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On the Modes of the Independence Polynomial of the Centipede

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Abstract

The independence polynomial of the graph called the centipede has only real zeros. It follows that this polynomial is log-concave, and hence unimodal. Levit and Mandrescu gave a conjecture about the mode of this polynomial. In this paper, the exact value of the mode is determined, and some central limit theorems for the sequence of the coefficients are established.

1 Introduction

All graphs considered in this paper are assumed to be finite and simple. The terminology is taken from [6], or may be found in any other standard book on graph theory. A graph G is denoted by $G = (E, V)$ where V is the set of its vertices and E is the set of its edges. A *tree* is a connected, cycle-free graph. A *spider* is a tree having at most one vertex of degree greater than 3. A *centipede* is a tree; it is denoted by $W_n = (A, B, E)$ $n \geq 1$, where $A \bigcup B$ is its vertex set, $A = (a_1, \ \ldots, \ a_n)$; $B = (b_1, \ \ldots, b_n)$ and the edge set $E = \{a_i b_i : 1 \leq i \leq n\} \bigcup \{b_i b_{i+1} \ : 1 \leq i \leq n\}$ (see Figure 1 shown below). A *stable set* is a set of pairwise nonadjacent vertices. The number of the stable sets having k elements is denoted by s_k. The *independence number* $\alpha(G)$ of a graph G is the cardinality of a maximum

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independent set. A graph is *well-covered* provided all its maximal stable sets have the same size; this notion was introduced by Plummer [21]. A graph is *very well-covered* if G has no isolated vertex and $|V| = 2\alpha(G)$. The independence polynomial of the graph G [13] is the polynomial $I(G,x) = \sum_{k=0}^{\alpha(G)} s_k x^k$. The study of these polynomials is important, since they supply many information about the graph itself. There are many open questions concerning these polynomials and/or their coefficients, especially unimodality and real zeros. Let us recall the following notions:

A real positive sequence $(a_k)_{k=0}^n$ is said to be *unimodal*, if there exist integers $k_0,\ k_1,\ k_0 \leq k_1$, such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k_0} = a_{k_0+1} = \cdots = a_{k_1} \geq a_{k_1+1} \geq \cdots \geq a_n.$$

The integers $k_0 \leq k \leq k_1$ are the *modes* of the sequence. A sequence is *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1},$ for $1 \leq k \leq n-1$. A real sequence (a_k) is said to *have no internal zeros* (NIZ) if $i < j, a_i \neq 0,\ a_j \neq 0$ then $a_l \neq 0$ for every $l, i \leq l \leq j$. A (NIZ) log-concave sequence is obviously unimodal, but the converse is not true. The sequence 1, 1, 4, 5, 4, 2,1 is unimodal but not log-concave. Note the importance of (NIZ): the sequence 0, 1, 0, 0, 2, 1 is log-concave but not unimodal. A real polynomial is *unimodal* (log-concave, symmetric, respectively) provided that the sequence of its coefficients is unimodal (log-concave, symmetric, respectively). If the inequalities in the log-concavity definition are strict, then the sequence is called *strictly log-concave* (SLC for short), and in this case, it has at most two consecutive modes. The next result (due to Newton) may be helpful in proving unimodality:

If the polynomial $\sum_{k=0}^{n} a_k x^k$ associated with the sequence $(a_k)_{k=0}^n$ has only real zeros then

$$a_k^2 \geq \frac{k+1}{k}\frac{n-k+1}{n-k}a_{k-1}a_{k+1},\ \text{for } 1 \leq k \leq n-1.$$

If the sequence $(a_k)_{k=0}^n$ is positive then it is SLC.

The determination of the mode rests heavily on the following result.

Theorem 1 (Darroch [10])**.** *Let $(a_k)_{k=0}^n$ be a positive real sequence. Suppose that the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros. Then every mode of the sequence $(a_k)_{k=0}^n$ satisfies*

$$\left\lfloor \frac{\sum_{k=0}^{n} k a_k}{\sum_{k=0}^{n} a_k} \right\rfloor \leq k_0 \leq \left\lceil \frac{\sum_{k=1}^{n} k a_k}{\sum_{k=0}^{n} a_k} \right\rceil. \tag{1}$$

Darroch's result was also proved independently in [3, 4]. A famous conjecture about the unimodality of the independence polynomial of a tree, stated by Erdős et al. [1] is

Conjecture 2. The independence polynomial of a tree is unimodal.

An important result concerning the independence polynomials of graphs is Hamidoune's result [15].

Theorem 3. *The independence polynomial of a claw free graph is log-concave.*

Recall that a *claw-free graph* is a graph which does not contain an induced subgraph isomorphic to $K_{1,3}$. A stronger result was proved recently by Chudnovsky and Seymour [9]:

Theorem 4. *The independence polynomial of a claw free graph has only real zeros.*

The centipedes are well-covered trees (see Figure 1 below). The independence polynomial of the centipede is log-concave [18, 19], in fact $I(W_n, x)$ has only real zeros and

$$I(W_n, x) = (1+x)^{\lfloor \frac{n}{2} \rfloor} H(x),$$

where $H(x)$ is the independence polynomial of a claw-free graph G, where $G \in \{M_n, L_n\}$ (see Figure 2 and Figure 3 below). The sequence $A_n = I(W_n, 1)$ is known and counts the number of words of length n, without adjacent $0'$s from the alphabet $\{0, 1, 2\}$. This is sequence A028859 in Sloane's online Encyclopedia. The unimodality of the polynomial $I(W_n, x)$ was proved by Levit and Mandrescu [20]. There it was conjectured that the mode k_n of $I_n(W, x)$ is

$$\begin{aligned} k_n &= n - f(n) \\ f(n) &= \begin{cases} 1 + \left\lfloor \frac{n}{5} \right\rfloor, & \text{if } 2 \le n \le 6; \\ f\left(2 + ((n-2) \bmod 5)\right) + 2\left\lfloor \frac{n-2}{5} \right\rfloor, & \text{if } n \ge 7. \end{cases} \end{aligned}$$

Wang and Zhu [24] proved that the zeros of $I_n(W, x)$ are real; they also determine the zeros explicitly. Using this fact, and Darroch's result, [10], they gave a counterexample for $n = 142$, by showing that $k_{142} = 85$, as stated by the conjecture, is not a mode of $I_n(W, x)$.

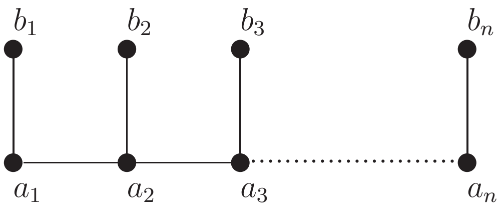

Figure 1: The centipede W_n

In this paper, using Theorems 1 and 5, we estimate, up to an additive factor of 1, the mode of the polynomial $I(W_n, x)$. We evaluate $I'(W_n, 1)/I(W_n, 1)$. The explicit form of the polynomial given by Wang and Zhu is explicit, but not suitable for our calculations. We use the reciprocal polynomial of $H(x)$, making the manipulation of $I'(W_n, 1)$ and $I(W_n, 1)$ easier. Finally, we prove that the sequence s_k is asymptotically normal.

2 The independence polynomial of the centipede

Wang and Zhu [24] proved that $I(W_n,x)$ has only real zeros. Although in their result the zeros are given explicitly, for our purposes, we will need a closed form that will enable us to evaluate $\left\lfloor \dfrac{I'(W_n,1)}{I(W_n,1)} \right\rfloor$. The independence polynomial $I(W_n,x)$ satisfies the recursion [18, 19, 20]

$$I(W_n,x) = (x+1)\left(I(W_{n-1},x) + xI(W_{n-2},x)\right), \tag{2}$$

with $W_0 = 1,\ I(W_1,x) = 1+2x$.

Theorem 5. *[24] The independence polynomial of the centipede is given by*

$$\begin{aligned} I(W_n,x) &= \frac{(x+1)^{\lfloor \frac{n}{2} \rfloor}}{\sqrt{\Delta}} \left(\left(\frac{3x+1+\sqrt{\Delta}}{2} \right)^{\frac{n+2}{2}} - \left(\frac{3x+1-\sqrt{\Delta}}{2} \right)^{\frac{n+2}{2}} \right) \\ &= (x+1)^{\lfloor \frac{n}{2} \rfloor} H(x), \end{aligned}$$

where $\Delta = 5x^2+6x+1$.

Proof. By [24, Lemma 2.3], we have

$$I(W_n,x) = \frac{1}{(x+1)\sqrt{\Delta}} \left(\left(\frac{1+x+\sqrt{\Delta}}{2} \right)^{n+2} - \left(\frac{x+1-\sqrt{\Delta}}{2} \right)^{n+2} \right).$$

The last formula is explicit but not convenient for calculations, especially for the localization the mode of the polynomial $I(W_n,x)$. A more explicit one is given below. Let $n = 2l$. Then

$$I(W_{2l},x) =$$

$$\begin{aligned} & \frac{1}{(x+1)\sqrt{\Delta}} \left(\left(\frac{1+x+\sqrt{\Delta}}{2} \right)^{2l+2} - \left(\frac{x+1-\sqrt{\Delta}}{2} \right)^{2l+2} \right) \\ =\ & \frac{1}{(x+1)\sqrt{\Delta}} \left(\left(\left(\frac{1+x+\sqrt{\Delta}}{2} \right)^{2} \right)^{l+1} - \left(\left(\frac{x+1-\sqrt{\Delta}}{2} \right)^{2} \right)^{l+1} \right) \\ =\ & \frac{1}{(x+1)\sqrt{\Delta}} \left(\left(\frac{6x^2+8x+2+2(x+1)\sqrt{\Delta}}{4} \right)^{l+1} - \left(\frac{6x^2+8x++2-2(x+1)\sqrt{\Delta}}{4} \right)^{l+1} \right) \\ =\ & \frac{1}{(x+1)\sqrt{\Delta}} \left(\left(\frac{3x^2+4x+1+(x+1)\sqrt{\Delta}}{2} \right)^{l+1} - \left(\frac{3x^2+4x+1-(x+1)\sqrt{\Delta}}{2} \right)^{l+1} \right) \\ =\ & \frac{(x+1)^l}{\sqrt{\Delta}} \left(\left(\frac{3x+1+\sqrt{\Delta}}{2} \right)^{l+1} - \left(\frac{3x+1-\sqrt{\Delta}}{2} \right)^{l+1} \right) \\ =\ & (x+1)^l H_l(x) \end{aligned}$$

For $n = 2l+1$, $(j = l+1)$ we have

$$I(W_{2l+1}, x) =$$

$$\begin{aligned}
& \frac{1}{(x+1)\sqrt{\Delta}}\left(\left(\frac{1+x+\sqrt{\Delta}}{2}\right)^{2l+3} - \left(\frac{x+1-\sqrt{\Delta}}{2}\right)^{2l+3}\right) \\
= & \frac{1}{(x+1)\sqrt{\Delta}}\left(\left(\frac{1+x+\sqrt{\Delta}}{2}\right)\left(\left(\frac{1+x+\sqrt{\Delta}}{2}\right)^{2}\right)^{j} - \left(\frac{x+1-\sqrt{\Delta}}{2}\right)\left(\left(\frac{x+1-\sqrt{\Delta}}{2}\right)^{2}\right)^{j}\right) \\
= & \frac{(x+1)^{j-1}}{\sqrt{\Delta}}\left(\left(\frac{1+x+\sqrt{\Delta}}{2}\right)\left(\frac{3x+1+\sqrt{\Delta}}{2}\right)^{j} - \left(\frac{x+1-\sqrt{\Delta}}{2}\right)\left(\frac{3x+1-\sqrt{\Delta}}{2}\right)^{j}\right) \\
= & \frac{(x+1)^{j-1}}{\sqrt{\Delta}}\left(\left(\frac{3x+1+\sqrt{\Delta}}{2}\right)^{j+\frac{1}{2}} - \left(\frac{3x+1-\sqrt{\Delta}}{2}\right)^{j+\frac{1}{2}}\right) \\
= & \frac{(x+1)^{l}}{\sqrt{\Delta}}\left(\left(\frac{3x+1+\sqrt{\Delta}}{2}\right)^{\frac{2l+3}{2}} - \left(\frac{3x+1-\sqrt{\Delta}}{2}\right)^{\frac{2l+3}{2}}\right) \\
= & (x+1)^{l} H_{l+1}(x)
\end{aligned}$$

Gathering all this together, we obtain the desired formula:

$$I(W_n, x) = \frac{(x+1)^{\lfloor \frac{n}{2} \rfloor}}{\sqrt{\Delta}}\left(\left(\frac{3x+1+\sqrt{\Delta}}{2}\right)^{\frac{n+2}{2}} - \left(\frac{3x+1-\sqrt{\Delta}}{2}\right)^{\frac{n+2}{2}}\right) = (x+1)^{\lfloor \frac{n}{2} \rfloor} H(x) \tag{3}$$

□

The polynomial $H(x)$ is of degree $\lfloor \frac{n+1}{2} \rfloor$. Also, these polynomials are the independence polynomials of the claw-free graphs M_n (if n is even, and L_n (if n is odd). The fact that the polynomial $I(W_n, x)$ has only real zeros, follows from the general result of Chudnovsky and Seymour[9]. Now we can determine the mode of the centipede.

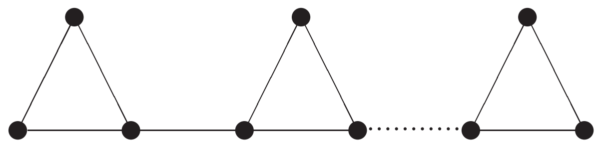

Figure 2: The graph L_n

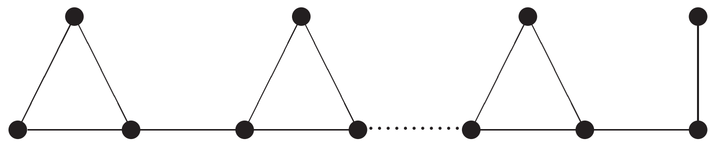

Figure 3: The graph M_n

Theorem 6. *Every mode of the independence polynomial of the centipede satisfies*

$$\left\lfloor n - \frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor - \frac{\sqrt{3}}{12}(n+2) + \frac{1}{3}\right\rfloor \le k_0 \le \left\lfloor n - \frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor - \frac{\sqrt{3}}{12}(n+2) + \frac{1}{3}\right\rfloor + 1.$$

Proof. Our aim is to evaluate $\frac{I'(W_n,1)}{I(W_n,1)}$. We just need to evaluate $H'(x)$. Unfortunately, it turns out that $H'(x)$ is not easy to handle, and then, the form of $\frac{H'(1)}{H(1)}$ is also cumbersome. In order to avoid this, consider the reciprocal polynomial:

$$\begin{aligned} H_r(x) &= x^{\lfloor\frac{n+1}{2}\rfloor} H(\frac{1}{x}) \\ &= \frac{1}{\sqrt{B}}\left(\left(\frac{x+3+\sqrt{B}}{2}\right)^{\frac{n+2}{2}} - \left(\frac{x+3-\sqrt{B}}{2}\right)^{\frac{n+2}{2}}\right), \end{aligned}$$

where $B = x^2+6x+5$. Now

$$H_r{}'(x) = \frac{(n+2)}{2B}\left(\left(\frac{x+3+\sqrt{B}}{2}\right)^{\frac{n+2}{2}} + \left(\frac{x+3-\sqrt{B}}{2}\right)^{\frac{n+2}{2}}\right) - \frac{(x+3)}{B}H_r(x).$$

Note that

$$\frac{I'(W_n,1)}{I(W_n,1)} = \frac{1}{2}\left\lfloor\frac{n}{2}\right\rfloor + \frac{H'(1)}{H(1)}.$$

But

$$\frac{H'(1)}{H(1)} = \left\lfloor\frac{n+1}{2}\right\rfloor - \frac{H_r'(1)}{H_r(1)},$$

now

$$\begin{aligned} \frac{H_r'(1)}{H_r(1)} &= \sqrt{3}\frac{(n+2)}{12}\frac{1+a^{n+2}}{1-a^{n+2}} - \frac{1}{3}, \qquad (a = 7-4\sqrt{3} = 0.07179\cdots) \\ &\ge \sqrt{3}\frac{(n+2)}{12} - \frac{1}{3}. \end{aligned}$$

On the other hand,

$$\frac{H_r'(1)}{H_r(1)} < \sqrt{3}\frac{(n+2)}{12} - \frac{1}{3} + \frac{2}{3}$$

is equivalent to

$$\sqrt{3}\frac{(n+2)}{12}\frac{1+a^{n+2}}{1-a^{n+2}} \leq \sqrt{3}\frac{(n+2)}{12} + \frac{2}{3},$$

or

$$\frac{(n+2)a^{n+2}}{1-a^{n+2}} \leq \frac{4}{\sqrt{3}},$$

which is true, since the function $x(e^{\alpha x}-1)^{-1},\ \alpha > 0$, is decreasing for $x \geq 1$. So,

$$\begin{aligned}
\frac{I'(W_n,1)}{I(W_n,1)} &= \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor + \frac{H'(1)}{H(1)} \\
&= \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor - \frac{H_r'(1)}{H_r(1)} \\
&\leq n - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \frac{H_r'(1)}{H_r(1)} \leq n - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \sqrt{3}\frac{(n+2)}{12} + \frac{1}{3}.
\end{aligned}$$

We obtain

$$\left\lfloor \frac{I'(W_n,1)}{I(W_n,1)} \right\rfloor \leq \left\lfloor n - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \sqrt{3}\frac{(n+2)}{12} + \frac{1}{3} \right\rfloor. \tag{4}$$

Also

$$\begin{aligned}
\frac{I'(W_n,1)}{I(W_n,1)} &= \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor + \frac{H'(1)}{H(1)} \\
&= \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor - \frac{H_r'(1)}{H_r(1)} \\
&\geq n - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \frac{H_r'(1)}{H_r(1)} \geq n - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \sqrt{3}\frac{(n+2)}{12} + \frac{1}{3} - \frac{2}{3} \\
&> n - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \sqrt{3}\frac{(n+2)}{12} + \frac{1}{3} - 1.
\end{aligned}$$

It follows then that

$$\left\lfloor \frac{I'(W_n,1)}{I(W_n,1)} \right\rfloor \geq \left\lfloor n - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \sqrt{3}\frac{(n+2)}{12} + \frac{1}{3} \right\rfloor - 1. \tag{5}$$

Equations (4) and (5) give the desired result. □

Corollary 7. *For every $l \in \mathbb{N}$, $l \geq 1$, there exists an integer n_0 such that*

$$\frac{I'(W_n,1)}{I(W_n,1)} - k_n > l,\ for\ n \geq n_0.$$

In other words, the conjecture of Levit and Mandrescu is false.

Proof. Let $l > 0$, be a fixed integer. Then

$$\frac{I'(W_n,1)}{I(W_n,1)} \geq n - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor - \sqrt{3}\frac{(n+2)}{12} + \frac{1}{3} - \frac{2}{3} \geq \frac{(9-\sqrt{3})}{12}n - 1.$$

Also

$$k_n \leq \frac{3}{5}n + 3.$$

But for $n \geq 801$, say, we have

$$\frac{I'(W_n,1)}{I(W_n,1)} - k_n \geq \frac{(9-\sqrt{3})}{12}n - \frac{3}{5}n - 4 \geq .0005n - 4 > 0,$$

and then, for $n \geq 200(l+4)$ we obtain the desired result. $\square$

The calculations agove are not very accurate, and $n \geq 800$ may be highly improved. For example, for $n = 202$ we have

$$\frac{I'(W_{402},1)}{I(W_{402},1)} - k_{402} = 243.5209\cdots - 241 = 2.5209\cdots$$

For $n = 1000$,

$$\frac{I'(W_{1000},1)}{I(W_{1000},1)} - k_{1000} = 605.707\cdots - 600 = 5.707\cdots$$

The constant term in $H_r(x)$ is F_n, the Fibonacci number. Using the results of [24], we may deduce some identities involving the roots and the coefficients of the polynomials of $H(x)$. For example, the sequence of the coefficient of x (=sum of the roots) in $H_r(x)$ is the sequence A129722. Also, we may deduce

$$\prod_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \left(1 + 4\cos^2\left(\frac{k\pi}{n+2}\right)\right) = F_{n+1}.$$

3 The sequence s_k is asymptotically normal

A positive real sequence $a(n,k)_{k=0}^{n}$, with $A_n = \sum_{k=0}^{n} a(n,k) \neq 0$, is said to satisfy a central limit theorem (or is *asymptotically normal*) with mean μ_n and variance σ_n^2 if

$$\lim_{n \longrightarrow +\infty} \sup_{x \in R} \left| \sum_{0 \leq k \leq \mu_n + x\sigma_n} \frac{a(n,k)}{A_n} - (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \right| = 0. \quad (6)$$

The sequence satisfies a local limit theorem on $B \subseteq \mathbb{R}$; with mean μ_n and variance σ_n^2 if

$$\lim_{n \longrightarrow +\infty} \sup_{x \in B} \left| \frac{\sigma_n a(n, \mu_n + x\sigma_n)}{A_n} - (2\pi)^{-1/2} e^{-\frac{x^2}{2}} \right| = 0. \quad (7)$$

Recall the following result (see Bender [2]).

Theorem 8. *Let $(Q_n)_{n\geq 1}$ be a sequence of real polynomials; with only real negative zeros. The sequence of the coefficients of the $(Q_n)_{n\geq 1}$ satisfies a central limit theorem; with $\mu_n = \frac{Q_n^{'}(1)}{Q_n(1)}$ and $\sigma_n^2 = \left(\frac{Q_n^{''}(1)}{Q_n(1)} + \frac{Q_n^{'}(1)}{Q_n(1)} - \left(\frac{Q_n^{'}(1)}{Q_n(1)}\right)^2\right)$ provided that $\lim_{n\longrightarrow+\infty} \sigma_n^2 = +\infty$. If, in addition, the sequence of the coefficients of each Q_n is with no internal zeros; then the sequence of the coefficients satisfies a local limit theorem on $\mathbb{R}$.*

Generally speaking, a central limit theorem for a sequence of random variables gives (6). Relation (7) is then deduced under the condition that the sequence has no internal zeros (see [2]). Relation (6) is nothing than pointwise convergence. We have the following result

Theorem 9. *The sequence s_k satisfies a central limit and a local limit theorem on $\mathbb{R}$, with mean*

$$\mu_n = \frac{I'(W_n,1)}{I(W_n,1)} \approx \frac{(9-\sqrt{3})n}{12}$$

and variance

$$\sigma_n^2 = \left(\frac{I''(W_n,1)}{I(W_n,1)} + \frac{I'(W_n,1)}{I(W_n,1)} - \left(\frac{I'(W_n,1)}{I(W_n,1)}\right)^2\right) \approx \frac{(15-2\sqrt{3})n}{24}.$$

Proof. In order to prove that the sequence of the coefficients of $P_n(x)$ is asymptotically normal, let us evaluate

$$\left(\frac{I''(W_n,1)}{I(W_n,1)} + \frac{I'(W_n,1)}{I(W_n,1)} - \left(\frac{I'(W_n,1)}{I(W_n,1)}\right)^2\right).$$

We have

$$\begin{aligned}
I(W_n,x) &= (x+1)^{\lfloor\frac{n}{2}\rfloor}H(x)\\
I'(W_n,x) &= \left\lfloor\frac{n}{2}\right\rfloor(x+1)^{\lfloor\frac{n}{2}\rfloor-1}H(x) + (x+1)^{\lfloor\frac{n}{2}\rfloor}H^{'}(x)\\
I''(W_n,x) &= \left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)(x+1)^{\lfloor\frac{n}{2}\rfloor-2}H(x) + 2\left\lfloor\frac{n}{2}\right\rfloor(x+1)^{\lfloor\frac{n}{2}\rfloor-1}H^{'}(x) + (x+1)^{\lfloor\frac{n}{2}\rfloor}H^{''}(x).
\end{aligned}$$

We also have

$$\frac{I'(W_n,1)}{I(W_n,1)} = \frac{\lfloor\frac{n}{2}\rfloor\left(\lfloor\frac{n}{2}\rfloor-1\right)}{4} + \left\lfloor\frac{n}{2}\right\rfloor\frac{H^{'}(1)}{H(1)} + \frac{H^{''}(1)}{H(1)},$$

so

$$\begin{aligned}
\sigma_n^2 &= \frac{\lfloor\frac{n}{2}\rfloor}{4} + \frac{H"(1)}{H(1)} + \frac{H'(1)}{H(1)} - \left(\frac{H'(1)}{H(1)}\right)^2\\
&= \frac{\lfloor\frac{n}{2}\rfloor}{4} + (\sigma_n^H)^2\\
&= \frac{\lfloor\frac{n}{2}\rfloor}{4} + \frac{H'(1)}{H(1)}\left(1 + \frac{H''(1)}{H'(1)} - \frac{H'(1)}{H(1)}\right) > \frac{\lfloor\frac{n}{2}\rfloor}{4}.
\end{aligned}$$

It follows that $\lim_{n\longrightarrow\infty} \sigma_n = \infty$, and then the sequences s_k satisfies a central limit theorem. Now let $(-\alpha_i)$, $(-\beta_i)$, be respectively the zeros of $H(x)$ and $H^{'}(x)$. Then by Rolle's theorem we get

$$-\alpha_1 \leq \beta_1 \leq -\alpha_2 \leq \beta_2 \leq \cdots \leq -\beta_{\lfloor \frac{n+1}{2} \rfloor -1} \leq -\alpha_{\lfloor \frac{n+1}{2} \rfloor}.$$

We deduce

$$\begin{aligned}\left(1+\frac{H''(1)}{H'(1)}-\frac{H'(1)}{H(1)}\right) &= \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{\alpha_i}{1+\alpha_i} - \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor -1} \frac{\beta_i}{1+\beta_i}, \\ &= \frac{H_r'(1)}{H_r(1)} - \frac{H_r''(1)}{H_r'(1)} \approx 1\end{aligned}$$

It follows that

$$\sigma_n^2 \approx \frac{(15-2\sqrt{3})n}{24}$$

□

By Theorem 8, and because all the s_k are nonvanishing, we have a local limit theorem, from which we deduce the

Corollary 10. *Let $S_{k_0} = \max_k s_k$. Then we have the approximation of the maximum stable set*

$$S_{k_0} \approx \frac{I(W_n,1)}{\sigma_n} \approx 1.02\frac{(1+\sqrt{3})^n}{\sqrt{\pi n}}$$

Finally, we note that the same limit theorems, remain true for the sequences of the coefficients of the independence polynomials of M_n and L_n.

4 Acknowledgments

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