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Generating Functions for Wilf Equivalence Under Generalized Factor Order

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Abstract

Kitaev, Liese, Remmel, and Sagan recently defined generalized factor order on words comprised of letters from a partially ordered set $(P, \leq_P)$ by setting $u \leq_P w$ if there is a contiguous subword v of w of the same length as u such that the i-th character of v is greater than or equal to the i-th character of u for all i. This subword v is called an embedding of u into w. For the case where P is the positive integers with the usual ordering, they defined the weight of a word $w = w_1 \ldots w_n$ to be $\mathrm{wt}(w) = t^n x^{\sum_{i=1}^n w_i}$, and the corresponding weight generating function $F(u; t, x) = \sum_{w \geq_P u} \mathrm{wt}(w)$. They then defined two words u and v to be Wilf equivalent, denoted $u \backsim v$, if and only if $F(u; t, x) = F(v; t, x)$. They also defined the related generating function $S(u; t, x) = \sum_{w \in \mathcal{S}(u)} \mathrm{wt}(w)$ where $\mathcal{S}(u)$ is the set of all words w such that the only embedding of u into w is a suffix of w, and showed that $u \backsim v$ if and only if $S(u; t, x) = S(v; t, x)$. We continue this study by giving an explicit formula for $S(u; t, x)$ if u factors into a weakly increasing word followed by a weakly decreasing word. We use this formula as an aid to classify Wilf equivalence for all words of length 3. We also show that coefficients of related generating functions are well-known sequences in several special cases. Finally, we discuss a conjecture that if $u \backsim v$ then u and v must be rearrangements, and the stronger conjecture that there also must be a weight-preserving bijection f on words over the positive integers such that $f(w)$ is a rearrangement of w for all w, and w embeds u if and only if $f(w)$ embeds v.

1 Introduction and definitions

Kitaev, Liese, Remmel, and Sagan [2] recently introduced the generalized factor order on words comprised of letters from a partially ordered set (poset). That is, let $\mathcal{P} = (P, \leq_P)$ be a poset and let P^* be the Kleene closure of P so that

$$P^* = \{w = w_1 w_2 \ldots w_n \mid n \geq 0 \text{ and } w_i \in P \text{ for all } i\}.$$

For $w \in P^*$, let $|w|$ denote the number of characters in w. Then for any $u, w \in P^*$, u is less than or equal to w in the *generalized factor order* relative to $\mathcal{P}$, written $u \leq_{\mathcal{P}} w$, if there is a string v of $|u|$ consecutive characters in w such that the i-th character of v is greater than or equal to the i-th character of u under $\leq_{\mathcal{P}}$ for each i, $1 \leq i \leq |u|$. If $u \leq_{\mathcal{P}} w$, we will also say that w *embeds* u, and that v is an *embedding of* u *into* w. We will primarily be interested in the poset $\mathcal{P}_1 = (\mathbb{P}, \leq)$, where $\mathbb{P}$ is the set of positive integers and $\leq$ is the usual total order on $\mathbb{P}$. In this case, for example, $u = 321 \leq_{\mathcal{P}_1} w = 142322$, and 423 and 322 are embeddings of u into w. Kitaev, Liese, Remmel, and Sagan [2] noted that generalized factor order is related to generalized subword order, in which the characters of v are not required to be adjacent [3].

Kitaev, Liese, Remmel, and Sagan [2] defined Wilf equivalence under the generalized factor order on the positive integers in the following way. For $w = w_1 \ldots w_n \in \mathbb{P}^*$, let $\Sigma(w) = \sum_{i=1}^n w_i$ and define the *weight* of w to be $\mathrm{wt}(w) = t^n x^{\Sigma(w)}$. Then define

$$\mathcal{F}(u) = \{w \in \mathbb{P}^* \mid u \leq_{\mathcal{P}_1} w\},$$

and the related generating function

$$F(u; t, x) = \sum_{w \in \mathcal{F}(u)} \mathrm{wt}(w).$$

Two words $u, v \in \mathbb{P}^*$ are then said to be *Wilf equivalent*, denoted $u \backsim v$, if and only if $F(u; t, x) = F(v; t, x)$. Kitaev, Liese, Remmel, and Sagan [2] noted that this idea, while inspired by the notion of Wilf equivalence used in the theory of pattern avoidance, is different, since the partial order in question is not that of pattern containment. More information about Wilf equivalence in the pattern avoidance context is contained in the survey article by Wilf [4].

In proving results about Wilf equivalence, it is often convenient to study the sets

$$\begin{aligned}
\mathcal{S}(u) &= \{w \in \mathbb{P}^* \mid u \leq_{\mathcal{P}_1} w \text{ and the last } |u| \text{ characters of } w \text{ form the only} \\
&\qquad\qquad \text{embedding of } u \text{ into } w\}, \\
\mathcal{W}(u) &= \{w \in \mathbb{P}^* \mid u \leq_{\mathcal{P}_1} w \text{ and } |w| = |u|\}, \text{ and} \\
\mathcal{A}(u) &= \{w \in \mathbb{P}^* \mid u \not\leq_{\mathcal{P}_1} w\}
\end{aligned}$$

and the corresponding weight generating functions

$$\begin{aligned}
S(u;t,x) &= \sum_{w \in \mathcal{S}(u)} \mathrm{wt}(w), \\
W(u;t,x) &= \sum_{w \in \mathcal{W}(u)} \mathrm{wt}(w), \text{ and} \\
A(u;t,x) &= \sum_{w \in \mathcal{A}(u)} \mathrm{wt}(w).
\end{aligned}$$

Kitaev, Liese, Remmel, and Sagan [2] proved that $F(u;t,x)$, $S(u;t,x)$, and $A(u;t,x)$ are rational. They constructed a non-deterministic finite automaton for each $u \in \mathbb{P}^*$ that recognizes $\mathcal{S}(u)$, implying that $S(u;t,x)$ is rational. That the others are rational follows from the fact that the weight generating function for all words in $\mathbb{P}^*$ is

$$\begin{aligned}
\sum_{w \in \mathbb{P}^*} \mathrm{wt}(w) &= \frac{1}{1 - \sum_{n \geq 1} tx^n} \\
&= \frac{1}{1 - tx/(1-x)} \\
&= \frac{1-x}{1-x-tx},
\end{aligned}$$

and therefore

$$F(u;t,x) = S(u;t,x)\frac{1-x}{1-x-tx} \tag{1}$$

and

$$F(u;t,x) = \frac{1-x}{1-x-tx} - A(u;t,x).$$

We also note that $W(u;t,x)$ is rational since

$$W(u;t,x) = \frac{t^{|u|}x^{\Sigma(u)}}{(1-x)^{|u|}}.$$

From (1), we have that $F(u;t,x) = F(v;t,x)$ if and only if $S(u;t,x) = S(v;t,x)$, and therefore $u \backsim v$ if and only if $S(u;t,x) = S(v;t,x)$. Much of our work will be centered around computing explicit formulas for $S(u;t,x)$ for certain words u. In particular, Kitaev, Liese, Remmel and Sagan [2] gave two examples of classes of words u such that $S(u;t,x)$ has a simple form. That is, they proved that if $u = 1\ 2\ 3 \ldots n-1\ n$ or $u = 1^k b^\ell$ for some $k \geq 0$, $\ell \geq 1$, and $b \geq 2$, then $S(u;t,x) = \frac{x^s t^r}{P(u;t,x)}$ for some polynomial $P(u;t,x)$, and produced an explicit expression for $P(u;t,x)$ in each case. We shall show that there is a much richer class of of words u such that $S(u;t,x)$ has this same form. Specifically, for any word u, let u_{inc} be the longest weakly increasing prefix of u. If $u = u_{inc}v$ and v is weakly decreasing, then we shall say that u has an *increasing/decreasing factorization* and denote v as u_{dec}. Thus if $u = u_1 u_2 \ldots u_n$ has an increasing/decreasing factorization, then either $u_1 \leq \cdots \leq u_n$, in which case $u_{inc} = u$ and u_{dec} is the empty string ε, or there is a $k < n$ such that $u_1 \leq \cdots \leq u_k > u_{k+1} \geq \cdots \geq u_n$, in which case $u_{inc} = u_1 \ldots u_k$ and $u_{dec} = u_{k+1} \ldots u_n$. For the theorem that follows, we define

$$D^{(i)}(u) = \{n - i + j : 1 \leq j \leq i \text{ and } u_j > u_{n-i+j}\}$$

and $d_i(u) = \sum_{n-i+j \in D^{(i)}(u)} (u_j - u_{n-i+j})$. For example, if $u = 1\ 2\ 3\ 4\ 4\ 3\ 1\ 1$ and $i = 5$, then by considering the diagram

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 4 & 3 & 1 & 1 \\ & & & 1 & 2 & 3 & \underline{4} & \underline{4} \end{array}$$

we see that $D^{(5)}(u) = \{7, 8\}$ and $d_5(u) = (4-1) + (4-1) = 6$. One of our main results is the following theorem.

Theorem 1. *Let $u = u_1 u_2 \ldots u_n \in \mathbb{P}^*$ have an increasing/decreasing factorization. For $1 \leq i \leq n-1$, let $s_i = u_{i+1} u_{i+2} \ldots u_n$ and $d_i = d_i(u)$. Also let $s_n = \varepsilon$ and $d_n = 0$. Then*

$$S(u;t,x) = \frac{t^n x^{\Sigma(u)}}{t^n x^{\Sigma(u)} + (1 - x - tx) \sum_{i=1}^{n} t^{n-i} x^{d_i + \Sigma(s_i)} (1-x)^{i-1}}.$$

Since the words $u = 1\ 2\ 3 \ldots n-1\ n$ or $u = 1^k b^\ell$ for some $k \geq 0$, $\ell \geq 1$, and $b \geq 2$ clearly have increasing/decreasing factorizations, Theorem 1 covers both of the cases proved by Kitaev, Liese, Remmel and Sagan [2].

Theorem 1 will lead us to the other main results in our work. First, we will use Theorem 1, as well as a slight modification in a special case, to completely classify the Wilf equivalence classes of $\mathcal{P}_1$ for all words of length 3. We will also compute $S(u;t,x)$, along with $F(u;t,x)$ and $A(u;t,x)$, for some simple words and show that the coefficients in these generating functions are often well-known sequences. Next, Theorem 1 will allow us to show that if u and v are words with increasing/decreasing factorizations, then $u \backsim v$ if and only if v is a rearrangement of the letters of u. This shows that words with increasing/decreasing factorizations satisfy the following conjecture of Kitaev, Liese, Remmel, and Sagan [2].

Conjecture 2 (Kitaev, Liese, Remmel, Sagan)**.** If $u \backsim v$, then v is a rearrangement of u.

We shall call this conjecture the *weak rearrangement conjecture*. In fact, we conjecture something much stronger is true.

Conjecture 3. If $u \backsim v$, then there is a weight preserving bijection $f : \mathbb{P}^* \to \mathbb{P}^*$ such that for all $w \in \mathbb{P}^*$, $f(w)$ is a rearrangement of w and $w \in \mathcal{F}(u) \iff f(w) \in \mathcal{F}(v)$.

We will call such a bijection f a *rearrangement map* that *witnesses* $u \backsim v$ and refer to this conjecture as the *strong rearrangement conjecture*. All the Wilf equivalences proved by Kitaev, Liese, Remmel, and Sagan in Section 4 of [2] were proved by a constructing a rearrangement map that witnessed the given Wilf equivalence.

We investigate the rearrangement conjectures by considering the class of finite posets $\mathcal{P}_{[m]} = ([m], \leq)$, where $[m] = \{1, \ldots, m\}$ and $\leq$ is the usual total order on $\mathbb{P}$. For any word $w \in [m]^*$ and $i \in [m]$, let $c_i(w)$ equal the number of occurrences of i in w. Then we introduce variables $x_1, x_2, \ldots, x_m$, and define the weight of w, $W_{[m]}(w)$, to be $W_{[m]}(w) = \prod_{i=1}^m x_i^{c_i(w)}$. To define Wilf equivalence in this context, we set

$$F(u; x_1, \ldots, x_m) = \sum_{w \in \mathcal{F}(u) \cap [m]^*} W_{[m]}(w),$$

and define $u, v \in [m]^*$ to be Wilf equivalent with respect to the poset $\mathcal{P}_{[m]}$, denoted $u \backsim_{[m]} v$, if and only $F(u; x_1, \ldots, x_m) = F(v; x_1, \ldots, x_m)$. We will also have use for the related generating functions

$$\begin{aligned} W(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{W}(u) \cap [m]^*} W_{[m]}(w), \\ S(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{S}(u) \cap [m]^*} W_{[m]}(w), \text{ and} \\ A(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{A}(u) \cap [m]^*} W_{[m]}(w). \end{aligned}$$

Note that we have dropped the t dependence in these generating functions since length is recorded by the number of variables in a monomial. We now have

$$\sum_{w \in [m]^*} W_{[m]}(w) = \frac{1}{1 - \sum_{i=1}^m x_i}.$$

Thus since $\mathcal{F}(u) \cap [m]^* = (\mathcal{S}(u) \cap [m]^*)[m]^*$ and $A(u) \cap [m]^* = [m]^* - (\mathcal{F}(u) \cap [m]^*)$, we have that

$$\begin{aligned} F(u; x_1, \ldots, x_m) &= S(u; x_1, \ldots, x_m) \frac{1}{1 - \sum_{i=1}^m x_i} \text{ and} \\ A(u; x_1, \ldots, x_m) &= \frac{1}{1 - \sum_{i=1}^m x_i} - F(u; x_1, \ldots, x_m) \end{aligned}$$

so that if any one of $F(u; x_1, \ldots, x_m)$, $A(u; x_1, \ldots, x_m)$, or $S(u; x_1, \ldots, x_m)$ is rational, then so are the other two. It follows from Theorem 8.2 of [2] that $S(u; x_1, \ldots, x_m)$ is rational for all $m \geq 1$, so $F(u; x_1, \ldots, x_m)$ and $A(u; x_1, \ldots, x_m)$ are also rational for all $m \geq 1$. Also note that if $u = u_1 \ldots u_n$, then

$$W(u; x_1, \ldots, x_m) = \prod_{r=1}^n \sum_{s=u_r}^m x_s,$$

so $W(u; x_1, \ldots, x_m)$ is rational. We will show that if $u \backsim_{[m]} v$ for some m, then there is a rearrangement map that witnesses the equivalence $u \backsim v$. This gives us a way to test the strong rearrangement conjecture for any particular pair of words $u, v \in \mathbb{P}^*$. We will also give an analogue of Theorem 1 for these posets.

The outline of this paper is as follows. In Section 2, we prove Theorem 1 and show that the weak rearrangement conjecture holds for words with increasing/decreasing factorizations. In Section 3, we compute $F(u; t, x)$, $S(u; t, x)$ and $A(u; t, x)$ for some simple words. In particular, we show that the sequences of coefficients that arise in the expansions around $x = 0$ of $F(k; 1, x)$, $S(k; 1, x)$, $A(k; 1, x)$, $S(1k1; 1, x)$ and $A(1k1; 1, x)$ as k varies have appeared in the On-line Encyclopedia of Integer Sequences (OEIS). We follow this with the classification of the Wilf equivalence classes of words of length 3 in Section 4. The results of Sections 2 and 4 allow us to compute $S(\sigma; t, x)$ and $A(\sigma; t, x)$ for all permutations in the symmetric group S_3 as there are only two Wilf equivalences classes for such permutations. In these cases, the coefficients that arise in the expansions of $S(\sigma; 1, x)$ and $A(\sigma; 1, x)$ around $x = 0$ do not correspond to any sequences that have appeared in the OEIS. We discuss the strong rearrangement conjecture in Section 5, as well as the analogue of Theorem 1 for the posets $\mathcal{P}_{[m]}$. We conclude with a few remarks about further work in Section 6.

2 Words such that $S(u; t, x) = \frac{x^s t^r}{P(u;t,x)}$ where $P(u; t, x)$ is a polynomial.

In this section we prove Theorem 1, and show that Conjecture 2 holds for words with an increasing/decreasing factorization.

Proof of Theorem 1. Let $u = u_1 u_2 \ldots u_n \in \mathbb{P}^*$ have an increasing/decreasing factorization. If $w = w_1 \ldots w_m \in \mathcal{S}(u)$, then $w_1 \ldots w_{m-n} \in \mathcal{A}(u)$ and $u \le w_{m-n+1} \ldots w_m$. However if $v \in \mathcal{A}(u)$ and $z = z_1 \ldots z_n$ is such that $u \le z$, then it may not be the case that $w = vz \in \mathcal{S}(u)$ because there might be another embedding of u in the last $2n - 1$ letters of w, starting in v and ending in z. Of course, there can be no embedding of u which starts to the left of the last $2n - 1$ letters of w since $v \in \mathcal{A}(u)$. For each $1 \le i \le n - 1$, we define $\mathcal{S}^{(i)}(u)$ to be set of all words $w = w_1 \ldots w_m$ such that

(i) $u \le w_{m-n+1} \ldots w_m$ (so that u embeds into the suffix of length n of w) and

(ii) the left-most embedding of u into w starts at position $m - 2n + i + 1$.

Note that $\mathcal{S}^{(i)}(u)$ is empty when $m - 2n + i + 1$ is non-positive. We then let

$$S^{(i)}(u; t, x) = \sum_{w \in \mathcal{S}^{(i)}(u)} \mathrm{wt}(w) = \sum_{w \in \mathcal{S}^{(i)}(u)} x^{\Sigma(w)} t^{|w|}.$$

Thus

$$\mathcal{S}(u) = \mathcal{A}(u)\mathcal{W}(u) - \bigcup_{i=1}^{n-1} \mathcal{S}^{(i)}(u). \tag{2}$$

Now,

$$\begin{aligned}\sum_{w \in \mathcal{A}(u)\mathcal{W}(u)} x^{\Sigma(w)} t^{|w|} &= A(u;t,x)\frac{t^n x^{\Sigma(u)}}{(1-x)^n} \\ &= \frac{(1-x)}{(1-x-tx)}(1-S(u;t,x))\frac{t^n x^{\Sigma(u)}}{(1-x)^n}. \end{aligned} \tag{3}$$

We claim that we have the following lemma.

Lemma 4. *Let $u = u_1 u_2 \ldots u_n \in \mathbb{P}^*$ have an increasing/decreasing factorization, and define d_i and s_i as in Theorem 1. Then for $1 \le i \le n-1$,*

$$S^{(i)}(u;t,x) = S(u;t,x) t^{n-i} x^{d_i + \Sigma(s_i)} \left(\frac{1}{1-x}\right)^{n-i}.$$

Given Lemma 4, it is easy to complete the proof of Theorem 1. That is, our definitions ensure that $\mathcal{S}^{(1)}(u), \mathcal{S}^{(2)}(u), \ldots, \mathcal{S}^{(n-1)}(u)$ are pairwise disjoint, so that

$$\begin{aligned}\sum_{w \in \bigcup_{i=1}^{n-1} \mathcal{S}^{(i)}(u)} x^{\Sigma(w)} t^{|w|} &= \sum_{i=1}^{n-1} S^{(i)}(u;t,x) \\ &= S(u;t,x) \sum_{i=1}^{n-1} t^{n-i} x^{d_i+\Sigma(s_i)} \left(\frac{1}{1-x}\right)^{n-i}. \end{aligned}$$

Thus it follows from (2) and (3) that

$$\begin{aligned} S(u;t,x) &= \frac{(1-x)}{(1-x-tx)}(1-S(u;t,x))\frac{t^n x^{\Sigma(u)}}{(1-x)^n} \\ &\quad - S(u;t,x)\sum_{i=1}^{n-1} \frac{x^{d_i+\Sigma(s)_i} t^{n-i}}{(1-x)^{n-i}}. \end{aligned}$$

Solving for $S(u;t,x)$ will yield the result in the theorem.

Thus we need only prove Lemma 4. To this end, fix i, $1 \le i \le n-1$, and suppose that $w = w_1 \ldots w_m \in \mathcal{S}^{(i)}(u)$. If $\bar{w} = w_1 \ldots w_{m-n+i}$, then our definitions ensure that

1. $\bar{w} \in \mathcal{S}(u)$,
2. $u_1 \ldots u_i \le w_{m-n+1} \ldots w_{m-n+i}$ and
3. $s_i = u_{i+1} \ldots u_n \le w_{m-n+i+1} \ldots w_m$.

Now, the generating function of all words v of length $n-i$ such that $s_i \le v$ is $\frac{x^{\Sigma(s_i)} t^{n-i}}{(1-x)^{n-i}}$. So let $\bar{\mathcal{S}}^{(i)}(u)$ denote the set of all words $\bar{w}$ that satisfy conditions 1 and 2, and let

$$\bar{S}^{(i)}(u;t,x) = \sum_{\bar{w} \in \bar{\mathcal{S}}^{(i)}(u)} x^{\Sigma(\bar{w})} t^{|\bar{w}|}.$$

Then

$$S^{(i)}(u;t,x) = \bar{S}^{(i)}(u;t,x)\frac{x^{\Sigma(s_i)}t^{n-i}}{(1-x)^{n-i}}.$$

Thus we need only show that

$$\bar{S}^{(i)}(u;t,x) = x^{d_i}S(u;t,x). \tag{4}$$

Now suppose that $v = v_1 \dots v_p \in \bar{\mathcal{S}}^{(i)}(u)$. Then let $\tilde{v} = \tilde{v}_1 \dots \tilde{v}_p$ be the word that results from v by decrementing v_{p-i+j} by $u_j - u_{n-i+j}$ if $n-i+j \in D^{(i)}(u)$ and leaving all other letters the same. If $n-i+j \in D^{(i)}(u)$, then $v_{p-i+j} \geq u_j$, and hence $\tilde{v}_{p-i+j} \geq u_{n-i+j}$. Thus it will still be the case that u embeds in the final segment of $\tilde{v}$ of length n so that $\tilde{v} \in \mathcal{S}(u)$. Thus to complete the proof of (4), we need only show that if we start with a word $\tilde{v} = \tilde{v}_1 \dots \tilde{v}_p$ in $\mathcal{S}(u)$ and create a new word $v = v_1 \dots v_p$ by incrementing $\tilde{v}_{p-i+j}$ by $u_j - u_{n-i+j}$ if $n-i+j \in D^{(i)}(u)$ and leaving all other letters the same, then $v \in \bar{\mathcal{S}}^{(i)}(u)$. Clearly v satisfies condition (2) above. The only question is whether v is still in $\mathcal{S}(u)$. That is, since we have incremented some letters in $\tilde{v}$ to get v, we might have created a new embedding of u which starts to the left of position $p-n+1$. If so, any such embedding must contain at least one position of the form $p-i+j$ where $n-i+j \in D^{(i)}(u)$. However if u_r is the letter in this new embedding of u into v which corresponds to position $p-i+j$, then r must be strictly greater than $n-i+j$. But if $u = u_1 \leq \cdots \leq u_k > u_{k+1} \geq \cdots \geq u_n$, then it must always be the case that $D^{(i)}(u) \subseteq \{k+1, \dots, n\}$. That is, if $j < n-i+j \leq k$, then $u_j \leq u_{n-i+j}$ and hence $n-i+j \notin D^{(i)}(u)$. Hence $u_{n-i+j} \geq u_r$. But then $\tilde{v}_{p-i+j} \geq u_{n-i+j} \geq u_r$ which would mean that there would have been an embedding of u into $\tilde{v}$ which started to the left of $p-n+1$. Since $\tilde{v}$ was assumed to be in $\mathcal{S}(u)$, there can be no such embedding and hence $v \in \mathcal{S}(u)$. Thus (4) holds and the lemma is proved. $\square$

To illustrate the ideas in the proof, consider $u = 1\,2\,6\,5\,3\,2$, so that $u_{inc} = 1\,2\,6$ and $u_{dec} = 5\,3\,2$, and let $i = 5$. Then elements of $\bar{\mathcal{S}}^{(5)}(u)$ must end in an embedding of u in the final six characters and an embedding of $1\,2\,6\,5\,3$ in the final five characters, as shown:

$$\begin{array}{cccccccc}
\tilde{v} = & \cdots & \bullet & \bullet & \bullet & \star & \star & \star \\
 & & 1 & 2 & 6 & 5 & 3 & 2\ , \\
 & & & 1 & 2 & 6 & 5 & 3
\end{array}$$

where the stars indicate the positions in $\tilde{v}$ that must be increased to form v. Note that the stars all embed characters of u_{dec}, and that $d_5 = (6-5)+(5-3)+(3-2) = 4$. If v were to contain a new embedding of u to the left of the first original embedding, that new embedding must end in the second or third position from the end:

$$\begin{array}{ccccccccc}
v = & \cdots & \bullet & \bullet & \bullet & \bullet & \star & \star & \star \\
 & & & 1 & 2 & 6 & 5 & 3 & 2 \\
 & & 1 & 2 & 6 & 5 & 3 & 2 &
\end{array}$$

or

$$\begin{array}{cccccccccc}
v = & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \star & \star & \star \\
 & & & & 1 & 2 & 6 & 5 & 3 & 2\ . \\
 & & 1 & 2 & 6 & 5 & 3 & 2 & &
\end{array}$$

But in both cases, the characters below the stars are decreasing, so such an embedding would have already existed in $\tilde{v}$.

It is worth noting here that the condition that u has an increasing/decreasing factorization is necessary for the technique in the proof of Lemma 4 to be valid. That is, if u does not have an increasing/decreasing factorization, there is always at least one index i where words counted by $\bar{S}^{(i)}(u;t,x)$ can not be formed by simply starting with a word $\tilde{v} \in \mathcal{S}(u)$ and creating a word $v = v_1 \ldots v_p$ by incrementing $\tilde{v}_{p-i+j}$ by $u_j - u_{n-i+j}$ if $n-i+j \in D^{(i)}(u)$ and leaving all other letters the same. For example, consider $u = 2112$ with $i = 2$. Then $D^{(2)}(u) = \{3\}$ and $d_2(u) = 1$. However if we start with $\tilde{v} = 122112 \in \mathcal{S}(u)$ and increment $\tilde{v}_5$ to obtain v, then $v = 122122$ which is not in $\mathcal{S}(u)$ because there is an embedding of u which starts at position 2. The problem here is that the second 1 in u is followed by a larger character, and also has a larger character to its left. A similar situation will always occur for at least one i when u does not have an increasing/decreasing factorization. Experimental evidence suggests the following conjecture.

Conjecture 5. For $u \in \mathbb{P}^*$, $S(u;t,x) = \frac{x^s t^r}{P(u;t,x)}$ where $P(u;t,x)$ is a polynomial if and only if u has an increasing/decreasing factorization.

It is a consequence of Corollary 4.2 in [2] that if u and v have increasing/decreasing factorizations and u is a rearrangement of v, then $u \backsim v$. We shall give a new proof of that fact here, as well as prove the converse. That is, if u and v both have increasing/decreasing factorizations and $u \backsim v$, then u and v are rearrangements, showing that the weak rearrangement conjecture holds for words with increasing/decreasing factorizations.

We begin with the following lemma.

Lemma 6. *Suppose $u = u_1 \ldots u_n$ is a rearrangement of $v = v_1 \ldots v_n$ and that u and v have increasing/decreasing factorizations. For each i, $1 \leq i \leq n-1$, let $s_i(u) = u_{i+1} \ldots u_n$, $s_i(v) = v_{i+1} \ldots v_n$, $d_i(u) = \sum_{n-i+j \in D^{(i)}(u)} (u_j - u_{n-i+j})$, and $d_i(v) = \sum_{n-i+j \in D^{(i)}(v)} (v_j - v_{n-i+j})$. Then for all $1 \leq i \leq n-1$,*

$$d_i(u) + \Sigma(s_i(u)) = d_i(v) + \Sigma(s_i(v)).$$

Proof. First suppose that $u = u_1 \ldots u_n$ where $u_1 \leq \cdots \leq u_n$. Then for each i, $1 \leq i \leq n-1$, $d_i(u) = 0$ and $\Sigma(s_i(u)) = \sum_{j=i+1}^n u_j$. So it suffices to show that $d_i(v) + \Sigma(s_i(v)) = \Sigma(s_i(u))$ for all $1 \leq i \leq n-1$ whenever v has an increasing/decreasing factorization and v is a rearrangement of u. So fix i, $1 \leq i \leq n-1$, and let $\sigma = \sigma_1 \ldots \sigma_n$ be a permutation of $\{1, \ldots, n\}$ such that

$$v = u_{\sigma_1} \leq \cdots \leq u_{\sigma_j} > u_{\sigma_{j+1}} \geq \cdots \geq u_{\sigma_n}.$$

Then let

$$\begin{aligned}
A^i &= \{s : s \leq i \text{ and } u_s \in \{u_{\sigma_1}, \ldots, u_{\sigma_j}\}\}, \\
B^i &= \{s : s > i \text{ and } u_s \in \{u_{\sigma_1}, \ldots, u_{\sigma_j}\}\}, \\
C^i &= \{s : s > i \text{ and } u_s \in \{u_{\sigma_{j+1}}, \ldots, u_{\sigma_n}\}\}, \text{ and} \\
D^i &= \{s : s \leq i \text{ and } u_s \in \{u_{\sigma_{j+1}}, \ldots, u_{\sigma_n}\}\}.
\end{aligned}$$

For example, suppose $u = 1\ 2\ 3\ 3\ 4\ 5\ 5\ 6\ 7\ 7$ and $\sigma = 2\ 3\ 4\ 9\ 10\ 8\ 7\ 6\ 5\ 1$ so that

$$\begin{array}{rcccccccccccc} v & = & u_2 & u_3 & u_4 & u_9 & u_{10} & u_8 & u_7 & u_6 & u_5 & u_1 \\ & = & 2 & 3 & 3 & 7 & 7 & 6 & 5 & 5 & 4 & 1 \end{array}$$

and $j = 5$. Then for $i = 6$, $A^6 = \{2, 3, 4\}$, $B^6 = \{9, 10\}$, $C^6 = \{7, 8\}$, and $D^6 = \{1, 5, 6\}$. Let $a_i = |A^i|$, $b_i = |B^i|$, $c_i = |C^i|$, and $d_i = |D^i|$. Then our definitions force $a_i + d_i = i$, $b_i + c_i = n - i$, $a_i + b_i = j$, and $c_i + d_i = n - j$. For any set $D = \{d_1 < \cdots < d_r\} \subseteq \{1, \ldots, n\}$, let

$$\begin{array}{rcl} D(u)\uparrow & = & u_{d_1} u_{d_2} \ldots u_{d_r} \text{ and} \\ D(u)\downarrow & = & u_{d_r} u_{d_{r-1}} \ldots u_{d_1}. \end{array}$$

Thus $v = A^i(u)\uparrow\ B^i(u)\uparrow\ C^i(u)\downarrow\ D^i(u)\downarrow$ and $\Sigma(B^i(u)\uparrow) + \Sigma(C^i(u)\downarrow) = \Sigma(s_i(u))$. We then have four cases to consider depending on whether $v_i \in A^i(u)\uparrow$, $v_i \in B^i(u)\uparrow$, $v_i \in C^i(u)\downarrow$, or $v_i \in D^i(u)\downarrow$.

Case 1. $v_i \in A^i(u)\uparrow$.

In this case, it must be that $i = a_i$ and $A^i(u)\uparrow = u_1 \ldots u_i$. But then $D^i = \emptyset$ and $s_i(v) = B^i(u)\uparrow\ C^i(u)\downarrow$, a rearrangement of $s_i(u)$. Moreover, it will be the case that $v_j \leq v_{n-i+j}$ for $j = 1, \ldots, i$ so that $d_i(v) = 0$. Thus $d_i(v) + \Sigma(s_i(v)) = \Sigma(s_i(u))$ as desired. As an example, with u as in the previous example, consider

$$\begin{array}{rcccccc|cc|cc} v & = & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_9 & u_{10} & u_8 & u_7 \\ & = & 1 & 2 & 3 & 3 & 4 & 5 & 7 & 7 & 6 & 5 \end{array}$$

so that $j = 8$, and again let $i = 6$. Then as indicated by the dividers, $A^6(u)\uparrow = u_1 \ldots u_6$ so that $a_i = i = 6$, $B^6(u)\uparrow = u_9 u_{10}$ and $C^6(u)\downarrow = u_8 u_7$.

Case 2. $v_i \in B^i(u)\uparrow$.

In this case $a_i < i \leq a_i + b_i$. For example, with the same u as above, let

$$\begin{array}{rcccc|ccc|c|cc} v & = & u_2 & u_3 & u_4 & u_6 & u_7 & u_9 & u_{10} & u_8 & u_5 & u_1 \\ & = & 2 & 3 & 3 & 5 & 5 & 7 & 7 & 6 & 4 & 1 \end{array}$$

so that $j = 7$, and again let $i = 6$. Then $A^6(u)\uparrow = u_2 u_3 u_4 u_6$, $B^6(u)\uparrow = u_7 u_9 u_{10}$, $C^6(u)\downarrow = u_8$, and $D^6(u)\downarrow = u_5 u_1$, so $v_6 \in B^6(u)\uparrow$ and $a_6 = 4 < i \leq 7 = a_6 + b_6$.

Now let $B_1^i(u) = v_{a_i+1} \ldots v_i$ and $B_2^i(u) = v_{i+1} \ldots v_{a_i+b_i}$. Then $s_i(v) = B_2^i(u) C^i(u)\downarrow\ D^i(u)\downarrow$. When we compare the first i letters of v with the last i letters of v, we see that the letters in $B_1^i(u)$ are compared with the letters in $D^i(u)\downarrow$ since $|B_1^i(u)| = i - a_i = |D^i(u)\downarrow|$. But the letters in $D^i(u)\downarrow$ come from $\{u_s : s \leq i\}$ and the letters from $B_1^i(u)$ come from $\{u_s : s > i\}$. Thus any letter in $B_1^i(u)$ is greater than or equal to every letter in $D^i(u)\downarrow$ so that such letters will contribute $\Sigma(B_1^i(u)) - \Sigma(D^i(u)\downarrow)$ to $d_i(v)$. However the letters in

$A^i(u)\uparrow$ will be compared to letters that lie in either $C^i(u)\downarrow$, $B^i(u)\uparrow$, or later letters in $A^i(u)\uparrow$, and hence they will contribute 0 to $d_i(v)$. Thus

$$\begin{aligned} d_i(v) + \Sigma(s_i(v)) &= \Sigma(B_1^i(u)) - \Sigma(D^i(u)\downarrow) + \Sigma(B_2^i(u)) + \Sigma(C^i(u)\downarrow) + \Sigma(D^i(u)\downarrow) \\ &= \Sigma(B^i(u)\uparrow) + \Sigma(C^i(u)\downarrow) = \Sigma(s_i(u)). \end{aligned}$$

Case 3. $v_i \in C^i(u)\downarrow$.

In this case $a_i + b_i < i \le a_i + b_i + c_i$. For example, with the same u as above let

$$\begin{array}{rcll|ll|ll|llll} v & = & u_2 & u_3 & u_9 & u_{10} & u_8 & u_7 & u_6 & u_5 & u_4 & u_1 \\ & = & 2 & 3 & 7 & 7 & 6 & 5 & 5 & 4 & 3 & 1 \end{array}$$

so that $j = 4$, and again let $i = 6$. Then $A^6(u)\uparrow = u_2u_3$, $B^6(u)\uparrow = u_9u_{10}$, $C^6(u)\downarrow = u_8u_7$, and $D^6(u)\downarrow = u_6u_5u_4u_1$, so $v_6 \in C^6(u)\uparrow$ and $a_6 + b_6 = 4 < i \le 6 = a_6 + b_6 + c_6$.

Now let $C_1^i(u) = v_{a_i+b_i+1} \dots v_i$ and $C_2^i(u) = v_{i+1} \dots v_{a_i+b_i+c_i}$. Then $s_i(v) = C_2^i(u)D^i(u)\downarrow$. When we compare the first i letters of v with the last i letters of v, we see that the letters in $B^i(u)\uparrow C_1^i(u)$ are compared with the letters in $D^i(u)\downarrow$ since $|B^i(u)\uparrow| + |C_1^i(u)| = i - a_i = |D^i(u)\downarrow|$. But the letters in $D_i^i(u)\downarrow$ come from $\{u_s : s \le i\}$ and the letters from $B^i(u)\uparrow C_1^i(u)$ come from $\{u_s : s > i\}$. Thus any letter in $B^i(u)\uparrow C_1^i(u)$ is greater than or equal to every letter in $D^i(u)\downarrow$ so that such letters will contribute $\Sigma(B^i(u)\uparrow) + \Sigma(C_1^i(u)) - \Sigma(D^i(u)\downarrow)$ to $d_i(v)$. However the letters in $A^i(u)\uparrow$ will be compared to letters that lie in either $C^i(u)\downarrow$, $B^i(u)\uparrow$, or later letters in $A^i(u)\uparrow$, and hence they will contribute 0 to $d_i(v)$. Thus

$$\begin{aligned} d_i(v) + \Sigma(s_i(v)) &= \Sigma(B^i(u)\uparrow) + \Sigma(C_1^i(u)) - \Sigma(D^i(u)\downarrow) + \Sigma(C_2^i(u)) + \Sigma(D^i(u)\downarrow) \\ &= \Sigma(B^i(u)\uparrow) + \Sigma(C^i(u)\downarrow) = \Sigma(s_i(u)). \end{aligned}$$

Case 4. $v_i \in D^i(u)\uparrow$.

In this case $a_i + b_i + c_i < i$. For example, with the same u as above, now let

$$\begin{array}{rcl|ll|ll|lllll} v & = & u_2 & u_9 & u_{10} & u_8 & u_7 & u_6 & u_5 & u_4 & u_3 & u_1 \\ & = & 2 & 7 & 7 & 6 & 5 & 5 & 4 & 3 & 3 & 1 \end{array}$$

so that $j = 3$, and once again let $i = 6$. Then $A^6(u)\uparrow = u_2$, $B^6(u)\uparrow = u_9u_{10}$, $C^6(u)\downarrow = u_8u_7$, and $D^6(u)\downarrow = u_6u_5u_4u_3u_1$, so $v_6 \in D^6(u)\uparrow$ and $a_6 + b_6 + c_6 = 5 < i$.

Now let $D_1^i(u) = v_{a_i+b_i+c_i+1} \dots v_i$ and $D_2^i(u) = v_{i+1} \dots v_n$. Then $s_i(v) = D_2^i(u)$. When we compare the first i letters of v with the last i letters of v, we see that the letters in $B^i(u)\uparrow C^i(u)\downarrow D_1^i(u)$ are compared with the letters in $D^i(u)\downarrow$ since $|B^i(u)\uparrow| + |C^i(u)\downarrow| + |D_1^i(u)| = i - a_i = |D^i(u)\downarrow|$. But each letter in $B^i(u)\uparrow C^i(u)\downarrow D_1^i(u)$ will be greater than or equal to its corresponding letter in $D^i(u)\downarrow$, so that such letters will contribute

$$\begin{aligned} &\Sigma(B^i(u)\uparrow) + \Sigma(C^i(u)\downarrow) + \Sigma(D_1^i(u)) - (\Sigma(D_1^i(u)) + \Sigma(D_2^i(u))) = \\ &\Sigma(B^i(u)\uparrow) + \Sigma(C^i(u)\downarrow) - \Sigma(D_2^i(u)) \end{aligned}$$

to $d_i(v)$. However the letters in $A^i(u)\uparrow$ will be compared to letters that lie in either $C^i(u)\downarrow$, $B^i(u)\uparrow$, or later letters in $A^i(u)\uparrow$ and hence they will contribute 0 to $d_i(v)$. Thus

$$\begin{aligned} d_i(v) + \Sigma(s_i(v)) &= \Sigma(B^i(u)\uparrow) + \Sigma(C^i(u)\downarrow) - \Sigma(D_2^i(u)) + \Sigma(D_2^i(u)) \\ &= \Sigma(B^i(u)\uparrow) + \Sigma(C^i(u)\downarrow) = \Sigma(s_i(u)). \end{aligned}$$

□

We are now ready for the result referred to immediately before Lemma 6.

Theorem 7. *If $u, v \in \mathbb{P}^*$ have increasing/decreasing factorizations, then $u \backsim v$ if and only if u is a rearrangement of v.*

Proof. Suppose $u, v \in \mathbb{P}^*$ have increasing/decreasing factorizations. If u is a rearrangement of v, then $S(u; t, x) = S(v; t, x)$ by Theorem 1 and Lemma 6. Hence $u \backsim v$.

For the converse, suppose $u \backsim v$. Since we have just shown that a word with an increasing/decreasing factorization is Wilf equivalent to any rearrangement of itself with an increasing/decreasing factorization, it suffices to consider the case when u and v are both nondecreasing, and to show that $u = v$. So let $u = u_1u_2\cdots u_n$ and $v = v_1v_2\cdots v_n$ be nondecreasing. First note that $u \backsim v$ implies $\mathrm{wt}(u) = \mathrm{wt}(v)$ since for any word w, the minimum powers of x and t in $F(w; t, x)$ are $\Sigma(u)$ and $|u|$, respectively. So the numerators of the expressions for $S(u; t, x) = S(v; t, x)$ in Theorem 1 are equal. Equating the denominators, and noting that $d_i = 0$ for all i for both u and v, we have

$$\begin{aligned} &t^n x^{\Sigma(v)} + (1 - x - tx) \sum_{i=1}^{n} t^{n-i} x^{\sum_{j=i+1}^{n} v_j} (1 - x)^{i-1} \\ &= t^n x^{\Sigma(u)} + (1 - x - tx) \sum_{i=1}^{n} t^{n-i} x^{\sum_{j=i+1}^{n} u_j} (1 - x)^{i-1}. \end{aligned}$$

Simplifying, this becomes

$$\sum_{i=1}^{n} t^{n-i} x^{\sum_{j=i+1}^{n} v_j} (1 - x)^{i-1} = \sum_{i=1}^{n} t^{n-i} x^{\sum_{j=i+1}^{n} u_j} (1 - x)^{i-1}.$$

Hence for each i, $1 \le i \le n$, we have

$$x^{\sum_{j=i+1}^{m} v_j} = x^{\sum_{j=i+1}^{n} u_j},$$

and therefore $u = v$. □

Since the values of $d_i + \Sigma(s_i)$ determine equivalence for those words with increasing/decreasing factorizations, it is natural to ask the same question about those that do not. Unfortunately, equality of $d_i + \Sigma(s_i)$ for all i is not enough to determine equivalence in general. For example, it was shown in [2] that 24153 and 24315 are not Wilf equivalent, but both have the following values:

i	$d_i + \Sigma(s_i)$
1	13
2	10
3	9
4	8

.

However, we have not found two words that are Wilf equivalent that have different values of $d_i + \Sigma(s_i)$. In particular, the equivalences proved by Kitaev, Liese, Remmel, and Sagan [2] all preserve equality between the $d_i + \Sigma(s_i)$'s.

3 Connections with some known sequences

In this section we show that some well-known sequences occur as coefficients in the generating functions $S(u; 1, x)$, $A(u; 1, x)$, and $F(u; 1, x)$ for some simple words u. For those sequences that appear in the OEIS, we provide the sequence number as well as a combinatorial proof as to why the entries of the sequence count particular words.

3.1 Words consisting of a single digit

The first family of words we will examine is that of words consisting of a single digit $i \geq 1$. From Theorem 1 we obtain that

$$S(i; t, x) = \frac{tx^i}{tx^i + (1 - x - tx)}$$

and

$$A(i; t, x) = \frac{1-x}{1-x-tx}(1 - S(i; t, x)) = \frac{1}{1 - t\sum_{j=1}^{i-1} x^j}.$$

If we let $t = 1$ and $i = 3, 4 \ldots$, we obtain some familiar generating functions:

$$\begin{aligned}
A(3; 1, x) &= \frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \cdots \\
A(4; 1, x) &= \frac{1}{1-x-x^2-x^3} = 1 + x + 2x^2 + 4x^3 + 7x^4 + 13x^5 + 24x^6 + \cdots \\
A(5; 1, x) &= \frac{1}{1-x-x^2-x^3-x^4} = 1 + x + 2x^2 + 4x^3 + 8x^4 + 15x^5 + 29x^6 + \cdots \\
A(6; 1, x) &= \frac{1}{1-x-x^2-x^3-x^4-x^5} = 1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + 31x^6 + \cdots ..
\end{aligned}$$

The coefficients of $A(3; 1, x)$, $A(4; 1, x)$, $A(5; 1, x)$, $A(6; 1, x)$ are the Fibonacci numbers (sequence A000045 in OEIS), Tribonacci numbers (A000073) , Tetranacci numbers (A000078), and Pentanacci numbers (A001591), respectively. In general, the coefficient of x^j in $A(i; 1, x)$

is F_{j+1}^{i-1},the $(i-1)$-step Fibonnaci number. The n-step Fibonacci number is defined by $F_k^n = 0$ for $k \leq 0$, $F_1^n = F_2^n = 1$, and all other terms by the recurrence

$$F_k^n = \sum_{i=1}^{n} F_{k-i}^n.$$

This fact is easily verified by classifying words in $\mathcal{A}(i)$ by their last digit. If we expand $S(i;1,x)$ as a series we obtain

$$\begin{aligned}
S(3;1,x) &= \frac{x^3}{1-2x+x^3} = x^3(1+2x+4x^2+7x^3+12x^4+20x^5+33x^6+\cdots) \\
S(4;1,x) &= \frac{x^4}{1-2x+x^4} = x^4(1+2x+4x^2+8x^3+15x^4+28x^5+52x^6+\cdots) \\
S(5;1,x) &= \frac{x^5}{1-2x+x^5} = x^5(1+2x+4x^2+8x^3+16x^4+31x^5+60x^6+\cdots) \\
S(6;1,x) &= \frac{x^6}{1-2x+x^6} = x^6(1+2x+4x^2+8x^3+16x^4+32x^5+63x^6+\cdots).
\end{aligned}$$

The coefficients of $S(3;1,x)$, $S(4;1,x)$, $S(5;1,x)$, and $S(6;1,x)$ are partial sums of Fibonacci (A000071), Tribonacci (A008937), Tetranacci (A107066), and Pentanacci (A001949) numbers, respectively. In fact, the coefficients of $S(i;1,x)$ are the partial sums of the $(i-1)$-step Fibonacci numbers and can be found in i-th column of the array defined in A172119. It is also easy to verify this fact by classifying words in $\mathcal{S}(i)$ by their last digit.

Lastly, using the relationship

$$F(u;t,x) = \frac{1-x}{1-x-tx} - A(u;t,x),$$

we obtain

$$F(i;t,x) = \frac{1-x}{1-x-tx} - \frac{1}{1-t\sum_{j=1}^{i-1} x^j}.$$

With $t=1$ this simplifies to

$$F(i;1,x) = \frac{x^i}{1-3x+x^2+\cdots+x^{i-1}+2x^i}.$$

Expanding $F(i;1,x)$ as a series we then obtain

$$\begin{aligned}
F(3;1,x) &= \frac{x^3}{1-3x+x^2+2x^3} = x^3(1+3x+8x^2+19x^3+43x^4+94x^5+201x^6+\cdots) \\
F(4;1,x) &= \frac{x^4}{1-3x+x^2+x^3+2x^4} = x^4(1+3x+8x^2+20x^3+47x^4+107x^5+ \\
&\qquad 238x^6+\cdots) \\
F(5;1,x) &= \frac{x^5}{1-3x+x^2+x^3+x^4+2x^5} = x^5(1+3x+8x^2+20x^3+48x^4+ \\
&\qquad 111x^5+251x^6+\cdots) \\
F(6;1,x) &= \frac{x^6}{1-3x+x^2+x^3+x^4+x^5+2x^6} = x^6(1+3x+8x^2+20x^3+48x^4+ \\
&\qquad 112x^5+255x^6+\cdots).
\end{aligned}$$

The coefficient of x^j for $F(3;1,x)$, $F(4;1,x)$, $F(5;1,x)$, and $F(6;1,x)$ is 2^{j-1} minus F^2_{j+1} (A008466), 2^{j-1} minus F^3_{j+1} (A050231), 2^{j-1} minus F^4_{j+1} (A050232) and 2^{j-1} minus F^5_{j+1} (A050233) respectively. In general, the coefficient of x^j in $F(i;1,x)$ is 2^{j-1} minus F^{i-1}_{j+1}. This fact is also easily verified as the total number of words of weight j is simply 2^{j-1} (the number of compositions of j) and we can subtract the number of words that avoid u, which has already been shown to be an $(i-1)$-step Fibonacci number, to obtain the number of words that embed u.

3.2 Words of the form $1\ (1+s)\ 1$

We now turn to a different family of words. Suppose that $u = r\ (r+s)\ r$ where $r, s \geq 1$. Then in the notation of Theorem 1, $s_1 = (r+s)\ r$, $s_2 = r$ and $s_3 = \varepsilon$. To compute $d_1(u)$ and $d_2(u)$, consider the arrays

$$\begin{array}{cccccc} r & r+s & r & & & \\ & & r & r+s & r & \end{array} \qquad \begin{array}{ccccc} r & r+s & r & & \\ & r & & r+s & r. \end{array}$$

It is easy to see from these arrays that $d_1(u) = 0$ and $d_s(u) = s$. Thus $d_1(u) + \Sigma(s_1) = 2r + s$ and $d_2(u) + \Sigma(s_2) = r + s$. By definition $d_3(u) = 0$ so that $d_3(u) + \Sigma(s_3) = 0$. Thus by Theorem 1

$$S(r\ (r+s)\ r;t,x) = \frac{t^3x^{3r+s}}{t^3x^{3r+s} + (1-x-xt)(t^2x^{2r+s} + tx^{r+s}(1-x) + (1-x)^2)}$$

and

$$A(r\ (r+s)\ r;t,x) = \frac{1-x}{1-x-xt}(1 - S(r\ (r+s)\ r;t,x)).$$

Now, when $t = 1$ and $r = 1$ these simplify to

$$\begin{aligned}
S(1\ (1+s)\ 1;1,x) &= \frac{x^{3+s}}{(1-x)^3(1-\sum_{i=1}^{s} x^i)} \\
A(1\ (1+s)\ 1;1,x) &= \frac{1-2x+x^2+x^{s+1}}{(1-x)^2(1-\sum_{i=1}^{s} x^i)}.
\end{aligned}$$

We can expand these functions as power series around $x = 0$ and find that

$$\begin{aligned}
S(121;1,x) &= x^4 + 4x^5 + 10x^6 + 20x^7 + 35x^8 + 56x^9 + 84x^{10} + 120x^{11} + 165x^{12} + \\
&\quad 220x^{13} + 286x^{14} + 364x^{15} + 455x^{16} + 560x^{17} + 680x^{18} + \cdots, \\
S(131;1,x) &= x^5 + 4x^6 + 11x^7 + 25x^8 + 51x^9 + 97x^{10} + 176x^{11} + 309x^{12} + 530x^{13} + \\
&\quad 894x^{14} + 1490x^{15} + 2462x^{16} + 4043x^{17} + 6610x^{18} + \cdots, \\
S(141;1,x) &= x^6 + 4x^7 + 11x^8 + 26x^9 + 56x^{10} + 114x^{11} + 224x^{12} + 430x^{13} + 813x^{14} + \\
&\quad 1522x^{15} + 2831x^{16} + 5244x^{17} + 9688x^{18} + \cdots, \text{and} \\
S(151;1,x) &= x^7 + 4x^8 + 11x^9 + 26x^{10} + 57x^{11} + 119x^{12} + 241x^{13} + 479x^{14} + 941x^{15} + \\
&\quad 1835x^{16} + 3562x^{17} + 6895x^{18} + \cdots.
\end{aligned}$$

Now, the sequence $1, 4, 10, 20, 35, 56, \ldots$ of coefficients starting at x^4 for $S(121;1,x)$ is the sequence of tetrahedral numbers (A000292), defined by $a(n) = \binom{n+2}{3}$. Thus we obtain a new combinatorial interpretation of these numbers. That is, $a(n)$ equals the number of words u such that $\sum(u) = n+3$ and $u \in \mathcal{S}(121)$. In fact, there is a simple bijective proof of this fact. It is well known that the tetrahedral numbers count the number of weak compositions of $n-1$ into 4 parts. Given such a weak composition of $n-1$ into 4 parts, say $c = (c_1, c_2, c_3, c_4)$ we can define $f(c)$ to be the word

$$f(c) := (c_1 + 1)\ \underbrace{1\ 1\ \ldots\ 1}_{c_2}\ (c_3 + 2)\ (c_4 + 1).$$

Note that $\sum f(c) = n + 3$ and $f(c) \in \mathcal{S}(121)$. It is a simple verification that f is indeed a bijection.

Similarly the sequence of coefficients starting at x^5 for $S(131;1,x)$ appears in the OEIS as sequence A014162. In this case, with an off-set of 4, these numbers $b(n)$ count the number of 132-avoiding two-stack sortable permutations which contain exactly one subsequence of type 51234. See Egge and Mansour [1]. Again, we obtain a new combinatorial interpretation of these numbers. That is, $b(n)$ equals the number of words u such that $\sum(u) = n + 4$ and $u \in \mathcal{S}(131)$. A bijective proof of this connection is an interesting open problem.

The sequences of coefficients for $S(141, 1, x)$ and $S(151, 1, x)$ have not previously appeared in the OEIS.

Similarly, one can expand $A(1\ (1+s)\ 1; 1, x)$ as a power series about $x = 0$ and find that

$$\begin{aligned}
A(121;1,x) &= 1 + x + 2x^2 + 4x^3 + 7x^4 + 11x^5 + 16x^6 + 22x^7 + 29x^8 + 37x^9 + 46x^{10} + \\
&\quad 56x^{11} + 67x^{12} + 79x^{13} + 92x^{14} + 106x^{15} + \cdots, \\
A(131;1,x) &= 1 + x + 2x^2 + 4x^3 + 8x^4 + 15x^5 + 27x^6 + 47x^7 + 80x^8 + 134x^9 + 222x^{10} + \\
&\quad 365x^{11} + 597x^{12} + 973x^{13} + 1582x^{14} + 2568x^{15} + \cdots, \\
A(141;1,x) &= 1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + 31x^6 + 59x^7 + 111x^8 + 207x^9 + 384x^{10} + \\
&\quad 710x^{11} + 1310x^{12} + 2414x^{13} + 4445x^{14} + 8181x^{15} + \cdots, \text{ and} \\
A(151;1,x) &= 1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + 32x^6 + 63x^7 + 123x^8 + 239x^9 + 463x^{10} + \\
&\quad 895x^{11} + 1728x^{12} + 3334x^{13} + 6430x^{14} + 12398x^{15} + \cdots.
\end{aligned}$$

The coefficient of x^n in $A(121; 1, x)$ is $a(n) = \binom{n}{2}+1$. These numbers are the central polygonal numbers (A000124) and one interpretation of $a(n+1)$ is the number of length n binary words that have no 0-digits between any pair of consecutive 1-digits. There is a simple bijective proof of this fact which we will present at the end of this subsection. Another interpretation is the maximal number of pieces obtained when slicing a pancake with n cuts.

The sequence of coefficients in the expansion of $A(131; 1, x)$ starting at x is A000126 and counts the number of length n binary words with fewer than two 0-digits between any pair of consecutive 1-digits. It also counts the number of ternary numbers with no 0-digit and at least one 2-digit.

The sequence of coefficients in the expansion of $A(141; 1, x)$ starting at x is A007800 counts the number of length n binary words with fewer than three 0-digits between any pair of consecutive 1-digits. It is also said to have come from a problem in AI planning and satisfies a recurrence $a(n) = 4+a(n-1)+a(n-2)+a(n-3)+a(n-4)-a(n-5)-a(n-6)-a(n-7)$ for $n > 7$.

The sequences of coefficients in the expansion of $A(151; 1, x)$ and $A(161; 1, x)$ starting at x are A145112 and A145113 respectively, and count the number of length n binary words with fewer than four (resp. five) 0-digits between any pair of consecutive 1-digits.

In general, the sequence of coefficients in the expansion of $A(1i1; 1, x)$ for $i \geq 2$ starting at x counts the number of length n binary words with fewer than $i-1$ 0-digits between any pair of consecutive 1-digits. We then obtain a new combinatorial interpretation of these numbers as the number of words of u such that $\sum(u) = n + 1$ and u does not embed $1i1$. There is a simple bijective proof of this fact. Given a binary word w of length n with fewer than $i - 1$ 0-digits between any pair of consecutive 1-digits, we define $f(w)$ to be the word obtained by adding a 1 to the end of w and then replacing every maximal run of k consecutive 0-digits followed by a 1 by the single digit $k + 1$. For example, suppose $w = 01000111010000100100$ then

$$f(w) = 2\ 4\ 1\ 1\ 2\ 5\ 3\ 3.$$

Note that $\sum f(w) = n+1$ and the condition of having fewer than $i-1$ 0-digits in w guarantees that $f(w) \in \mathcal{A}(1i1)$. It is a simple verification that f is indeed a bijection between length n binary words having fewer than $i - 1$ 0-digits between any pair of consecutive 1-digits and words of weight $n + 1$ in $\mathcal{A}(1i1)$.

Finally, we note that as in the previous subsection, we can also compute the coefficients in the expansion of $F(1\ (1 + s)\ 1; 1, x)$ for various s. However, these do not appear in the OEIS, and therefore we omit them here.

3.3 The word 123

Another simple example is $S(123; t, x)$. In this case it is easy to see that $d_1(123) = d_2(123) = d_3(123) = 0$. Thus it follows that

$$\begin{aligned} S(123; t, x) &= \frac{t^3x^6}{t^3x^6 + (1 - x - xt)(t^2x^5 + tx^3(1 - x) + (1 - x)^2)} \quad \text{and} \\ A(123; t, x) &= \frac{1 - x}{1 - x - xt}(1 - S(123; t, x)). \end{aligned}$$

One can compute that

$$\begin{aligned} S(123;t,x) &= \frac{x^6}{(1-x)^2(x^4-x^3+2x-1)} \text{ and} \\ A(123;t,x) &= \frac{1-2x+x^2+x^3-x^4+x^5}{(1-x)^2(x^4-x^3+2x-1)}. \end{aligned}$$

Expanding these functions as power series about $x = 0$ and letting $t = 1$, we obtain that

$$\begin{aligned} S(123;1,x) &= x^6+4x^7+11x^8+25x^9+52x^{10}+103x^{11}+199x^{12}+\cdots \text{and} \\ A(123;1,x) &= 1+x+2x^2+4x^3+8x^4+16x^5+31x^6+59x^7+111x^8+ \\ &\quad 208x^9+389x^{10}+727x^{11}+1358x^{12}+\cdots. \end{aligned}$$

In this case, neither sequence of coefficients have appeared in the OEIS.

4 Wilf equivalence for words of length 3

We now turn to the classification of the Wilf equivalence classes for all words of length 3 in $\mathcal{P}_1 = (\mathbb{P}, \leq)$. Theorem 1 will provide the necessary information for words with increasing/decreasing factorizations. The only words of length 3 without increasing/decreasing factorizations are words of the form bac or cab where $a < b \leq c$. But Kitaev, Liese, Remmel, and Sagan (Lemma 4.1 of [2]) show that any word is Wilf equivalent to its reverse, so it suffices to consider bac. To that end, we give an explicit formula for $S(bac;t,x)$ where $a < b \leq c$.

Theorem 8. *For positive integers $a < b \leq c$,*

$$S(bac;t,x) = \frac{t^3x^{a+b+c}(1+tx^c(1+x+\cdots+x^{b-a-1}))}{(1-x-tx)\psi_{a,b,c}(t,x)+t^3x^{a+b+c}(1+tx^c(1+x+\cdots+x^{b-a-1}))}$$

where

$$\psi_{a,b,c}(t,x) = (1-x)^2+tx^c(1-x)+t^2x^{a+c}+t^3x^{a+2c}(1+x+\cdots+x^{b-a-1}).$$

Proof. We start with the following expression, which follows from (2) and Lemma 4, with extra terms to account for the fact that bac does not have an increasing/decreasing factorization:

$$\begin{aligned} S(bac;t,x) &= A(bac;t,x)\,t^3\frac{x^{a+b+c}}{(1-x)^3} - S(bac;t,x)\,t^2\frac{x^{a+c}}{(1-x)^2} - S(bac;t,x)\,t\frac{x^c}{1-x} \\ &\quad +A(bac;t,x)\,t^4\frac{x^{b+2c}}{(1-x)^3}(x^a+\cdots+x^{b-1}) \\ &\quad -S(bac;t,x)\,t^3\frac{x^{2c}}{(1-x)^2}(x^a+\cdots+x^{b-1}). \end{aligned}$$

The first term,

$$A(bac;t,x)\,t^3\frac{x^{a+b+c}}{(1-x)^3},$$

is the generating function for words consisting of an embedding of bac appended to a word that avoids bac. This includes the words in $\mathcal{S}(bac)$, but also includes words that end in overlapping embeddings of bac, either in the last four or five characters (and do not embed bac prior to those embeddings). So we need to remove the terms associated with these words. First consider words w that end in overlapping embeddings in the last five characters, as shown:

$$\begin{array}{cccccccc} w = & \cdots & b^+ & a^+ & c^+ & a^+ & c^+ \\ \hline & & b & a & c & & \\ & & & & b & a & c \end{array},$$

where for a positive integer m, m^+ represents any integer greater than or equal to m. Since $b \le c$, these words can be formed by appending an embedding of ac to words in $\mathcal{S}(bac)$. So the second term on the right hand side,

$$S(bac;t,x)\,t^2\frac{x^{a+c}}{(1-x)^2},$$

accounts for these words.

Now consider words that end in overlapping embeddings in the final four characters only:

$$\begin{array}{cccccc} w = & \cdots & b^+ & b^+ & c^+ & c^+ \\ \hline & & b & a & c & \\ & & & b & a & c \end{array}.$$

The third term,

$$S(bac;t,x)\,t\frac{x^{c}}{1-x},$$

removes the terms associated with these words by appending an embedding of c to words in $\mathcal{S}(bac)$. However, because $a < b$, this also includes terms associated with words of the following form:

$$\begin{array}{cccccc} w = & \cdots & b^+ & [a,b) & c^+ & c^+ \\ \hline & & b & a & c & \\ & & & b & a & c \end{array},$$

that is, words that first embed bac starting at the fourth character from the end, and end in an embedding of $bacc$ but not $bbcc$. To correct for these terms, consider the fourth term on the right hand side:

$$A(bac;t,x)\,t^4\frac{x^{b+2c}}{(1-x)^3}(x^a+\cdots+x^{b-1}).$$

This is the generating function for those words that end in an embedding of $bacc$ but not $bbcc$ (hence the $x^a+\cdots+x^{b-1}$ term), appended to words that avoid bac. So this includes the words that we want, but also may include words that first embed bac beginning at the fifth or sixth character from the end. However, the first of these situations is impossible, as shown,

$$\begin{array}{ccccccc} w = & \cdots & \bullet & b^+ & [a,b) & c^+ & c^+ \\ \hline & & b & a & c & & \\ & & & b & a & c & \\ & & & & b & a & c \end{array},$$

since a character in $[a,b)$ cannot embed c. The final term,

$$S(bac;t,x)\,t^3\frac{x^{2c}}{(1-x)^2}(x^a+\cdots+x^{b-1}),$$

accounts for the second possibility,

$$\begin{array}{cccccccc} w = & \cdots & b^+ & a^+ & c^+ & [a,b) & c^+ & c^+ \\ \hline & & b & a & c & & & \\ & & & & b & a & c & \\ & & & & & b & a & c \end{array},$$

by appending an embedding of acc, whose first character is in $[a,b)$, to words in $\mathcal{S}(bac)$.

We can now solve for $S(bac;t,x)$:

$$\begin{aligned} S(bac;t,x) &= \frac{t^3\frac{x^{a+b+c}}{(1-x)^3}+t^4\frac{x^{b+2c}}{(1-x)^3}(x^a+\cdots+x^{b-1})}{1+t\frac{x^c}{1-x}+t^2\frac{x^{a+c}}{(1-x)^2}+t^3\frac{x^{2c}}{(1-x)^2}(x^a+\cdots+x^{b-1})}A(bac;t,x) \\ &= \frac{t^3x^{a+b+c}+t^4x^{b+2c}(x^a+\cdots+x^{b-1})}{(1-x)^3+tx^c(1-x)^2+t^2x^{a+c}(1-x)+t^3x^{2c}(1-x)(x^a+\cdots+x^{b-1})} \\ &\quad \cdot A(bac;t,x). \end{aligned}$$

Substituting $A(bac;t,x)=\frac{1-x}{1-x-tx}(1-S(bac;t,x))$, we obtain

$$\begin{aligned} S(bac;t,x) &= \frac{t^3x^{a+b+c}+t^4x^{b+2c}(x^a+\cdots+x^{b-1})}{(1-x)^2+tx^c(1-x)+t^2x^{a+c}+t^3x^{2c}(x^a+\cdots+x^{b-1})} \\ &\quad \cdot\frac{1}{1-x-tx}(1-S(bac;t,x)). \end{aligned}$$

Solving for $S(bac;t,x)$ and factoring appropriate terms gives the result. □

As a corollary, we can now classify Wilf equivalence for words of the form bac with $a<b\le c$.

Corollary 9. *For positive integers $a<b\le c$, the only words Wilf equivalent to bac are bac and cab.*

Proof. As noted previously, $bac \backsim cab$ since they are reverses of each other. So it remains to show that no other words of length three are Wilf equivalent to these two. First, note that $1+tx^c(1+x+\cdots+x^{b-a-1})$ does not divide the denominator of the expression for $S(bac;t,x)$

in Theorem 8 since, for example, it does not divide it when $x = 1$. So $S(bac; t, x)$ does not have a single monomial in the numerator, and therefore is not equal to $S(u; t, x)$ for any u that has an increasing/decreasing factorization by Theorem 1. So suppose $bac \backsim b'a'c'$ with $a' < b' \leq c'$. We will show that $a = a'$, $b = b'$ and $c = c'$.

Equating $S(bac; t, x)$ and $S(b'a'c'; t, x)$ from Theorem 8, we have

$$\begin{aligned}&\phi_{a,b,c}(t,x)\left[(1-x-tx)\psi_{(a',b',c')}(t,x)+\phi_{a',b',c'}(t,x)\right]\\&=\phi_{a',b',c'}(t,x)\left[(1-x-tx)\psi_{(a,b,c)}(t,x)+\phi_{a,b,c}(t,x)\right],\end{aligned}$$

where $\phi_{a,b,c}(t, x) = t^3x^{a+b+c}(1+tx^c(1+x+\cdots+x^{b-a-1}))$, and similarly for $\phi_{a',b',c'}(t, x)$. Since $bac \backsim b'a'c'$, we have $a + b + c = a' + b' + c'$, so we may simplify to

$$\begin{aligned}&(1+tx^c(1+x+\cdots+x^{b-a-1}))\psi_{(a',b',c')}(t,x)\\&=(1+tx^{c'}(1+x+\cdots+x^{b'-a'-1}))\psi_{(a,b,c)}(t,x).\end{aligned}$$

Recalling that

$$\psi_{(a,b,c)}(t,x)=(1-x)^2+tx^c(1-x)+t^2x^{a+c}+t^3x^{2c}(x^a+\cdots+x^{b-1}),$$

and equating powers of t^2 on both sides, we have

$$x^{a'+c'}+x^{c+c'}(1-x)(1+x+\cdots+x^{b-a-1})=x^{a+c}+x^{c+c'}(1-x)(1+x+\cdots+x^{b'-a'-1})$$

Since $a < c$, the smallest power of x on the left is $a' + c'$, and the smallest on the right is $a + c$. So

$$a'+c'=a+c.$$

Since we know $a + b + c = a' + b' + c'$, this gives $b = b'$. Now equating the largest powers of x, we have

$$c+c'+b-a=c+c'+b'-a',$$

which gives $a = a'$, and therefore $c = c'$. □

We are now ready to completely classify Wilf equivalence of words of length 3.

Theorem 10. *Wilf equivalence relative to $\mathcal{P}_1$ partitions $\mathbb{P}^3$ into the following equivalence classes.*

1. $\{aaa\}$ *for any* $a \in \mathbb{P}$
2. $\{aab, aba, baa\}$ *if* $a < b$
3. $\{aab, baa\}$ *and* $\{aba\}$ *if* $a > b$
4. $\{bac, cab\}$ *and* $\{abc, acb, cba, bca\}$ *if* $a < b < c$.

Proof. Theorem 7 establishes the equivalence among words in the sets $\{aab, aba, baa\}$ in case 2, the sets $\{aab, baa\}$ in case 3, and $\{abc, acb, cba, bca\}$ in case 4, as well as the fact that these sets are all in distinct Wilf equivalences classes. The remaining cases, $\{bac, cab\}$ in case 4 and $\{aba\}$ in case 3, follow from Corollary 9. □

Note that when we consider the permutations of S_3, there are only two Wilf equivalence classes, namely, $\{123, 132, 321, 231\}$ and $\{213, 312\}$. We computed $S(123; t, x)$ and $A(123; t, x)$ in the previous section. Thus to complete the possibilities for $S(\sigma; t, x)$ for $\sigma \in S_3$, we need only compute $S(213; t, x)$ and $A(213; t, x)$. In this case, we must use Theorem 8 from which we obtain that

$$\begin{aligned} S(213; t, x) &= \frac{t^3x^6(1 + tx^3)}{(1 - x - xt)((1 - x)^2 + tx^3(1 - x) + t^2x^4 + t^3x^7) + t^3x^6(1 + tx^3)} \text{ and} \\ A(213; t, x) &= \frac{1 - x}{1 - x - xt}(1 - S(213; t, x)). \end{aligned}$$

One can compute that

$$\begin{aligned} S(213; 1, x) &= \frac{x^6 + x^8}{1 - 4x + 5x^2 - x^3 - 2x^4 + 2x^6 + x^7 - 2x^8} \text{ and} \\ A(213; 1, x) &= \frac{(1 - x)(1 - 4x + 5x^2 - x^3 - 2x^4 + x^6 + x^7 - 3x^8)}{(1 - 2x)(1 - 4x + 5x^2 - x^3 - 2x^4 + 2x^6 + x^7 - 2x^8)}. \end{aligned}$$

In this case, if one expands these functions as power series about $x = 0$, one obtains

$$\begin{aligned} S(213; 1, x) &= x^6 + 4x^7 + 11x^8 + 26x^9 + 55x^{10} + 109x^{11} + 207x^{12} + 381x^{13} + \\ &\quad 684x^{14} + 1201x^{15} + O[x]^{16} \text{ and} \\ A(213; 1, x) &= 1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + 31x^6 + 59x^7 + 111x^8 + 207x^9 + \\ &\quad 385x^{10} + 716x^{11} + 1334x^{12} + 2494x^{13} + 4685x^{14} + 8853x^{15} + O[x]^{16}. \end{aligned}$$

However, neither of the two sequence of coefficients have appeared in the OEIS.

5 The strong rearrangement conjecture

In this section we discuss the strong rearrangement conjecture and its connection to the family of finite posets $\mathcal{P}_{[m]} = ([m]^*, \leq)$. We also give an analogue of Theorem 1 for $S(u; x_1, \ldots, x_n)$. Our first result relates Wilf equivalence in $[m]^*$ to Wilf equivalences in $\mathbb{P}^*$ that are witnessed by rearrangement maps.

Theorem 11. *Suppose $u, v \in [m]^*$ for some positive integer m. Then $u \backsim_{[m]} v$ if and only if there exists a rearrangement map $f : \mathbb{P}^* \to \mathbb{P}^*$ that witnesses the Wilf equivalence $u \backsim v$.*

Proof. First note that if there is a rearrangement map $f : \mathbb{P}^* \to \mathbb{P}^*$ that witnesses the Wilf equivalence $u \backsim v$, then the restriction of f to $[m]^*$ is a $W_{[m]}$-preserving bijection that shows $u \backsim_{[m]} v$.

For the converse, suppose $u, v \in [m]^*$ and $u \backsim_{[m]} v$, so that $F(u; x_1, \ldots, x_m) = F(v; x_1, \ldots, x_m)$. Then there is a $W_{[m]}$-preserving bijection $g : \mathcal{F}(u) \cap [m]^* \to \mathcal{F}(v) \cap [m]^*$. So $g(w)$ is a rearrangement of w for all w. This bijection can then be lifted to the desired rearrangement f, as follows. Suppose $w = w_1 \cdots w_n \in \mathbb{P}^*$ and $1 \leq i_1 < \cdots < i_l \leq n$ is the sequences of indices i such that $w_i \geq m$. Then let $\overline{w}$ be the word in $[m]^*$ that results by replacing each w_{i_k} with m. Then $u \leq w$ if, and only if, $u \leq \overline{w}$. Now apply g to $\overline{w}$. Then since $z = g(\overline{w})$ is a rearrangement of $\overline{w}$, there is a sequence $1 \leq j_1 < \cdots < j_l \leq n$ consisting of all the indices j such that $z_j = m$. Then let $f(w)$ be the result of replacing z_{j_k} by w_{i_k} for $k = 1, \ldots, l$. $\square$

Theorem 11 shows that the question of whether $u \backsim v$ implies a rearrangement witnessing the equivalence can be answered by restricting to a finite alphabet. We have computed $S(u; x_1, \ldots, x_5)$ for all permutations in S_n for $n \leq 5$ and indeed, if $u \backsim v$ in this case, then $S(u; x_1, \ldots, x_5) = S(v; x_1, \ldots, x_5)$. Thus the strong rearrangement conjecture holds for these words.

Next we consider an analogue of Theorem 1 for the more refined generating functions $S(u; x_1, \ldots, x_m)$. It is still the case that

$$\mathcal{S}(u) \cap [m]^* = (\mathcal{A}(u) \cap [m]^*)(\mathcal{W}(u) \cap [m]^*) - \left(\bigcup_{i=1}^{n-1} (\mathcal{S}^{(i)}(u) \cap [m]^*) \right). \tag{5}$$

It is easy to see that

$$\begin{aligned} \sum_{w \in \mathcal{A}(u)\mathcal{W}(u) \cap [m]^*} W_{[m]}(w) &= A(u; x_1, \ldots, x_m) \prod_{r=1}^{n} \sum_{s=u_i}^{m} x_j \\ &= \frac{1}{1 - \sum_{i=1}^{m} x_i} (1 - S(u; x_1, \ldots, x_m)) \prod_{r=1}^{n} \sum_{s=u_r}^{m} x_s. \end{aligned} \tag{6}$$

We also have that

$$\mathcal{S}^{(i)}(u) \cap [m]^* = \bar{\mathcal{S}}^{(i)}(u)\mathcal{W}(s_i(u)) \cap [m]^*.$$

Thus if

$$\begin{aligned} S^{(i)}(u, x_1, \ldots, x_m) &= \sum_{w \in \mathcal{S}^{(i)}(u) \cap [m]^*} W_{[m]}(w) \text{ and} \\ \bar{S}^{(i)}(u, x_1, \ldots, x_m) &= \sum_{w \in \bar{\mathcal{S}}^{(i)}(u) \cap [m]^*} W_{[m]}(w), \end{aligned}$$

then we will have

$$S^{(i)}(u, x_1, \ldots, x_m) = \bar{S}^{(i)}(u, x_1, \ldots, x_m) \prod_{r=i+1}^{n} \sum_{s=u_r}^{m} x_s.$$

The only step in our proof of Theorem 1 which does not have an analogue in this case is the fact that

$$\bar{S}^{(i)}(u; t, x) = x^{d_i(u)} S(u; t, x).$$

It will no longer be the case that $\bar{S}^{(i)}(u; x_1, \ldots, x_m)$ is a multiple of $S(u; x_1, \ldots, x_m)$ if $d_i(u) > 0$. However, if $d_i(u) = 0$, then it will be the case that $\bar{S}^{(i)}(u) \cap [m]^* = \mathcal{S}^{(i)}(u) \cap [m]^*$ so that

$$\bar{S}^{(i)}(u, x_1, \ldots, x_m) = S(u, x_1, \ldots, x_m).$$

Thus if $d_i(u) = 0$ for all $i = 1, \ldots, n-1$, then we will have

$$S^{(i)}(u, x_1, \ldots, x_m) = S(u, x_1, \ldots, x_m) \sum_{r=i+1}^{n} \sum_{s=u_r}^{m} x_s \tag{7}$$

for all i. However, it is easy to see that $d_i(u) = 0$ for all $i = 1, \ldots, n-1$ if and only if $u_1 \leq \cdots \leq u_n$. In that case, we can see from (5), (6), and (7) that

$$\begin{aligned} S(u, x_1, \ldots, x_m) &= \frac{1}{1 - \sum_{i=1}^m x_i}(1 - S(u; x_1, \ldots, x_m)) \sum_{r=1}^{n} \sum_{s=u_r}^{m} x_s \\ &\quad - \sum_{i=1}^{n-1} S(u, x_1, \ldots, x_m) \sum_{r=i+1}^{n} \sum_{s=u_r}^{m} x_s. \end{aligned}$$

Solving for $S(u, x_1, \ldots, x_m)$ will then result in the following theorem.

Theorem 12. *Suppose $u = u_1 \ldots u_n \in [m]^*$ is weakly increasing. Then*

$$S(u; x_1, \ldots, x_m) = \frac{\prod_{i=1}^n \sum_{j=u_i}^m x_j}{\left(1 + \sum_{i=1}^{n-1} \prod_{j=i+1}^n \sum_{l=u_j}^m x_l\right)(1 - \sum_{i=1}^m x_i) + \prod_{i=1}^n \sum_{j=u_i}^m x_j}.$$

6 Further work

Many of the ideas in this paper can be extended to generalized factor order on $\mathbb{P}^*$ with other partial orders. In particular, in [5] we consider the mod k partial order on $\mathbb{P}^*$ defined by setting $m \leq_k n$ if $m \leq n$ and $m = n \bmod k$. For example, the Hasse diagram for the mod 3 partial order consists of the three chains

$$\begin{array}{l} 1 \leq_3 4 \leq_3 7 \leq_3 \cdots, \\ 2 \leq_3 5 \leq_3 8 \leq_3 \cdots, \text{ and} \\ 3 \leq_3 6 \leq_3 9 \leq_3 \cdots. \end{array}$$

An generalization of Theorem 1 in this context applies to a rich class of words that generalizes the set of words in $\mathbb{P}^*$ with increasing/decreasing factorizations. One interesting result of this theorem is that the rearrangement conjectures do not hold in general for the mod k partial order with $k \geq 2$, however we can identify those words for which we believe the rearrangement conjectures do hold. We refer the reader to [5] for details.

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