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Counting Toroidal Binary Arrays

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Abstract

A formula for the number of toroidal $m \times n$ binary arrays, allowing rotation of the rows and/or the columns but not reflection, is known. Here we find a formula for the number of toroidal $m \times n$ binary arrays, allowing rotation and/or reflection of the rows and/or the columns.

1 Introduction

The number of *necklaces* with n beads of two colors when turning over is not allowed is

$$\frac{1}{n}\sum_{d\,|\,n}\varphi(d)\,2^{n/d}, \tag{1}$$

where φ is Euler's phi function. When turning over is allowed, the number becomes

$$\frac{1}{2n}\sum_{d\,|\,n}\varphi(d)\,2^{n/d}+\begin{cases}2^{(n-1)/2}, & \text{if } n \text{ is odd;}\\ 3\cdot 2^{n/2-2}, & \text{if } n \text{ is even.}\end{cases} \tag{2}$$

These are the core sequences A000031 and A000029, respectively, in *The On-Line Encyclopedia of Integer Sequences* [3].

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Our concern here is with two-dimensional versions of these formulas. We consider an $m \times n$ binary array. When opposite edges are identified, it becomes what we will call a *toroidal* binary array. Just as we can rotate a necklace without effect, we can rotate the rows and/or the columns of such an array without effect. The number of (distinct) toroidal $m \times n$ binary arrays is

$$\frac{1}{mn}\sum_{c\,|\,m}\sum_{d\,|\,n}\varphi(c)\varphi(d)\,2^{mn/\operatorname{lcm}(c,d)}, \tag{3}$$

where lcm stands for least common multiple. This is A184271 in the *OEIS* [3]. The diagonal is A179043. Rows (or columns) 2–8 are A184264–A184270. Row (or column) 1 is of course A000031.

Our aim here is to find the formula that is related to (3) in the same way that (2) is related to (1). More precisely, we wish to count the number of toroidal $m \times n$ binary arrays allowing rotation and/or reflection of the rows and/or the columns. This is A222188 in the *OEIS* [3]. The diagonal is A209251. Rows (or columns) 2–5 are A222187 and A222189–A222191. Row (or column) 1 is of course A000029.

For an alternative description, we could define a group action on the set of $m \times n$ binary arrays, which has 2^{mn} elements. If the group is $C_m \times C_n$, where C_m denotes the cyclic group of order m, then the number of orbits is given by (3) (see Theorem 1 below). If the group is $D_m \times D_n$, where D_m denotes the dihedral group of order $2m$, then the number of orbits is given in Theorem 2 below.

Both theorems are proved using Pólya's enumeration theorem (actually, the simplified unweighted version; see, e.g., van Lint and Wilson [4, Theorem 37.1, p. 524]). Gilbert and Riordan [1] gave other applications of Pólya's theorem in which the group was, as it is here, a direct product.

To help clarify the distinction between the two group actions, we provide an example. There is no distinction in the 2×2 case, so we consider the 3×3 case. When the group is $C_3 \times C_3$ (allowing rotation of the rows and/or the columns but not reflection), there are 64 orbits, as shown in Table 1. When the group is $D_3 \times D_3$ (allowing rotation and/or reflection of the rows and/or the columns), there are 36 orbits, as shown in Table 2. Both tables were generated by *Mathematica* programs.

Our interest in the number of toroidal $m \times n$ binary arrays allowing rotation and/or reflection of the rows and/or the columns derives from the fact that this is the size of the state space of the projection of the Markov chain of Mihailović and Rajković [2] under the mapping that takes a state to the orbit containing it. This reduction of the state space, from 512 states to 36 states in the 3×3 case for example, makes it easier to evaluate the stationary distribution.

2 Rotations of rows and columns

Let $X_{m,n} := \{0,1\}^{\{0,1,\ldots,m-1\}\times\{0,1,\ldots,n-1\}}$ be the set of $m \times n$ arrays of 0s and 1s, which has 2^{mn} elements. Let $a(m,n)$ denote the number of orbits of the group action on $X_{m,n}$ by the

Table 1: A list of the 64 orbits of the group action in which the group $C_3 \times C_3$ acts on the set of 3×3 binary arrays. (Rows and/or columns can be rotated but not reflected.) Each orbit is represented by its minimal element in 9-bit binary form . Subscripts indicate orbit size. Bars separate different numbers of 1s.

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}_1 \Bigg| \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}_9 \Bigg| \begin{pmatrix}0&0&0\\0&0&0\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&0&1\\0&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix}_9 \Bigg|$$

$$\begin{pmatrix}0&0&0\\0&0&0\\1&1&1\end{pmatrix}_3 \begin{pmatrix}0&0&0\\0&0&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&0&1\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&0&1\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&1&1\\0&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&1&1\\0&1&0\end{pmatrix}_9$$

$$\begin{pmatrix}0&0&0\\0&1&1\\1&0&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix}_3 \begin{pmatrix}0&0&1\\0&0&1\\0&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&0&1\\1&0&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}_3 \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}_3 \Bigg|$$

$$\begin{pmatrix}0&0&0\\0&0&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&1&1\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&1&1\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&0\\1&1&1\\0&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&0&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&0&1\\1&0&1\end{pmatrix}_9$$

$$\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&1\\0&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&0&0\\1&1&0\end{pmatrix}_9 \Bigg|$$

$$\begin{pmatrix}0&0&0\\0&1&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\1&1&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&1\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix}_9$$

$$\begin{pmatrix}0&0&1\\1&0&0\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&0&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&0&1\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&0&1\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&1&0\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&1&0\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&1&0\\1&1&0\end{pmatrix}_9 \Bigg|$$

$$\begin{pmatrix}0&0&0\\1&1&1\\1&1&1\end{pmatrix}_3 \begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&0&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&1&0\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&1&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\1&1&1\\1&0&1\end{pmatrix}_9$$

$$\begin{pmatrix}0&0&1\\1&1&1\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&1&1\\0&1&1\\0&1&1\end{pmatrix}_3 \begin{pmatrix}0&1&1\\0&1&1\\1&0&1\end{pmatrix}_9 \begin{pmatrix}0&1&1\\0&1&1\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}_3 \begin{pmatrix}0&1&1\\1&1&0\\1&0&1\end{pmatrix}_3 \Bigg|$$

$$\begin{pmatrix}0&0&1\\1&1&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&1&1\\0&1&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&1&1\\1&0&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&1&1\\1&1&0\\1&1&1\end{pmatrix}_9 \Bigg| \begin{pmatrix}0&1&1\\1&1&1\\1&1&1\end{pmatrix}_9 \Bigg| \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}_1$$

Table 2: A list of the 36 orbits of the group action in which the group $D_3 \times D_3$ acts on the set of 3×3 binary arrays. (Rows and/or columns can be rotated and/or reflected.) Each orbit is represented by its minimal element in 9-bit binary form. Subscripts indicate orbit size. Bars separate different numbers of 1s.

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}_1 \Bigg| \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}_9 \Bigg| \begin{pmatrix}0&0&0\\0&0&0\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&0&1\\0&0&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}_{18} \Bigg|$$

$$\begin{pmatrix}0&0&0\\0&0&0\\1&1&1\end{pmatrix}_3 \begin{pmatrix}0&0&0\\0&0&1\\0&1&1\end{pmatrix}_{36} \begin{pmatrix}0&0&0\\0&0&1\\1&1&0\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix}_3 \begin{pmatrix}0&0&1\\0&0&1\\0&1&0\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}_6 \Bigg|$$

$$\begin{pmatrix}0&0&0\\0&0&1\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&0\\0&1&1\\1&0&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&0&1\\0&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\0&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}_{36} \Bigg|$$

$$\begin{pmatrix}0&0&0\\0&1&1\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&0&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&0&1\\0&1&0\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&1&1\\0&1&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&1&1\\1&0&1\end{pmatrix}_{18} \begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix}_{36} \begin{pmatrix}0&0&1\\1&1&0\\1&1&0\end{pmatrix}_9 \Bigg|$$

$$\begin{pmatrix}0&0&0\\1&1&1\\1&1&1\end{pmatrix}_3 \begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}_{36} \begin{pmatrix}0&0&1\\1&1&0\\1&1&1\end{pmatrix}_{18} \begin{pmatrix}0&1&1\\0&1&1\\0&1&1\end{pmatrix}_3 \begin{pmatrix}0&1&1\\0&1&1\\1&0&1\end{pmatrix}_{18} \begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}_6 \Bigg|$$

$$\begin{pmatrix}0&0&1\\1&1&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&1&1\\0&1&1\\1&1&1\end{pmatrix}_9 \begin{pmatrix}0&1&1\\1&0&1\\1&1&1\end{pmatrix}_{18} \Bigg| \begin{pmatrix}0&1&1\\1&1&1\\1&1&1\end{pmatrix}_9 \Bigg| \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}_1$$

group $C_m \times C_n$. In other words, $a(m, n)$ is the number of (distinct) toroidal $m \times n$ binary arrays, allowing rotation of the rows and/or the columns but not reflection.

Theorem 1.

$$a(m, n) = \frac{1}{mn} \sum_{c\,|\,m} \sum_{d\,|\,n} \varphi(c)\varphi(d)\, 2^{mn/\operatorname{lcm}(c,d)}. \tag{4}$$

Proof. By Pólya's enumeration theorem,

$$a(m, n) = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} 2^{A_{ij}}, \tag{5}$$

where A_{ij} is the number of cycles in the permutation $\sigma^i\tau^j$; here σ rotates the rows (row 0 becomes row 1, row 1 becomes row 2, ..., row $m-1$ becomes row 0) and τ rotates the columns. For example, $A_{00} = mn$ because the identity permutation has mn fixed points, each of which is a cycle of length 1.

It is well known that, if d divides n, then the number of elements of C_n that are of order d is $\varphi(d)$. So if c divides m and d divides n, then the number of pairs $(i, j) \in \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$ such that σ^i is of order c and τ^j is of order d is $\varphi(c)\varphi(d)$. For such (i, j), $\sigma^i\tau^j$ is of order $\operatorname{lcm}(c, d)$ because σ^i and τ^j commute, hence each cycle of $\sigma^i\tau^j$ has length $\operatorname{lcm}(c, d)$ and $A_{ij} = mn/\operatorname{lcm}(c, d)$. We are using the fact that each permutation in $C_m \times C_n$ has the property that all of its cycles are of equal length. Therefore, (4) follows from (5). □

Clearly, $a(1, n)$ reduces to (1); also, $a(m, n) = a(n, m)$ for all $m, n \geq 1$. Table 3 provides numerical values of $a(m, n)$ for small m and n.

Table 3: The number $a(m, n)$ of toroidal $m \times n$ binary arrays, allowing rotation of the rows and/or the columns but not reflection, for $m, n = 1, 2, \ldots, 8$.

2	3	4	6	8	14	20	36
3	7	14	40	108	362	1182	4150
4	14	64	352	2192	14624	99880	699252
6	40	352	4156	52488	699600	9587580	134223976
8	108	2192	52488	1342208	35792568	981706832	27487816992
14	362	14624	699600	35792568	1908897152	104715443852	5864063066500
20	1182	99880	9587580	981706832	104715443852	11488774559744	1286742755471400
36	4150	699252	134223976	27487816992	5864063066500	1286742755471400	288230376353050816

3 Rotations and reflections of rows and columns

Let $b(m, n)$ denote the number of orbits of the group action on $X_{m,n}$ by the group $D_m \times D_n$. In other words, $b(m, n)$ is the number of (distinct) toroidal $m \times n$ binary arrays, allowing rotation and/or reflection of the rows and/or the columns.

Theorem 2.

$$b(m, n) = b_1(m, n) + b_2(m, n) + b_3(m, n) + b_4(m, n),$$

where

$$b_1(m, n) = \frac{1}{4mn} \sum_{c \,|\, m} \sum_{d \,|\, n} \varphi(c)\varphi(d)\, 2^{mn/\operatorname{lcm}(c,d)},$$

$$\begin{aligned} &b_2(m, n) \\ &= \begin{cases} (4n)^{-1} 2^{(m+1)n/2}, & \text{if } m \text{ is odd;} \\ (8n)^{-1} [2^{mn/2} + 2^{(m+2)n/2}], & \text{if } m \text{ is even,} \end{cases} + \frac{1}{4n} \sum_{d \geq 2 :\, d \,|\, n} \varphi(d)\, 2^{mn/d} \\ &\quad + \begin{cases} (4n)^{-1} \sum' [2^{(m+1)\gcd(j,n)/2} - 2^{m \gcd(j,n)}], & \text{if } m \text{ is odd;} \\ (8n)^{-1} \sum' [2^{m \gcd(j,n)/2} + 2^{(m+2)\gcd(j,n)/2} - 2^{m \gcd(j,n)+1}], & \text{if } m \text{ is even,} \end{cases} \end{aligned}$$

with $\sum' := \sum_{1\le j\le n-1:\, n/\gcd(j,n) \text{ is odd}}$,

$$b_3(m,n) = b_2(n,m),$$

and

$$b_4(m,n) = \begin{cases} 2^{(mn-3)/2}, & \text{if } m \text{ and } n \text{ are odd;} \\ 3\cdot 2^{mn/2-3}, & \text{if } m \text{ and } n \text{ have opposite parity;} \\ 7\cdot 2^{mn/2-4}, & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

Proof. Again by Pólya's enumeration theorem,

$$b(m,n) = \frac{1}{4mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[2^{A_{ij}} + 2^{B_{ij}} + 2^{C_{ij}} + 2^{D_{ij}}],$$

where A_{ij} (resp., B_{ij}, C_{ij}, D_{ij}) is the number of cycles in the permutation $\sigma^i\tau^j$ (resp., $\sigma^i\tau^j\rho$, $\sigma^i\tau^j\theta$, $\sigma^i\tau^j\rho\theta$); here σ rotates the rows (row 0 becomes row 1, row 1 becomes row 2, ..., row $m-1$ becomes row 0), τ rotates the columns, ρ reflects the rows (rows 0 and $m-1$ are interchanged, rows 1 and $m-2$ are interchanged, ..., rows $\lfloor m/2\rfloor - 1$ and $m - \lfloor m/2\rfloor$ are interchanged), and θ reflects the columns.

By the proof of Theorem 1, we know the form of A_{ij}, and this gives the formula for $b_1(m,n)$.

Next we find (B_{i0}), the entries in the 0th column of matrix B. For $i = 0,1,\ldots,m-1$, the permutation $\sigma^i\rho$ can be described by its effect on the rows of $\{0,1,\ldots,m-1\}\times\{0,1,\ldots,n-1\}$. It reverses the first $m-i$ rows and reverses the last i rows. Since the reversal of k consecutive integers has $k/2$ transpositions if k is even and $(k-1)/2$ transpositions and one fixed point if k is odd, the permutation of $\{0,1,\ldots,m-1\}$ induced by $\sigma^i\rho$ has $(m-1)/2$ transpositions and one fixed point if m is odd, and $m/2$ transpositions if i is even and m is even, and $(m-2)/2$ transpositions and two fixed points if i is odd and m is even. These numbers must be multiplied by n for the permutation $\sigma^i\rho$ of $\{0,1,\ldots,m-1\}\times\{0,1,\ldots,n-1\}$. The results are that $B_{i0} = (m+1)n/2$ if m is odd, $B_{i0} = mn/2$ if i is even and m is even, and $B_{i0} = (m+2)n/2$ if i is odd and m is even. Therefore, $(4mn)^{-1}\sum_{i=0}^{m-1} 2^{B_{i0}} = (4n)^{-1}2^{(m+1)n/2}$ if m is odd, whereas $(4mn)^{-1}\sum_{i=0}^{m-1} 2^{B_{i0}} = (8n)^{-1}[2^{mn/2} + 2^{(m+2)n/2}]$ if m is even, and this gives the first term in the formula for $b_2(m,n)$.

We turn to B_{ij} for $i = 0,1,\ldots,m-1$ and $j = 1,2,\ldots,n-1$. First, by a property of cyclic groups, τ^j has order $d := n/\gcd(j,n)$. If d is even, then, since $\sigma^i\rho$ has order 2 (see the preceding paragraph), $\sigma^i\tau^j\rho$ has order d and all of its cycles have length d. In this case, $B_{ij} = mn/d = m\gcd(j,n)$. Suppose then that d is odd. There are three cases: (*i*) m odd, (*ii*) i even and m even, and (*iii*) i odd and m even. Recall that $\sigma^i\rho$ reverses the first $m-i$ rows and reverses the last i rows. In case (*i*), one row is fixed by $\sigma^i\rho$, so cycles of $\sigma^i\tau^j\rho$ in this row have length d and all others have length $2d$. We find that $B_{ij} = n/d + (m-1)n/(2d) = (m+1)n/(2d) = (m+1)\gcd(j,n)/2$. In case (*ii*), no rows are fixed by $\sigma^i\rho$, so all cycles of $\sigma^i\tau^j\rho$ have length $2d$. It follows that $B_{ij} = mn/(2d) = m\gcd(j,n)/2$. In case (*iii*), two rows are fixed by $\sigma^i\rho$, so cycles of $\sigma^i\tau^j\rho$ in these rows have length d and all others have length $2d$. We conclude that $B_{ij} = 2n/d + (m-2)n/(2d) = (m+2)\gcd(j,n)/2$. If the formula for B_{ij} that holds when d is even were valid generally, we would have the second term in the formula for $b_2(m,n)$. The third term in the formula for $b_2(m,n)$ is a correction to the second term to treat the cases (i)–(iii) in which d is odd.

Next, the formula for $b_3(m,n)$ follows by symmetry. More explicitly,

$$b_3(m,n) = \frac{1}{4mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} 2^{C_{ij}} = \frac{1}{4nm}\sum_{j=0}^{n-1}\sum_{i=0}^{m-1} 2^{B_{ji}} = b_2(n,m)$$

because the number of cycles C_{ij} of $\sigma^i\tau^j\theta$ acting on $m\times n$ arrays is equal to the number of cycles B_{ji} of $\sigma^j\tau^i\rho$ acting on $n\times m$ arrays.

Finally, we consider $b_4(m,n)$. For $i=0,1,\ldots,m-1$ and $j=0,1,\ldots,n-1$, $\sigma^i\tau^j\rho\theta$ has the effect of reversing the first $m-i$ rows, reversing the last i rows, reversing the first $n-j$ columns, and reversing the last j columns. If m and n are odd, then there is one fixed point and $(mn-1)/2$ transpositions, so $D_{ij} = (mn+1)/2$ for all i and j, hence $b_4(m,n) = 2^{(mn-3)/2}$. If m is odd and n is even, then $D_{ij} = mn/2$ for all i and even j and $D_{ij} = mn/2+1$ for all i and odd j. This leads to $b_4(m,n) = (1/8)[2^{mn/2} + 2^{mn/2+1}] = 3\cdot 2^{mn/2-3}$. If m is even and n is odd, then $D_{ij} = mn/2$ for even i and all j and $D_{ij} = mn/2+1$ for odd i and all j. This leads to the same formula for $b_4(m,n)$. Finally, if m and n are even, then $D_{ij} = mn/2$ unless i and j are both odd, in which case $D_{ij} = mn/2+2$. This implies that $b_4(m,n) = (1/4)[(3/4)2^{mn/2} + (1/4)2^{mn/2+2}] = 7\cdot 2^{mn/2-4}$.

This completes the proof. □

It is easy to check that $b(1,n)$ reduces to (2) and that $b(m,n) = b(n,m)$ for all $m,n\ge 1$. Table 4 provides numerical values of $b(m,n)$ for small m and n.

Table 4: The number $b(m,n)$ of toroidal $m\times n$ binary arrays, allowing rotation and/or reflection of the rows and/or the columns, for $m,n = 1,2,\ldots,8$.

2	3	4	6	8	13	18	30
3	7	13	34	78	237	687	2299
4	13	36	158	708	4236	26412	180070
6	34	158	1459	14676	184854	2445918	33888844
8	78	708	14676	340880	8999762	245619576	6873769668
13	237	4236	184854	8999762	478070832	26185264801	1466114420489
18	687	26412	2445918	245619576	26185264801	2872221202512	321686550498774
30	2299	180070	33888844	6873769668	1466114420489	321686550498774	72057630729710704

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