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Dedekind Sums with Arguments Near Euler's Number e

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Abstract

We study the asymptotic behaviour of the classical Dedekind sums $s(m/n)$ for convergents m/n of e, e^2, and $(e+1)/(e-1)$, where $e = 2.71828\ldots$ is Euler's number. Our main tool is the Barkan-Hickerson-Knuth formula, which yields a precise description of what happens in all cases.

1 Introduction and results

Dedekind sums have quite a number of interesting applications in analytic number theory (modular forms), algebraic number theory (class numbers), lattice point problems and algebraic geometry (for instance [1, 7, 9, 12]).

Let n be a positive integer and $m \in \mathbb{Z}$, $(m, n) = 1$. The classical Dedekind sum $s(m/n)$ is defined by

$$s(m/n) = \sum_{k=1}^{n} ((k/n))((mk/n))$$

where $((\ldots))$ is the usual sawtooth function (for example, [9, p. 1]). In the present setting it is more natural to work with

$$S(m/n) = 12s(m/n)$$

instead.

In the previous paper [3] we used the Barkan-Hickerson-Knuth-formula to study the asymptotic behaviour of $S(s_k/t_k)$ for the convergents s_k/t_k of a periodic simple continued fraction $\alpha = [a_0, a_1, a_2, \ldots]$, i. e., for a quadratic irrational α. In this situation two cases are possible: The sequence $S(s_k/t_k)$ either remains bounded with a finite number of cluster points or it essentially behaves like $C \cdot k$ for some constant C depending on α. In the latter case $S(s_k/t_k) - C \cdot k$ remains bounded with finitely many cluster points. The former case occurs, for instance, if the period length of α is odd.

Since the order of magnitude of $|S(m/n)|$ is $\log^2 n$ on average [4], quadratic irrationalities produce Dedekind sums of a considerably smaller size. In fact, the inequality $k \leq 2\log t_k/\log 2 + 1$ was already proved in 1841 [11]. Accordingly, if $|S(s_k/t_k)|$ is not bounded, we have $|S(s_k/t_k)| = O(\log t_k)$ for a quadratic irrational α.

Because the structure of the continued fraction expansions of transcendental numbers like e or e^2 is similar to that of quadratic irrationals [8, p. 123 ff.], nothing prevents us from applying the Barkan-Hickerson-Knuth-formula ((2) below) to these cases. It turns out that the asymptotic behaviour of Dedekind sums is quite similar to the case of quadratic irrationals. Only the case "$S(s_k/t_k)$ bounded" cannot occur, as the said formula shows, since the continued fraction expansions of these numbers have unbounded digits. We shall show

Theorem 1. *For a nonnegative integer k put*

$$L(k) = \begin{cases} \frac{k}{3}, & \text{if } k \equiv 0, 1, 5 \pmod 6; \\ -\frac{k}{3}, & \text{otherwise.} \end{cases}$$

Then we have, for the convergents s_k/t_k of Euler's number e,

$$S(s_k/t_k) - L(k) = O\left(\frac{1}{k}\right) + \begin{cases} e - 3, & \text{if } k \equiv 3 \pmod 6; \\ e - 3 - \frac{5}{6}, & \text{if } k \equiv 1 \pmod 6; \\ e - 3 + \frac{2}{3}, & \text{if } k \equiv 2 \pmod 6; \\ e - 3 + \frac{5}{6}, & \text{if } k \equiv 4 \pmod 6; \\ e - 3 - \frac{2}{3}, & \text{if } k \equiv 5 \pmod 6. \end{cases}$$

The continued fraction expansion of $e^2 = 7.38905\ldots$ is more complicated than that of e. This has the effect that the analogue of Theorem 1 also looks more complicated. We obtain

Theorem 2. *For a nonnegative integer k put*

$$L(k) = \begin{cases} -\frac{3k}{5}, & \text{if } k \equiv 1, 2, 3 \pmod{10}; \\ \frac{3k}{5}, & \text{if } k \equiv 6, 7, 8 \pmod{10}; \\ -\frac{6k}{5}, & \text{if } k \equiv 0, 4 \pmod{10}; \\ \frac{6k}{5}, & \text{if } k \equiv 5, 9 \pmod{10}. \end{cases}$$

Then we have, for the convergents s_k/t_k of the number e^2,

$$S(s_k/t_k) - L(k) = O\left(\frac{1}{k}\right) + \begin{cases} e^2 - 7, & \text{if } k \equiv 0 \pmod{10}; \\ e^2 - \frac{37}{5}, & \text{if } k \equiv 1 \pmod{10}; \\ e^2 - \frac{29}{5}, & \text{if } k \equiv 2 \pmod{10}; \\ e^2 - \frac{31}{5} + \frac{1}{2}, & \text{if } k \equiv 3 \pmod{10}; \\ e^2 - \frac{16}{5}, & \text{if } k \equiv 4 \pmod{10}; \\ e^2 + 1, & \text{if } k \equiv 5 \pmod{10}; \\ e^2 + \frac{7}{5}, & \text{if } k \equiv 6 \pmod{10}; \\ e^2 - \frac{1}{5}, & \text{if } k \equiv 7 \pmod{10}; \\ e^2 - \frac{4}{5} + \frac{1}{2}, & \text{if } k \equiv 8 \pmod{10}; \\ e^2 - \frac{14}{5}, & \text{if } k \equiv 9 \pmod{10}. \end{cases}$$

Finally, we consider the case of $e^* = (e+1)/(e-1)$, which is fairly simple.

Theorem 3. *For a nonnegative integer k put*

$$L(k) = \begin{cases} -2k, & \text{if } k \text{ is even;} \\ 2k, & \text{if } k \text{ is odd.} \end{cases}$$

Then we have, for the convergents s_k/t_k of e^*,

$$S(s_k/t_k) - L(k) = O\left(\frac{1}{k}\right) + \begin{cases} e^* - 2, & \text{if } k \text{ is even;} \\ e^* - 1, & \text{if } k \text{ is odd.} \end{cases}$$

2 Proofs

We start with the continued fraction expansion $[a_0, a_1, a_2, \ldots]$ of an arbitrary irrational number. The numerators and denominators of its convergents s_k/t_k are defined by the recursion formulas

$$\begin{aligned} &s_{-2} = 0, \quad s_{-1} = 1, \quad s_k = a_k s_{k-1} + s_{k-2} \text{ and} \\ &t_{-2} = 1, \quad t_{-1} = 0, \quad t_k = a_k t_{k-1} + t_{k-2}, \text{ for } k \geq 0. \end{aligned} \tag{1}$$

The Barkan-Hickerson-Knuth formula says that for $k \geq 0$

$$S(s_k/t_k) = \sum_{j=1}^{k} (-1)^{j-1} a_j + \begin{cases} (s'_k + t'_{k-1})/t'_k - 3, & \text{if } k \text{ is odd;} \\ \\ (s'_k - t'_{k-1})/t'_k, & \text{if } k \text{ is even;} \end{cases} \tag{2}$$

[2], [5], [6]. Here s'_k and t'_k are defined as in (1), but for the number $[0, a_1, a_2, \ldots]$ instead of $[a_0, a_1, a_2, \ldots]$. We prove the simplest case first.

Proof of Theorem 3. The digits a_j of the continued fraction expansion of e^* are $a_j = 4j+2$, $j = 0, 1, 2, \ldots$ [8, p. 124]. An easy calculation shows that for $k \ge 0$

$$\sum_{j=1}^{k} (-1)^{j-1} a_j = \begin{cases} -2k, & \text{if } k \text{ is even;} \\ 2k+4, & \text{if } k \text{ is odd.} \end{cases} \tag{3}$$

Now s'_k/t'_k converges against $[0, a_1, a_2, \ldots] = e^* - 2$, and $|e^* - 2 - s'_k/t'_k| < 1/t'^2_k$ [8, p. 37]. We remarked in the Introduction that $k = O(\log t'_k)$. Hence we also have $|e^* - 2 - s'_k/t'_k| = O(1/k)$. Finally, (1) gives $t'_{k-1}/t'_k = t'_{k-1}/(a_k t'_{k-1} + t'_{k-2}) \le 1/a_k = O(1/k)$. These observations, together with (2) and (3), prove the theorem. □

Proof of Theorem 1. In the case of $e = [a_0, a_1, a_2, \ldots]$ one easily derives from [8, p. 124] that

$$a_j = \begin{cases} 2, & \text{if } j = 0; \\ 2(j-1)/3 + 2, & \text{if } j \equiv 2 \pmod 3; \\ 1, & \text{otherwise.} \end{cases}$$

An elementary computation with arithmetic series (which is more laborious than that of the proof of Theorem 3) yields

$$\sum_{j=1}^{k} (-1)^{j-1} a_j = \begin{cases} \frac{k}{3}, & \text{if } k \equiv 0 \pmod 6; \\ -\frac{k}{3} + 1, & \text{if } k \equiv 3 \pmod 6; \\ \frac{k-1}{3} + 1, & \text{if } k \equiv 1 \pmod 6; \\ -\frac{k-1}{3}, & \text{if } k \equiv 4 \pmod 6; \\ -\frac{k-2}{3} - 1, & \text{if } k \equiv 2 \pmod 6; \\ \frac{k-2}{3} + 2, & \text{if } k \equiv 5 \pmod 6. \end{cases} \tag{4}$$

In the same way as in the proof Theorem 3 we have $s'_k/t'_k \to e-2$ and $|e-2-s'_k/t'_k| = O(1/k)$. If $k \equiv 2 \pmod 3$, we note $t'_{k-1}/t'_k \le 1/a_k = O(1/k)$. If $k \equiv 0 \pmod 3$ and $k \ge 3$, we have

$$\frac{t'_{k-1}}{t'_k} = \frac{t'_{k-1}}{t'_{k-1} - t'_{k-2}} = \frac{1}{1 + t'_{k-2}/t'_{k-1}}. \tag{5}$$

Since $t'_{k-2}/t'_{k-1} = O(1/k)$, this shows $t'_{k-1}/t'_k = 1 + O(1/k)$. If $k \equiv 1 \pmod 3$ and $k \ge 4$, formula (5) also holds. Together with $t'_{k-2}/t'_{k-1} = 1 + O(1/k)$, it gives $t'_{k-1}/t'_k = 1/2 + O(1/k)$. These observations, combined with (2) and (4), prove the theorem. □

Proof of Theorem 2. The proof follows the above pattern. One obtains from [8, p. 125]

$$a_j = \begin{cases} 7, & \text{if } j = 0; \\ (3j+7)/5, & \text{if } j \equiv 1 \pmod 5; \\ (3j+3)/5, & \text{if } j \equiv 4 \pmod 5; \\ 12j/5 + 6, & \text{if } j \equiv 0 \pmod 5, j > 0; \\ 1, & \text{otherwise.} \end{cases}$$

Further,

$$\sum_{j=1}^{k}(-1)^{j-1}a_j = \begin{cases} -\frac{6k}{5}, & \text{if } k \equiv 0 \pmod{10}; \\ \frac{6k}{5}+11, & \text{if } k \equiv 5 \pmod{10}; \\ -\frac{3(k-1)}{5}+2, & \text{if } k \equiv 1 \pmod{10}; \\ \frac{3(k-1)}{5}+9, & \text{if } k \equiv 6 \pmod{10}; \\ -\frac{3(k-2)}{5}+1, & \text{if } k \equiv 2 \pmod{10}; \\ \frac{3(k-2)}{5}+10, & \text{if } k \equiv 7 \pmod{10}; \\ -\frac{3(k-3)}{5}+2, & \text{if } k \equiv 3 \pmod{10}; \\ \frac{3(k-3)}{5}+9, & \text{if } k \equiv 8 \pmod{10}; \\ -\frac{6(k-4)}{5}-1, & \text{if } k \equiv 4 \pmod{10}; \\ \frac{6(k-4)}{5}+12, & \text{if } k \equiv 9 \pmod{10}. \end{cases} \tag{6}$$

In the same way as in the proof of Theorem 1 we observe $|e^2 - 7 - s'_k/t'_k| = O(1/k)$ and

$$\frac{t'_{k-1}}{t'_k} = O\left(\frac{1}{k}\right) + \begin{cases} 0, & \text{if } k \equiv 0, 1, 4 \pmod{5}; \\ 1, & \text{if } k \equiv 2 \pmod{5}; \\ \frac{1}{2}, & \text{if } k \equiv 3 \pmod{5}. \end{cases}$$

Thereby, and by (6), we obtain the theorem. □

Remark 4. 1. It is easy to see that the error term $O(1/k)$ in the theorems cannot be made smaller. Accordingly, the convergence is rather slow, which is a further difference between the present cases and the case of quadratic irrationals.

2. The continued fraction expansions of $e^{2/q}$ and $(e^{2/q}+1)/(e^{2/q}-1)$ for integers $q \geq 1$ have a shape similar to that of e, e^2, and e^* [8, p. 124 f.]. The same holds for the the numbers $\tan(1/q)$. Therefore, similar theorems about Dedekind sums can be expected for the convergents of these numbers.

3. Due to a theorem of Hurwitz [8, p. 119] one may even hope for similar results for the numbers

$$\frac{ae^{2/q}+b}{ce^{2/q}+d},$$

where the integer q is ≥ 1 and $a, b, c, d \in \mathbb{Z}$ are such that $ad - bc \neq 0$. It seems, however, that not all continued fraction expansions of these numbers are explicitly known.

4. The continued fraction expansions of the numbers

$$\sum_{j=0}^{\infty} b^{-2^j},\ b \in \mathbb{Z}, b \geq 3,$$

are also known [10]. They are, however, much more involved than those considered here. Accordingly, the asymptotic behaviour of the corresponding Dedekind sums seems to be far more complicated.

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