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On Integers for Which the Sum of Divisors is the Square of the Squarefree Core

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Abstract

We study integers $n > 1$ satisfying the relation $\sigma(n) = \gamma(n)^2$, where $\sigma(n)$ and $\gamma(n)$ are the sum of divisors and the product of distinct primes dividing n, respectively. We show that the only solution n with at most four distinct prime factors is $n = 1782$. We show that there is no solution which is fourth power free. We also show that the number of solutions up to $x > 1$ is at most $x^{1/4+\epsilon}$ for any $\epsilon > 0$ and all $x > x_\epsilon$. Further, call n primitive if no proper unitary divisor d of n satisfies $\sigma(d) \mid \gamma(d)^2$. We show that the number of primitive solutions to the equation up to x is less than x^ϵ for $x > x_\epsilon$.

1 Introduction

At the Western Number Theory conference in 2000, the second author asked for all positive integer solutions n to the equation

$$\sigma(n) = \gamma(n)^2 \tag{1}$$

(denoted "De Koninck's equation"), where $\sigma(n)$ is the sum of all positive divisors of n, and $\gamma(n)$ is the product of the distinct prime divisors of n, the so-called "core" of n. It is easy to check that $n = 1$ and $n = 1782$ are solutions, but, as of the time of writing, no other solutions are known. A computer search for all $n \leq 10^{11}$ did not reveal any other solution. The natural conjecture (coined the "De Koninck's conjecture") is that there are no other solutions. It is included in Richard Guy's compendium [1, Section B11].

It is not hard to see, and we prove such facts shortly, that any non-trivial solution n must have at least three prime factors, must be even, and can never be squarefree. The fourth author [2] has a derivation that the number of solutions with a fixed number of prime factors is finite. Indeed, he did this for the broader class of positive solutions n to the equation $\sigma(n) = a\gamma(n)^K$ where $K \geq 2$ and $1 \leq a \leq L$ with K and L fixed parameters. Other than this, there has been little progress on De Koninck's conjecture.

Here, we show that the above solutions $n = 1,\ 1782$ are the only ones having $\omega(n) \leq 4$. As usual, $\omega(n)$ stands for the number of distinct prime factors of n. The method relies on elementary upper bounds for the possible exponents of the primes appearing in the factorization of n and then uses resultants to solve the resulting systems of polynomial equations whose unknowns are the prime factors of n.

We then show that if an integer n is fourth power free (i.e. $p^4 \nmid n$ for all primes p), then n cannot satisfy De Koninck's equation (1). We then count the number of potential solutions n up to x. Pollack and Pomerance [4], call a positive integer n to be *prime–perfect* if n and $\sigma(n)$ share the same set of prime factors. Obviously, any solution n to the De Koninck's equation is also prime–perfect. Pollack and Pomerance show that the set of prime–perfect numbers is infinite and the counting function of prime–perfects $n \leq x$ has cardinality at most $x^{1/3+o(1)}$ as $x \to \infty$. By using the results of Pollack and Pomerance, we show that the number of solutions $n \leq x$ to De Koninck's equation is at most $x^{1/4+\epsilon}$ for any $\epsilon > 0$ and all $x > x_\epsilon$.

By restricting to so-called "primitive" solutions, using Wirsing's method [5], we obtain an upper bound of $O(x^\epsilon)$ for all $\epsilon > 0$. The notion of primitive that is used is having no proper unitary divisor $d \mid n$ satisfying $\sigma(d) \mid \gamma(d)^2$. In a final section of comments, we make some remarks about the related problem of identifying those integers n such that $\gamma(n)^2 \mid \sigma(n)$.

In summary: the aim of this paper is to present items of evidence for the truth of De Koninck's conjecture, and to indicate the necessary structure of a possible counter example.

Any non-trivial solution other than 1782 must be even, have one prime divisor to power 1 and possibly one prime divisor to a power congruent to 1 modulo 4, with other odd prime divisors being to even powers. At least one prime divisor must appear with an exponent 4 or more. Finally, any counter example must be greater than 10^{11}.

We use the following notations, most of which have been recorded already: $\sigma(n)$ is the sum of divisors, $\gamma(n)$ is the product of the distinct primes dividing n, if p is prime $v_p(n)$ is the highest power of p which divides n, $\omega(n)$ is the number of distinct prime divisors of n, and $\mathcal{K}$ is the set of all solutions to $\sigma(n) = \gamma(n)^2$. The symbols p, q, p_i and q_i with $i = 1, 2, \ldots$ are reserved for odd primes.

2 Structure of solutions

First we derive the shape of the members of $\mathcal{K}$.

Lemma 1. *If $n > 1$ is in $\mathcal{K}$, then*

$$n = 2^e p_1 \prod_{i=2}^{s} p_i^{a_i},$$

where $e \geq 1$ and a_i is even for all $i = 3, \ldots, s$. Furthermore, either a_2 is even in which case $p_1 \equiv 3 \pmod 8$, or $a_2 \equiv 1 \pmod 4$ and $p_1 \equiv p_2 \equiv 1 \pmod 4$.

Proof. Firstly, we note that n must be even: indeed, if $n > 1$ satisfies $\sigma(n) = \gamma(n)^2$ and n is odd, then $\sigma(n)$ must be odd so that the exponent of each prime dividing n must be even, making n a perfect square. But then $n < \sigma(n) = \gamma(n)^2 \leq n$, a contradiction.

Secondly, since n is even, it follows that $2^2 \| \gamma(n)^2$. Write

$$n = 2^e \prod_{i=1}^{s} p_i^{a_i}$$

with distinct odd primes $p_1, \ldots, p_s$ and positive integer exponents $a_1, \ldots, a_s$, where the primes are arranged in such a way that the odd exponents appear at the beginning and the even ones at the end. Using the fact that $\sigma(2^e) = 2^{e+1} - 1$ is odd, we get that $2^2 \| \prod_{i=1}^{s} \sigma(p_i^{a_i})$. Thus, there are at most two indices i such that $\sigma(p_i^{a_i})$ is even, with all the other indices being odd. But if p is odd and $\sigma(p^a)$ is also odd, then a is even. Thus, either only a_1 is odd, or only a_1 and a_2 are odd. Now let us show that there is at least one exponent which is 1. Assuming that this is not so, the above argument shows that $a_1 \geq 3$ and that $a_i \geq 2$ for $i = 2, \ldots, s$. Thus,

$$4p_1^2 \prod_{i=2}^{s} p_i^2 = \gamma(n)^2 = \sigma(n) \geq \sigma(2)\sigma(p_1^3) \prod_{i=2}^{s} \sigma(p_i^2) > 3p_1^3 \prod_{i=2}^{s} p_i^2,$$

leading to $p_1 < 4/3$, which is impossible. Hence, $a_1 = 1$. Finally, if a_2 is even, then $2^2 \| \sigma(p_1)$ showing that $p_1 \equiv 3 \pmod 8$, while if a_2 is odd, then $2 \| \sigma(p_1)$ and $2 \| \sigma(p_2^{a_2})$, conditions which easily lead to the conclusion that $p_1 \equiv p_2 \equiv 1 \pmod 4$ and $a_2 \equiv 1 \pmod 4$. $\square$

3 Solutions with $\omega(n) \le 4$

Theorem 2. *Let $n \in \mathcal{K}$ with $\omega(n) \le 4$. Then $n = 1$ or $n = 1782$.*

Proof. Using Lemma 1, we write $n = 2^\alpha pm$, where $\alpha > 0$ and m is coprime to $2p$.

We first consider the case $p = 3$. If additionally $m = 1$, we then get that $\sigma(n) = 6^2$, and we get no solution. On the other hand, if $m > 1$, then $\sigma(m)$ is a divisor of $\gamma(n)^2/4$ and must therefore be odd. This means that every prime factor of m appears with an even exponent. Say $q^\beta \| m$. Then

$$\sigma(q^\beta) = q^\beta + \cdots + q + 1$$

is coprime to $2q$ and is larger than $3^2 + 3 + 1 > 9$. Thus, there exists a prime factor of m other than 3 or q, call it r, which divides $q^\beta + \cdots + q + 1$, implying that it also divides m and that it appears in the factorization of m with an even exponent. Since $\omega(n) \le 4$, we have $m = q^\beta r^\gamma$. Now

$$q^\beta + \cdots + q + 1 = 3^i r^j \qquad \text{and} \qquad r^\gamma + \cdots + r + 1 = 3^k q^\ell,$$

where $i + k \le 2$ and $j, \ell \in \{1, 2\}$. Thus,

$$(q^\beta + \cdots + q + 1)(r^\gamma + \cdots + r + 1) = 3^{i+k} q^\ell r^j.$$

The left–hand side of this equality is greater than or equal to $3q^\beta r^\gamma$. In the case where $\beta > 2$, we have $\beta \ge 4$, so that $q^4 r^2 \le q^\beta r^\alpha \le 9q^2 r^2$, giving $q \le 3$, which is a contradiction. The same contradiction is obtained if $\gamma > 2$.

Thus, $\beta = \gamma = 2$. If $l = j = 2$, we then get that

$$(q^2 + q + 1)(r^2 + r + 1) = 3^{i+k} q^2 r^2,$$

leading to $\sigma(2^\alpha) \mid 3^{2-i-j}$. The only possibility is $\alpha = 1$ and $i + j = 1$, showing that $i = 0$ or $j = 0$. Since the problem is symmetric, we treat only the case $i = 0$. In that case, we get $q^2 + q + 1 = r^2$, which is equivalent to $(2q + 1)^2 + 3 = (2r)^2$, which has no convenient solution (q, r).

If $j = \ell = 1$, we then get that

$$q^2 r^2 < (q^2 + q + 1)(r^2 + r + 1) < 9qr,$$

implying that $qr < 9$, which is false.

Hence, it remains to consider the case $j = 2$ and $\ell = 1$, and viceversa. Since the problem is symmetric in q and r, we only look at $j = 2$ and $\ell = 1$. In that case, we have

$$q^2 r^2 < (q^2 + q + 1)(r^2 + r + 1) = 3^{i+k} r^2 q,$$

so that $q < 3^{i+k}$. Since $q > 3$, this shows that $i = k = 1$ and $q \in \{5, 7\}$. Therefore, $r^2 + r + 1 = 75,\ 147$, and neither gives a convenient solution n.

From now on, we can assume that $p > 3$, so that $p + 1 = 2^u m_1$, where $u \in \{1, 2\}$ and $m_1 > 1$ is odd. Let q be the largest prime factor of m_1. Clearly, $p + 1 \ge 2q$, so that $q < p$. Moreover, since $\omega(n) \le 4$ we have

$$p < 4q^4 < q^6,$$

so that $q > p^{1/6}$. Let again β be such that $q^\beta \| n$. We can show that $\beta \le 77$. Indeed, assuming that $\beta \ge 78$, we first observe that

$$p^{13} < q^{78} \le q^\beta < \sigma(q^\beta),$$

and write

$$\sigma(q^\beta) = 2^v m_2,$$

where $v \in \{0,1\}$ and m_2 is coprime to $2q$. If m_2 divides p^2, we get that

$$p^{13} < \sigma(q^\beta) \le 2p^2,$$

which is a contradiction. Thus, there exists another prime factor r of n, and $m_2 \le p^2r^2$. Hence,

$$p^{13} < \sigma(q^\beta) < 2p^2r^2 < p^3r^2,$$

implying that $r > p^5$. Let γ be such that $r^\gamma \| n$. Then

$$r + 1 \le \sigma(r^\gamma) \le 2p^2q^2 < p^5,$$

which is a contradiction. Thus, $\beta \le 77$.

Say r doesn't appear in the factorization of $(p+1)\sigma(q^\beta)$. Then we need to solve the system of equations

$$p + 1 = 2^u q^w \qquad \text{and} \qquad q^\beta + \cdots + q + 1 = 2^v p^z,$$

where $\beta \in \{1, \ldots, 77\},\ u \in \{1,2\},\ 0 \le v \le 2-u,\ \{w,z\} \subseteq \{1,2\}$, which we can solve with resultants. This gives us a certain number of possibilities for the pair (p,q). If $\omega(n) = 3$, we have $\sigma(n) = 4p^2q^2$, and we find n. If $\omega(n) = 4$, then $\sigma(r^\gamma)$ is a divisor of $2p^2q^2$ and we find certain possibilities for the pair (r,γ). Then we extract n from the relation $\sigma(n) = 4p^2q^2r^2$.

Now say r appears in the factorization of $(p+1)\sigma(q^\beta)$. We then write

$$p + 1 = 2^u q^w r^\delta \qquad \text{and} \qquad \sigma(q^\beta) = 2^v p^z r^\eta, \tag{2}$$

where $u \in \{1,2\},\ w \in \{1,2\},\ 0 \le v \le 2-u,\ z \in \{0,1,2\},\ \delta + \eta \in \{1,2\}$. If $z = 0$, then since $q > p^{1/6}$, we have that

$$q < \sigma(q^\beta) \le 2r^2 < r^3,$$

so that $r > q^{1/3} > p^{1/18}$. Now $\gamma \le 89$, for if not, then

$$p^5 < r^{90} \le r^\gamma < \sigma(r^\gamma) < 2p^2q^2 < p^5,$$

which is false.

Suppose now that $z > 0$. Then

$$q^w r^\delta < p < 4q^w r^\delta \tag{3}$$

from the first relation of (2), while

$$\frac{q^\beta}{2r^\eta} < p^z < \frac{2q^\beta}{r^\eta} \tag{4}$$

from the second relation of (2). If $z = 1$, we get from (3) and (4) that

$$r^{\delta+\eta} < 2q^{\beta-w} \qquad \text{and} \qquad r^{\delta+\eta} > \frac{q^{\beta-w}}{8}.$$

From the above left inequality and the fact that $\delta + \eta \geq 1$, we read that $\beta - w \geq 1$, and then from the right one that $9r^2 > 8r^{\delta+\eta} > q^{\beta-w} \geq q$, and thus $r^2 \geq 3r > q^{1/2}$, so that $r > q^{1/4} > p^{1/24}$. It now follows easily that $\gamma \leq 119$, for if not, then $\gamma \geq 120$ would give

$$p^5 < r^{120} \leq r^{\gamma} < \sigma(r^{\gamma}) \leq 2p^2q^2 < p^5,$$

which is a contradiction. Finally, if $z = 2$, we get from (4) that

$$\frac{q^{\beta/2}}{\sqrt{2}r^{\eta/2}} < p < \frac{\sqrt{2}q^{\beta/2}}{r^{\eta/2}},$$

which combined with (3) yields

$$r^{\delta+\eta/2} < \sqrt{2}q^{\beta/2-w} \qquad \text{and} \qquad r^{\delta+\eta/2} > \frac{q^{\beta/2-w}}{4\sqrt{2}}.$$

From the above left inequality and because $\delta + \eta/2 \geq 1/2$, we read that $\beta/2 > w$, implying that $\beta/2 - w \geq 1/2$. Thus,

$$4\sqrt{2}r^2 \geq 4\sqrt{2}r^{\delta+\eta/2} > q^{\beta/2-w} \geq q^{1/2}$$

and therefore

$$r^8 > 32r^4 \geq (4\sqrt{2}r^{\delta+\eta/2})^2 > q > p^{1/6},$$

showing that $r > p^{1/48}$. This shows that $\gamma \leq 239$, for if $\gamma \geq 240$, then

$$p^5 < r^{240} \leq r^{\gamma} < \sigma(r^{\gamma}) < 2p^2q^2 < p^5,$$

which is a contradiction. Thus, we need to solve

$$\begin{aligned} p+1 &= 2^u q^w r^{\delta}; \\ \sigma(q^{\beta}) &= 2^v p^z r^{\eta}; \\ \sigma(r^{\gamma}) &= 2^{\lambda} p^s q^t, \end{aligned}$$

where $1 \leq \beta \leq 77$, $1 \leq \gamma \leq 239$, $u \in \{1, 2\}$, $u + v + \lambda \leq 2$, $1 \leq w \leq 2$, $w + t \leq 2$, $\delta + \eta \in \{1, 2\}$, $z \in \{0, 1, 2\}$ and $s \in \{0, 1, 2\}$. This can be solved with resultants and it gives us a certain number of possibilities for the triplet (p, q, r). From $\sigma(n) = 4p^2q^2r^2$, we extract n by solving the equation for α, given p, q and r. Failure to detect an integer value for α means the candidate solution fails. A computer program went through all these steps and confirmed the conclusion of Theorem 2.

□

4 The case of fourth power free n

Theorem 3. *If $n > 1$ is in $\mathcal{K}$, then n is not fourth power free.*

Proof. Let us assume that the result is false, that is, that there exists some $n \in \mathcal{K}$ which is fourth power free. By Lemma 1 we can write

$$n = 2^e p_1 p_2^{a_2} \prod_{i=1}^{k} q_i^2,$$

where $a_2 \in \{0, 1\}$. Let $\mathcal{Q} = \{q_1, \ldots, q_k\}$. The idea is to exploit the fact that there exist at most two elements $q \in \mathcal{Q}$ such that $q \equiv 1 \pmod 3$. If there were three or more such elements, then 3^3 would divide $\prod_{q \in \mathcal{Q}} \sigma(q^2)$ and therefore a divisor of $\gamma(n)^2$, which is a contradiction.

We begin by showing that $k \leq 8$. To see this, let

$$\mathcal{R} = \left\{ r \in \mathcal{Q} : \gcd\left(\sigma(r^2), \prod_{q \in \mathcal{Q}} q\right) = 1 \right\}.$$

Then $\prod_{r \in \mathcal{R}} \sigma(r^2)$ divides p_1^2 (if $a_2 = 0$) and $p_1^2 p_2^2$ if $a_2 > 0$. It follows that $\sigma(r^2)$ is either a multiple of p_1 or of p_2 for each $r \in \mathcal{R}$. Since there can be at most two r's for which $\sigma(r^2)$ is a multiple of p_1, and at most two r's for which $\sigma(r^2)$ is a multiple of p_2, we get that $\#\mathcal{R} \leq 4$. When $r \in \mathcal{Q} \setminus \mathcal{R}$, we have, since $\sigma(r^2) > 9$, that $\sigma(r^2) = r^2 + r + 1$ is a multiple of some prime $q_{i_r} > 3$ for some $q_{i_r} \in \mathcal{Q}$. Now, since q_{i_r} is a prime divisor of $r^2 + r + 1$ larger than 3, it must satisfy $q_{i_r} \equiv 1 \pmod 3$. Since i_r can take the same value for at most two distinct primes r, and there are at most two distinct values of the index i_r, we get that $k - \#\mathcal{R} \leq 4$, which implies that $k \leq 8$, as claimed.

Next rewrite the equation $\sigma(n) = \gamma(n)^2$ as

$$\left(\frac{2^{e+1} - 1}{4}\right) \prod_{i=1}^{k} \left(\frac{q_i^2 + q_i + 1}{q_i^2}\right) = \left(\frac{p_1^2}{p_1 + 1}\right) \left(\frac{p_2^{2\delta_2}}{\sigma(p_2^{a_2})}\right), \tag{5}$$

where $\delta_2 = 0$ if $a_2 = 0$ and $\delta_2 = 1$ if $a_2 > 0$. The left–hand side of (5) is at most

$$\left(\frac{2^{e+1} - 1}{4}\right) \left(\prod_{q \leq 23} \frac{q^2 + q + 1}{q^2}\right) < 0.73(2^{e+1} - 1). \tag{6}$$

First assume that $a_2 = 0$. Then the right–hand side of (5) is

$$\frac{p_1^2}{p_1 + 1} \geq \frac{9}{4} = 2.25. \tag{7}$$

If $e = 1$, then the left–hand side of inequality (5) is, in light of (6), smaller than $0.73(2^2 - 1) < 2.22$, which contradicts the lower bound provided in (7). Thus, $e \in \{2, 3\}$, and

$$\frac{p_1^2}{p_1 + 1} \leq 0.73(2^4 - 1) = 10.95,$$

so that $p_1 \le 11$. Since $p_1 \equiv 3 \pmod 8$, we get that $p_1 \in \{3, 11\}$. If $p_1 = 11$, then $3 \in \mathcal{Q}$. If $p_1 = 3$, then since $e \in \{2, 3\}$, we get that either 5 or 7 is in $\mathcal{Q}$.

If $3 \in \mathcal{Q}$, then $13 \mid 3^2 + 3 + 1$, $61 \mid 13^2 + 13 + 1$ and $97 \mid 61^2 + 61 + 1$ are all three in $\mathcal{Q}$ and are congruent to 1 modulo 3, a contradiction.

If $5 \in \mathcal{Q}$, then $31 \mid 5^2 + 5 + 1$, $331 \mid 31^2 + 31 + 1$ and $7 \mid 331^2 + 331 + 1$ are all in $\mathcal{Q}$, a contradiction.

If $7 \in \mathcal{Q}$, then 7, $19 \mid 7^2 + 7 + 1$ and $127 \mid 19^2 + 19 + 1$ are all in $\mathcal{Q}$, a contradiction.

Assume next that $a_2 > 0$. Then, by Lemma 1, $p_1 \equiv p_2 \equiv 1 \pmod 4$. Since $e \in \{1, 2, 3\}$, it follows that one of $3, 5, 7$ divides n.

If $3 \mid n$, then $3 \in \mathcal{Q}$.

If $5 \mid n$, and 5 is one of p_1 or p_2, then $3 \mid \sigma(p_1 p_2^{a_2}) \mid n$, while if $5 \in \mathcal{Q}$, then $31 = 5^2 + 5 + 1$ is not congruent to 1 modulo 4 and divides n, implying that it belongs to $\mathcal{Q}$, and thus $3 \mid 31^2 + 31 + 1 \mid n$.

Finally, if $7 \mid n$, then 7 cannot be p_1 or p_2, meaning that 7 is in $\mathcal{Q}$ and therefore that $3 \mid 7^2 + 7 + 1$, which implies that $3 \mid n$.

To sum up, it is always the case that when $a_2 > 0$, necessarily 3 divides n.

Hence, $13 = 3^2 + 3 + 1$ divides n, so that either $13 \in \mathcal{Q}$, or not. If $13 \notin \mathcal{Q}$, then $7 \mid 13 + 1$ is in $\mathcal{Q}$, in which case $19 \mid 7^2 + 7 + 1$ divides n and it is not congruent to 1 modulo 4, implying that $19 \in \mathcal{Q}$ and thus that $127 \mid 19^2 + 19 + 1$ divides n and is not congruent to 1 modulo 4, so that $127 \in \mathcal{Q}$. Hence, all three numbers $7, 19, 127$ are in $\mathcal{Q}$, which again is a contradiction.

If $13 \in \mathcal{Q}$, then $61 \mid 13^2 + 13 + 1$ divides n.

If 61 is one of p_1 or p_2, then $31 \mid \sigma(p_1 p_2^{a_2})$ and $31 \equiv 3 \pmod 4$, so that $31 \in \mathcal{Q}$. Next $331 \mid 31^2 + 31 + 1$ is a divisor of n and it is not congruent to 1 modulo 4, implying that it belongs to $\mathcal{Q}$ and therefore that $13, 31, 331$ are all in $\mathcal{Q}$, a contradiction.

Finally, if $61 \in \mathcal{Q}$, then $97 \mid 61^2 + 61 + 1$ is a divisor of n. If $97 \in \mathcal{Q}$ we get a contradiction since 13 and 61 are already in $\mathcal{Q}$, while if 97 is one of p_1 or p_2, then $7 \mid \sigma(p_1 p_2^{a_2})$ is a divisor of n and therefore necessarily in $\mathcal{Q}$, again a contradiction. ∎

5 Counting the elements in $\mathcal{K} \cap [1, x]$

Let $\mathcal{K}(x) = \mathcal{K} \cap [1, x]$.

Theorem 4. *The estimate*

$$\#\mathcal{K}(x) \le x^{1/4 + o(1)}$$

holds as $x \to \infty$.

Proof. By Theorem 1.2 in [4], we have $\#\mathcal{K}(x) = x^{1/3+o(1)}$ as $x \to \infty$. It remains to improve the exponent 1/3 to 1/4. We recall the following result from [4].

Lemma 5. *If $\sigma(n)/n = N/D$ with $(N, D) = 1$, then given $x \ge 1$ and $d \ge 1$*

$$\#\{n \le x : D = d\} = x^{o(1)}$$

as $x \to \infty$.

Now let $n \in \mathcal{K}(x)$, assume that $n > 1$ and write it in the form $n = A \cdot B$ with A squarefree, B squarefull and $(A, B) = 1$. By Lemma 1, we have $A \in \{1, p_1, 2p_1, p_1p_2, 2p_1p_2\}$. Then

$$\frac{N}{D} = \frac{\sigma(n)}{n} = \frac{\gamma(n)^2}{n} = \frac{\gamma(A)^2}{A} \cdot \frac{\gamma(B)^2}{B} = \frac{A}{B/\gamma(B)^2}, \tag{8}$$

and $(A, B/\gamma(B)^2) = 1$. Since $\sigma(n) > n$, it follows that $B/\gamma(B)^2 < A$. Thus,

$$B/\gamma(B)^2 < \sqrt{AB/\gamma(B)^2} \le \sqrt{n} \le \sqrt{x}.$$

By Lemma 1 again, we can write $B = \delta C^2 D$, where C is squarefree, D is 4-full, $\delta \in \{1, 2^3\}$, and where δ, C and D are pairwise coprime. Then $B/\gamma(B)^2 = \delta/\gamma(\delta)^2 \times D/\gamma(D)^2$, so therefore $D/\gamma(D)^2 \le B/\gamma(B)^2 < x^{1/2}$. Because D is 4-full it follows that $D/\gamma(D)^2$ is squarefull and so the number of choices for $D/\gamma(D)^2$ is $O(x^{1/4})$. Hence, the number of choices for $B/\gamma(B)^2 \in \{D/\gamma(D)^2, 2D/\gamma(D)^2\}$ is also $O(x^{1/4})$, which together with Lemma 5 and formula (8) implies the desired conclusion. □

A positive integer d is said to be a *unitary divisor* of n if $d \mid n$ and $(d, n/d) = 1$; it is said to be a *proper unitary divisor* of n if it also satisfies $1 < d < n$. We will say that an integer $n \in \mathcal{K}(x)$ is *primitive* if no proper unitary divisor d of n satisfies $\sigma(d) \mid \gamma(d)^2$. Let us denote this subset of $\mathcal{K}(x)$ by $\mathcal{H}(x)$. Elements of $\mathcal{H}(x)$ can be considered as the *primitive solutions* of $\sigma(n) = \gamma(n)^2$. For example, the number $n = 1782 \in \mathcal{H}(x)$ since, although the proper divisor $d = 6$ of n satisfies $\sigma(d) \mid \gamma(d)^2$, it fails to be unitary. Also, it is interesting to observe that the condition $\sigma(d) \mid \gamma(d)^2$ seems to be very restrictive: for instance, the only positive integers $d < 10^8$ satisfying this condition are 6 and 1782; this is already an indication that the set $\mathcal{H}(x)$ is very thin. As a matter of fact, we now prove the following result.

Theorem 6. *Let $\epsilon > 0$ be given. Then, given any $\epsilon > 0$,*

$$\#\mathcal{H}(x) = O(x^\epsilon).$$

Proof. Let $n \in \mathcal{H}(x)$ and assume that $x > 0$ is large. Let a be the largest divisor of n such that all prime factors $p \mid a$ satisfy $p \le \log x$. Write $n = a \cdot b$ and write down the standard factorization of b into primes as

$$b = p_1^{\beta_1} \cdots p_k^{\beta_k}, \qquad \text{where} \qquad p_1 < \cdots < p_k.$$

Set $M := \lceil \log x / \log\log x \rceil$. Then, since $b \le n \le x$ and since for each i, we have $\log x < p_i$, we get

$$(\log x)^{\beta_1 + \cdots + \beta_k} < p_1^{\beta_1} \cdots p_k^{\beta_k} = b \le x,$$

implying that

$$\beta_1 + \cdots + \beta_k < \frac{\log x}{\log\log x}, \quad \text{so that} \quad k \le M.$$

Now assume that the positive integer a is given and that there is some positive integer b such that $n = a \cdot b$ is a primitive element of $\mathcal{K}$. We will show how to find b from a using the knowledge of the exponents $\beta_1, \ldots, \beta_k$.

Firstly, if a is already primitive, we then have $b=1$. So, suppose that a is not primitive.

Since $\sigma(a)\sigma(b)=\gamma(a)^2\gamma(b)^2$, and the two factors on the right hand side are coprime, we must have

$$d := \frac{\sigma(a)}{(\sigma(a),\gamma(a)^2)} \mid \gamma(b)^2.$$

Hence, let p_1 be the least prime dividing the left–hand side of the above relation. Note that the left–hand side is not 1, since otherwise we would have $\sigma(a) \mid \gamma(a)^2$, which is not possible since n is primitive.

Now replace a by $ap_1^{\beta_1}$ and proceed. If at step $i<k$, we have $d=1$, then the choice of the β_i's for $i=1,\ldots,k$ fails to generate an element of $\mathcal{K}$. We can then move on to the next choice. With success at every step, we generate b from a by finding primes $p_1,\ldots,p_k$ such that $a\cdot p_1^{\beta_1}\cdots p_k^{\beta_k}\in\mathcal{K}$.

To complete the proof, we only need to find an upper bound for

$$\#\{\text{choices for } a\}\cdot\#\{\text{choices for } (\beta_1,\ldots,\beta_k)\},$$

and this is the same as in Wirsing's proof [5] or [3, Theorem 7.8, pp. 1008-1010] for the case of multiperfect numbers:

$$\begin{aligned}
\#\{\text{choices for } (\beta_1,\ldots,\beta_k)\} &\le \#\{(\beta_1,\ldots,\beta_k) : \beta_1+\cdots+\beta_k \le M\} \le 2^M,\\
\#\{\text{choices for } a\} &\le \#\{n\le x : p\mid n \Rightarrow p\le \log x\}\\
&\le \#\{n\le x : p\mid n \Rightarrow \log^{\frac{3}{4}} x < p \le \log x\}\\
&\quad \times\#\{n\le x : p\mid n \Rightarrow p\le \log^{\frac{3}{4}} x\}\\
&\le 2^{4M}\times 2^M = 2^{5M},
\end{aligned}$$

in which case we obtain the upper bound

$$2^{6M} = x^{\frac{6\log 2}{\log\log x}} = x^{o(1)} \qquad \text{as} \qquad x\to\infty,$$

for the number of primitive $n\in\mathcal{K}(x)$, which completes the proof of this theorem. $\square$

6 Final remarks

Here, we briefly consider another question related to the problem of De Koninck, namely the one which consists in identifying those integers n satisfying $\gamma(n)^2 \mid \sigma(n)$. There is an infinite set of solutions $n=2^i3^j$ with $i\equiv 5 \pmod 6$, $j\equiv 1 \pmod 2$. If $n=2^i3^j$ satisfies $\gamma(n)^2\mid\sigma(n)$, then these two congruence conditions are also satisfied.

Indeed, first let $i=5+6k$, $j=1+2m$ and $n=2^i3^j$. Then $\sigma(2^i)=2^{6(k+1)}-1\equiv 0 \pmod 9$ and $3^{2(m+1)}-1\equiv 0 \pmod 8$ so that $3^2\mid\sigma(2^i)$ and $2^2\mid\sigma(3^j)$.

Now assume that $n=2^i3^j$ and that the integers r and s are such that $2^2\mid\sigma(3^s)$ and $3^2\mid\sigma(2^r)$. It is well–known and quite easy to prove by elementary arguments that

$$\begin{aligned}
v_2(\sigma(3^s))+1 &= v_2((3+1)(s+1)) \ge 3, \text{ and}\\
v_3(\sigma(2^r)) &= v_3((2+1)(r+1)) \ge 2,
\end{aligned}$$

so by the first of these equations s is odd. By the second equation, we see that $r \equiv 2 \pmod 3$, so that $r \equiv 2 \pmod 6$ or $r \equiv 5 \pmod 6$. If the first of these was true, then r would be even, so that $3 \mid \sigma(2^r)$ would not be possible. Thus, we must have $r \equiv 5 \pmod 6$.

Observe that this infinite set $2^i 3^j$ does not exhaust all of the non-trivial solutions, even those with only two distinct prime factors. For example, $n = p^{q-2} q^{p-2}$ with $p = 2,\ q = 1093$ or $p = 83,\ q = 4871$ are both solutions, since in either case we have

$$p^2 \mid q^{p-1} - 1 \quad \text{and} \quad q^2 \mid p^{q-1} - 1,$$

and such divisibilities yield $p^2 q^2 \mid \sigma(p^{q-2} q^{p-2})$. Note also that there are many non-trivial solutions with 3 prime factors, for example 17 solutions up to 10^6 and 25 up to 4×10^6. Typical solutions have the form

$$\{2^3 3^3 5^5,\ 2^5 3^5 7^1,\ 2^9 3^4 11^1\}.$$

As a final note, let us mention that, given any arbitrary integer $k \geq 2$, one can easily check that the more general property $\gamma(n)^k | \sigma(n)$ is indeed satisfied by infinitely many positive integers n, namely those of the form

$$n = 2^{2i3^{k-1}-1} 3^{j2^{k-1}-1} \qquad (i \geq 1,\ j \geq 1).$$

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