A_2A_2A_1x^{75} + +A_2A_2A_2x^{78} \end{aligned}$$

Then $e_3^T \cdot VP(x^{27})P(x^9)P(x^3)$ will be a sum of row vectors of the various $\triangle(i_1, i_2, i_3)$, indexed by appropriate powers of x. Then

$$e_3^T \cdot VP(x^{27})P(x^9)P(x^3)\left(xe_1 + x^2e_2 + x^3e_3\right),$$

multiplied by x^{39} (since for $k = 3$ we have $3(3^k - 1)/2 = 39$), will give us the next 81 terms

$$a_{40}x^{40} + \cdots + a_{118}x^{118}.$$

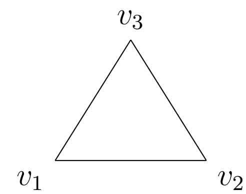
The rest of the series is similar.

$\square$

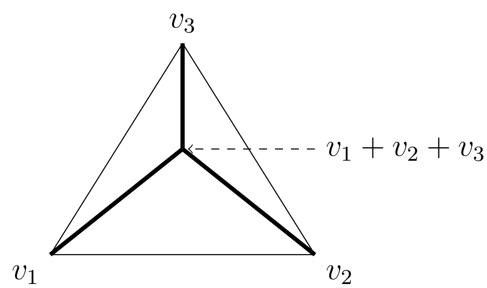
3 Stern's triatomic sequence and triangles

In this section we will show that Stern's triatomic sequence can be interpreted as a subdivision of a triangle, in analog to how Stern's diatomic sequence can be interpreted as systematically subdividing the unit interval.

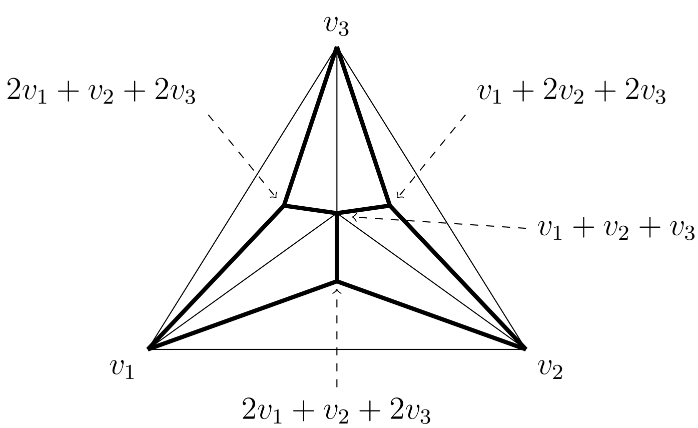
Start with a triangle $\triangle$



with vertices v_1, v_2, v_3. In this paper we will consider both the case when the vertices denote numbers and when the vertices denote vectors. From $\triangle$ form three new triangles



where $\triangle(0)$ has vertices $v_1, v_2, v_1+v_2+v_3$, $\triangle(1)$ has vertices $v_2, v_3, v_1+v_2+v_3$ and $\triangle(2)$ has vertices $v_3, v_1, v_1+v_2+v_3$. Each of these triangles can be divided into three more triangles, giving us nine triangles

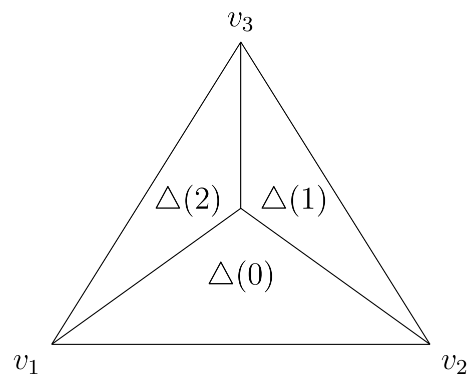


We will identify each of these triangles with their vertices. Thus we start with our initial triangle (which we will say is at level 0):

$$\triangle = (v_1, v_2, v_3).$$

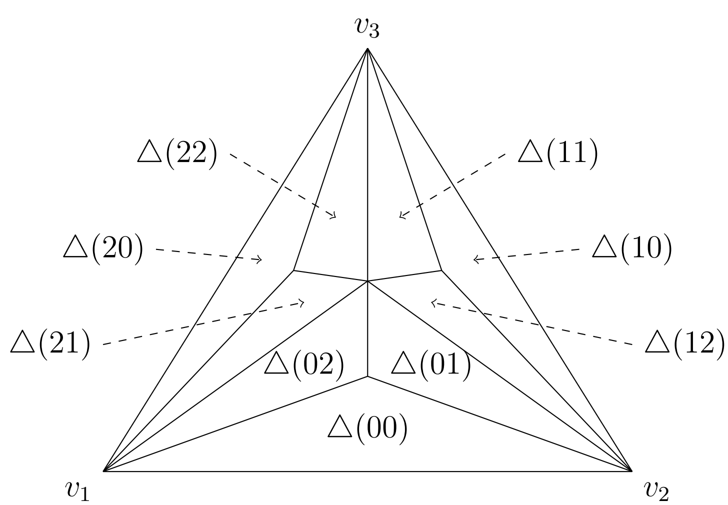
At level one we have

$$\begin{aligned}
\triangle(0) &= (v_1, v_2, v_1 + v_2 + v_3) \\
\triangle(1) &= (v_2, v_3, v_1 + v_2 + v_3) \\
\triangle(2) &= (v_3, v_1, v_1 + v_2 + v_3)
\end{aligned}$$



while for level two we have

$$
\begin{aligned}
\triangle(00) &= (v_1, v_2, 2v_1 + 2v_2 + v_3) \\
\triangle(01) &= (v_2, v_1 + v_2 + v_3, 2v_1 + 2v_2 + v_3) \\
\triangle(02) &= (v_1 + v_2 + v_3, v_1, 2v_1 + 2v_2 + v_3) \\
\triangle(10) &= (v_2, v_3, v_1 + 2v_2 + 2v_3) \\
\triangle(11) &= (v_3, v_1 + v_2 + v_3, v_1 + 2v_2 + 2v_3) \\
\triangle(12) &= (v_1 + v_2 + v_3, v_2, v_1 + 2v_2 + 2v_3) \\
\triangle(20) &= (v_3, v_1, 2v_1 + v_2 + 2v_3) \\
\triangle(21) &= (v_1, v_1 + v_2 + v_3, 2v_1 + v_2 + 2v_3) \\
\triangle(21) &= (v_1 + v_2 + v_3, v_3, 2v_1 + v_2 + 2v_3).
\end{aligned}
$$



Continuing this process produces a sequence of triples

$$\triangle, \triangle(0), \triangle(1), \triangle(2), \triangle(00), \triangle(01), \triangle(02), \triangle(10), \ldots,$$

which, in a natural way, will give us last section's Stern's triatomic sequence.

4 The Farey map

4.1 Definition

Stern's diatomic sequence is linked to continued fractions [34]. (This can also be seen in how the diatomic sequence can be interpreted via the Farey decomposition of the unit interval.) There is a multidimensional continued fraction algorithm which generates in an analogous fashion Stern's triatomic sequence. Beaver and the author [4] gave a (seemingly) new multidimensional continued fraction algorithm, with the goal of finding finding a generalization of the Minkowski $?(x)$ function. This algorithm, though, seems particularly well-suited to generalize Stern's diatomic sequence. (Panti [35] used the Monkmeyer algorithm to generalize the Minkowski $?(x)$ function.) Also, it is standard in the study of multidimensional continued fractions to pass back and forth between cones in $\mathbb{R}^3$ and triangle in $\mathbb{R}^2$, via the map, when $x \neq 0$, $(x, y, z) \to (y/x, z/x)$. In this subsection we define the Farey map as an iteration of a triangle, while in the next subsection we will think of the Farey map as an iteration of the corresponding cone.

Let

$$\triangle = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}.$$

Partition $\triangle$ into the three subtriangles

$$\begin{aligned} \triangle_0 &= \{(x, y) \in \triangle : 1 - 2y \geq x - y \geq y\} \\ \triangle_1 &= \{(x, y) \in \triangle : 2x - 1 \geq y \geq 1 - x\} \\ \triangle_2 &= \{(x, y) \in \triangle : 1 - 2x + 2y \geq 1 - x \geq x - y\}. \end{aligned}$$

The Farey map $T : \triangle \to \triangle$ is given by three one-to-one onto maps

$$T_i : \triangle_i \to \triangle$$

defined via

$$\begin{aligned} T_0(x, y) &= \left(\frac{x - y}{1 - 2y}, \frac{y}{1 - 2y} \right) \\ T_1(x, y) &= \left(\frac{y}{2x - 1}, \frac{1 - x}{2x - 1} \right) \\ T_2(x, y) &= \left(\frac{1 - x}{1 - 2x + 2y}, \frac{x - y}{1 - 2x + 2y} \right) \end{aligned}$$

These maps are the Farey analog of the Gauss map for continued fractions.

4.2 Farey map on vertices

Now we will identify triangles with their corresponding cones in $\mathbb{R}^3$. Let $\triangle$ be a cone with three vertices v_1, v_2, v_3, each written as a column vector in $\mathbb{R}^3$. Subdivide $\triangle$ into three sub-cones

$$\begin{array}{lll} \triangle(0) & \text{with vertices} & v_1, v_2, v_1+v_2+v_3 \\ \triangle(1) & \text{with vertices} & v_2, v_3, v_1+v_2+v_3 \\ \triangle(3) & \text{with vertices} & v_3, v_1, v_1+v_2+v_3 \end{array}$$

We have three maps

$$A_i : \triangle \to \triangle_i,$$

which we describe via matrix multiplication on the right.

Recall that

$$A_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

For any three vectors v_1 v_2 and v_3, we have

$$(v_1, v_2.v_3)A_0 = (v_1, v_2, v_1 + v_2 + v_3)$$

$$(v_1, v_2.v_3)A_1 = (v_2, v_3, v_1 + v_2 + v_3)$$

$$(v_1, v_2.v_3)A_2 = (v_3, v_1, v_1 + v_2 + v_3).$$

We now specify our initial three vectors, by setting, as before,

$$V = (v_1, v_2, v_3) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then VA_0, VA_1 and VA_2 is simply the Farey map's analog of the Farey subdivision of the unit interval and can be interpreted as the matrix approach for the inverse maps of the three Farey maps $T_i : \triangle_i \to \triangle$ of triangles in $\mathbb{R}^2$ from the previous subsection.

Then we have

Theorem 6. *For an n-tuple $I = (i_1, \ldots, i_n)$ where each i_j is a zero, one or two, we have that*

$$VA_{i_1} \ldots A_{i_n} = \begin{pmatrix} * & * & * \\ * & * & * \\ v_1(I) & v_2(I) & v_3(I) \end{pmatrix}.$$

Thus Stern's triatomic sequence is the analog to the fact that Stern's diatomic sequence can be interpreted as keeping track of the denominators for the Farey decomposition of the unit interval.

5 Combinatorial properties of triatomic sequences

We now develop some analogs of the combinatorial properties for Stern diatomic sequences for triatomic sequences.

5.1 Three-fold symmetry

Property 4 in Lehmer [29] is that there is a two-fold symmetry for Stern's diatomic sequences on each "level". We will see that for triatomic sequences, we have a three-fold symmetry at each level.

Proposition 7. *For any $N = \tau(0, j_2, \ldots, j_n; k)$, where each $j_i = 0, 1$ or 2 and k is as large as possible for $k \in \{1, 2, 3\}$, we have*

$$a_N = a_{N+3^k} = a_{N+2\cdot 3^N}.$$

Thus

$$a_{\tau(0,j_2,\ldots,j_n;k)} = a_{\tau(1,j_2,\ldots,j_n;k)} = a_{\tau(2,j_2,\ldots,j_n,;k)}.$$

Proof. At the 0th level, we have

$$a_1 = 1,\ a_2 = 1,\ a_3 = 1.$$

The 1th level is

$$a_4, \ldots, a_{12},$$

which by definition is

$$a_1, a_2, a_1 + a_2 + a_3, a_2, a_3, a_1 + a_2 + a_3, a_3, a_1, a_1 + a_2 + a_3,$$

which in turn is simply

$$1, 1, 3, 1, 1, 3, 1, 1, 3,$$

satisfying the desired relation.

The rest follows from the basic recursion relation.

□

5.2 Tribonacci numbers

Recall that the Tribonacci numbers are the terms in the sequence $\beta_1, \beta_2, \ldots$ given by the recursion relation

$$\beta_{k+3} = \beta_k + \beta_{k+1} + \beta_{k+2}.$$

Such sequences are intimately intwined with triatomic sequences.

Theorem 8. *For any n-tuple J, the following subsequence is a Tribonacci sequence:*

$$a_{\tau(J:1)}, a_{\tau(J:2)}, a_{\tau(J:3)}, a_{\tau(J,1:3)}, a_{\tau(J,1,1:3)}, a_{\tau(J,1,1,1:3)}, \ldots$$

Proof. Start with the triple

$$a_{\tau(J:1)}, a_{\tau(J:2)}, a_{\tau(J:3)}.$$

Then the triple $a_{\tau(J,1:1)}, a_{\tau(J,1:2)}, a_{\tau(J,1:3)}$ is

$$a_{\tau(J:2)}, a_{\tau(J:3)}, a_{\tau(J:1)} + a_{\tau(J:2)} + a_{\tau(J:3)}.$$

Next, the triple $a_{\tau(J,1,1:1)}, a_{\tau(J,1,1:2)}, a_{\tau(J,1,1:3)}$ is

$$a_{\tau(J:3)}, a_{\tau}(J, 1:3), a_{\tau(J:2)} + a_{(J:3)} + a_{(J,1:3)}.$$

This process continues, since by definition we have

$$a_{(J,1^s:3)} = a_{(J,1^{s-3}:3)} + a_{(J,1^{s-2}:3)} + a_{(J,1^{s-1}:3)},$$

where 1^s is notation for s ones in a row, giving us our result.

□

Corollary 9. *The standard Tribonacci sequence*

$$1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, \ldots$$

is

$$a_1, a_2, a_3, a_{\tau(1:3)}, a_{\tau(1,1:3)}, \ldots, a_{\tau(1^s:3)}, \ldots$$

5.3 Sums at levels

For Stern's diatomic sequence, Stern showed that the sum of all terms of level n is $3^n + 1$ (see property 2 in [29]). A similar sum exists for Stern's triatomic sequence.

Proposition 10. *The sum of all terms in Stern's triatomic sequence at the nth level is $5^n \cdot 3$. Thus for fixed n*

$$\sum a_{\tau(J:k)} = 5^n \cdot 3,$$

where the summation is over all n-tuples $J = (j_1, \ldots j_n)$, for $j_i = 0, 1$ or 2, and over $k = 1, 2, 3$.

Proof. The 0th level is the three numbers $1, 1, 1$, and hence the sum is 3. Now suppose we have at the n level the number $N = \tau(j_1, \ldots, j_n : 1)$. Then the triple of numbers

$$a_N, a_{N+1}, a_{N+2}$$

is in the nth level. What is important is that this triple generates the following nine terms in the $(n + 1)$st level:

$$a_N, a_{N+1}, a_N + a_{N+1} + a_{N+2},$$

$$a_{N+1}, a_{N+2}, a_N + a_{N+1} + a_{N+2}$$

and

$$a_{N+2}, a_N, a_N + a_{N+1} + a_{N+2},$$

whose sum is

$$5(a_N + a_{N+1} + a_{N+2}),$$

giving us our result.

□

5.4 Values at levels

We can write each positive integer N as $N = \tau(j_1, \ldots, j_n; k)$, where each $j_i \in \{0, 1, 2\}$ and each $k \in \{1, 2, 3\}$. We have determined what the corresponding a_N is at the nth level. We set

$$\delta_n(m) = \text{number of times } m \text{ occurs in the } n\text{th level.}$$

We further set

$$\delta_n^1(m) = \text{number of times } m \text{ occurs in the } n\text{th level when } k = 1,$$

$$\delta_n^2(m) = \text{number of times } m \text{ occurs in the } n\text{th level when } k = 2,$$

and

$$\delta_n^3(m) = \text{number of times } m \text{ occurs in the } n\text{th level when } k = 3.$$

At level 1, our sequence is

$$1, 1, 3, 1, 1, 3, 1, 1, 3,$$

in which case

$$\delta_1(1) = 6, \delta_1(3) = 3, \delta_1(n) = 0 \text{ for } n \neq 1, 3,$$
$$\delta_1^1(1) = 3, \delta_1^1(3) = 1, \delta_1^1(n) = 0 \text{ for } n \neq 1, 3,$$
$$\delta_1^2(1) = 3, \delta_1^2(3) = 1, \delta_1^2(n) = 0 \text{ for } n \neq 1, 3,$$
$$\delta_1^3(1) = 0, \delta_1^2(3) = 3, \delta_1^2(n) = 0 \text{ for } n \neq 1, 3.$$

We now discuss some formulas for these new functions.

First for a technical lemma:

Lemma 11. *Suppose we are at the nth level, for some $N = \tau(j_1, \ldots, j_n; 1)$. Then the triple*

$$a_N, a_{N+1}, a_{N+2}$$

will induce the following three triples on the $(n+1)$st level: The triple corresponding to

$$a_{\tau(j_1,\ldots,j_n,0;1)}, a_{\tau(j_1,\ldots,j_n,0;2)}, a_{\tau(j_1,\ldots,j_n,0;3)}$$

will be

$$(a_N, a_{N+1}, a_N + a_{N+1} + a_{N+2}),$$

the triple corresponding to

$$a_{\tau(j_1,\ldots,j_n,1;1)}, a_{\tau(j_1,\ldots,j_n,1;2)}, a_{(j_1,\ldots,j_n,1;3)}$$

will be

$$(a_{N+1}, a_{N+2}, a_N + a_{N+1} + a_{N+2}),$$

and the triple corresponding to

$$a_{\tau(j_1,\ldots,j_n,2;1)}, a_{\tau(j_1,\ldots,j_n,2;2)}, a_{\tau(j_1,\ldots,j_n,2;3)}$$

will be

$$(a_{N+2}, a_N, a_N + a_{N+1} + a_{N+2}).$$

We will say we have a triple a_N, a_{N+1}, a_{N+2} at level n if $N = \tau(j_n, \ldots, j_1; 1)$.

Proposition 12. *For all positive n and m,*

$$\delta_n(2m) = \delta_n^1(2m) = \delta_n^2(2m) = \delta_n^2(2m) = 0.$$

Proof. If $N = \tau(j_1, \ldots, j_n; 1)$ at the nth level and if each term in the triple a_N, a_{N+1}, a_{N+2} is odd, then all of their their descendants in the next level will still be odd, from the above lemma. Thus to prove the proposition, we just have to observe that at the initial level 0, our numbers are $1, 1, 1$, forcing all subsequent elements to be odd. □

Proposition 13. *For any odd number $2n + 1 > 1$, we have*

$$\delta_n^3(2n + 1) = 9.$$

Proof. We first show that if $N = \tau(0^n; 1)$, then the corresponding triple is $(1, 1, 2n+1)$. The proof is by induction. At the initial zero level, the initial triples $(1, 1, 1)$ as desired. Then suppose at the nth level the triple corresponding to $N = \tau(0^n; 1)$ is the desired $(1, 1, 2n+1)$. Then we have at the next level three triples descending from $(1, 1, 2n+1)$, namely $(1, 1, 2n+3), (1, 2n+1, 2n+3)$ and $(2n+1, 1, 2n+3)$. By the three-fold symmetry of the sequence, this means that

$$\delta_{n+1}(2n + 3) \geq 9.$$

Now to show that this number cannot be greater than 9. Suppose at level $n \geq 1$ there are only three triples of the form $(1, 1, 2n + 1)$. This is true at level one. For any triple (a_N, a_{N+1}, a_{N+2}) at level n with a_N or a_{N+1} greater than one, then none of this triple's descendants will contribute to $\delta_{n+1}^3(2n + 3)$. Then at the next level, we will only pick up three triples $(1, 1, 2n + 3)$. Putting all this together, means that the only triples at level n that will have descendants contributing to $\delta_{n+1}^3(2n + 3)$ must be of the form $(1, 1, 2n + 1)$, and each of these three triples will contribute three $(2n + 3)$s. Thus we have our result.

□

Proposition 14.

$$\delta_n(m) = \delta^1_{n+1}(m) = \delta^2_{n+1}(m).$$

Proof. This follows immediately from the construction of the sequence. □

Proposition 15.

$$\delta_{k+m}(2k+1) = 2^m \delta_k(2k+1).$$

Proof. It is at the kth level where $2k+1$ can last occur in a δ^3 term. Hence this follows from the previous proposition, since we have

$$\begin{aligned}\delta_{k+m}(2k+1) &= \delta^1_{k+m}(2k+1) + \delta^2_{k+m}(2k+1) + \delta^3_{k+m}(2k+1) \\ &= \delta_{k+m-1}(2k+1) + \delta_{k+m-1}(2k+1) + 0 \\ &= 2\delta_{k+m-1}(2k+1).\end{aligned}$$

□

Proposition 16.

$$\delta_n(2n+1) = 9 + 2\delta_{n-1}(2n+1)$$

Proof. We have already seen that $\delta^3_n(2n+1) = 9$. Since

$$\delta_n(2n+1) = \delta^1_n(2n+1) + \delta^2_n(2n+1) + \delta^3_n(2n+1) = \delta^1_n(2n+1) + \delta^2_n(2n+1) + 9,$$

all we need show is that $\delta^1_n(2n+1) + \delta^2_n(2n+1) = 2\delta_{n-1}(2n+1)$. But this follows from the fact that each triple a, b, c at the $k-1$st level generates for the kth level the triples $a, b, a+b+a$, or $b, c, a+b+c$, or $c, a, a+b+c$ and each number appearing in the $n-1$st level appears twice in the kth level.

□

Of course, it is easy to generate a table of various $\delta_K(n)$. The following is a list of some examples.

Table 1

(K,n)	$\delta_K(n)$	$\delta_K^1(n)$	$\delta_K^2(n)$	$\delta_K^3(n)$
$(0,1)$	3	1	1	1
$(1,1)$	6	3	3	0
$(1,3)$	3	0	0	3
$(2,1)$	12	6	6	0
$(2,3)$	6	3	3	0
$(2,5)$	9	0	0	9
$(3,1)$	24	12	12	0
$(3,1)$	24	12	12	0
$(3,3)$	12	6	6	0
$(3,5)$	18	9	9	0
$(3,7)$	9	0	0	9
$(3,9)$	18	0	0	18
$(4,1)$	48	24	24	0
$(4,3)$	24	12	12	0
$(4,5)$	36	18	18	0
$(4,7)$	18	9	9	0
$(4,9)$	45	18	18	9
$(4,13)$	36	0	0	36
$(4,15)$	18	0	0	18
$(4,17)$	18	0	0	18
$(5,1)$	96	48	48	0
$(5,3)$	48	24	24	0
$(5,5)$	72	36	36	0
$(5,7)$	36	18	18	0
$(5,9)$	90	45	45	0
$(5,11)$	9	0	0	9
$(5,13)$	72	36	36	0
$(5,15)$	36	18	18	0
$(5,17)$	72	18	18	36
$(5,19)$	18	0	0	18
$(5,21)$	18	0	0	18
$(5,23)$	18	0	0	18
$(5,25)$	72	0	0	72
$(5,29)$	36	0	0	36
$(5,31)$	18	0	0	18

6 Paths of a directed graph

There is a simple interpretation of Stern's triatomic sequence as the number of paths in a directed graph from three initial vertices. Recall the diagrams in Section 3. Starting with a triangle $\triangle$ with vertices v_1, v_2, v_3, we systematically constructed a subdivision of $\triangle$ so that at level n, we have 3^n triangles. It is quite easy to put the structure of a directed graph on this subdivision. Suppose we have one of the subtriangles $\triangle(I)$, where recall I is an n-tuple consisting of $0, 1$ or 2. Its vertices are denoted by $v_1(I), v_2(I)$ and $v_3(I)$. Then for level $n+1$, we add one more vertex, denoted by $v_1(I)+v_2(I)+v_3(I)$ and three directed paths, one from each $v_1(I)$ to the new vertex. Finally, note that there are no directed paths between any of the three initial vertices. Recall our earlier notation of writing any positive integer N as

$$N = \tau(j_1, j_2, \dots j_n; k)$$

if

$$N = 3(1 + 3 + 3^2 + +3^{n-1}) + j_1 3^n + j_2 3^{n-1} + \cdots + j_n 3 + k,$$

where each j_i is zero, one or two and l is chosen to be as large as possible from the set $\{1, 2, 3\}$.

Theorem 17. *For any positive integer N, the number a_N is the number of paths from v_1, v_2 and v_3 to the lth vertex of the $\triangle(j_1, \dots, j_n)$, save for when the corresponding vertex of $\triangle(j_n, \dots, j_1)$ is one of the initial v_1, v_2 or v_3.*

The proof is straightforward.

7 Questions

The Farey map is only one type of multidimensional continued fraction. (For background on multidimensional continued fractions, see Schweiger [39]). Recently, Dasaratha et al. [6] produced a family of multidimensional continued fractions which includes many previously well-known algorithms, though not the Farey map. Using this family, Dasaratha et al. [7] study analogs of Stern's diatomic sequence. Independently, Goldberg [15] has started finding analogs for the Monkemayer map. It would also be interesting to find the Stern analogs in the language of Lagarias [28].

Also, there are many properties of Stern's diatomic sequences whose analogs for the Farey map have yet to be discovered. Probably the most interesting would be to find the analog of the link between Stern's sequence and the number of hyperbinary representations there are for positive integers, as discussed by Northshield [34].

Another interesting problem is to attempt a multidimensional analog of the recently found link between the Tower of Hanoi graph and Stern's diatomic sequence by Hinz, Klavžar, Milutinović, Parisse and Petr [18].

Finally, it would be interesting to extend the polynomial analogs of Stern's diatomic sequence (as in the work of Dilcher and Stokarsky [8, 9], of Klavžar, Milutinović and Petr

[21], of Ulas [40, 41], of Vargas [42], and of Allouche and Mendès France [1]) to finding polynomial analogs for triatomic sequences.

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