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On Error Sum Functions Formed by Convergents of Real Numbers

Carsten Elsner and Martin Stein
Fachhochschule für die Wirtschaft Hannover
Freundallee 15
30173 Hannover
Germany
Carsten.Elsner@fhdw.de

Abstract

Let p_m/q_m denote the m-th convergent ($m \geq 0$) from the continued fraction expansion of some real number α. We continue our work on error sum functions defined by $\mathcal{E}(\alpha) := \sum_{m\geq 0} |q_m\alpha - p_m|$ and $\mathcal{E}^*(\alpha) := \sum_{m\geq 0}(q_m\alpha - p_m)$ by proving a new density result for the values of $\mathcal{E}$ and $\mathcal{E}^*$. Moreover, we study the function $\mathcal{E}$ with respect to continuity and compute the integral $\int_0^1 \mathcal{E}(\alpha)\,d\alpha$. We also consider generalized error sum functions for the approximation with algebraic numbers of bounded degrees in the sense of Mahler.

1 Introduction and statement of the main results

Recently the first author [2] introduced two error sums: Let $\alpha = [a_0; a_1, a_2, \ldots]$ be the continued fraction expansion of a real number α, which may be finite in the case of a rational number α. Let

$$\frac{p_m}{q_m} = [a_0; a_1, \ldots, a_m] \qquad (m \geq 0)$$

denote the convergents of α. The error sum functions $\mathcal{E}(\alpha)$ and $\mathcal{E}^*(\alpha)$ are defined by

$$\begin{aligned}\mathcal{E}(\alpha) &= \sum_{m\geq 0} |\alpha q_m - p_m| = \sum_{m\geq 0} (-1)^m(\alpha q_m - p_m)\,,\\ \mathcal{E}^*(\alpha) &= \sum_{m\geq 0} (\alpha q_m - p_m)\,.\end{aligned}$$

Both functions do not depend on the integer part a_0 of α. So we may restrict their domains on the interval $[0,1)$.

The first author [2] proved that

$$0 \le \mathcal{E}(\alpha) \le \rho = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad 0 \le \mathcal{E}^*(\alpha) \le 1 \qquad (\alpha \in \mathbb{R}).$$

The series $\sum_{m\ge 0} |q_m\alpha - p_m| \in [0,\rho]$ measures the approximation properties of α on average. The smaller this series is, the better rational approximations α has. Nevertheless, α can be a Liouville number and $\sum_{m\ge 0} |q_m\alpha - p_m|$ takes a value close to ρ. So, it may be interesting to question on the average value of $\mathcal{E}$ and $\mathcal{E}^*$, respectively. We compute the average value of $\mathcal{E}$, see Theorem 5. The error sum functions $\mathcal{E}$ and $\mathcal{E}^*$ have various interesting properties. In [2], applications are discussed for certain transcendental numbers and for quadratic irrational numbers. For instance, we have

$$\begin{aligned}
\mathcal{E}(\exp(1)) &= \sum_{m\ge 0} |q_m e - p_m| = 2e\int_0^1 \exp(-t^2)\,dt - e = 1.3418751\ldots,\\
\mathcal{E}^*(\exp(1)) &= \sum_{m\ge 0} (q_m e - p_m) = 2\int_0^1 \exp(t^2)\,dt - 2e + 3 = 0.4887398\ldots,\\
\mathcal{E}(\sqrt{7}) &= \sum_{m\ge 0} |q_m\sqrt{7} - p_m| = \frac{7+5\sqrt{7}}{14} = 1.444911182\ldots,\\
\mathcal{E}^*(\sqrt{7}) &= \sum_{m\ge 0} (q_m\sqrt{7} - p_m) = \frac{21-5\sqrt{7}}{14} = 0.555088817\ldots.
\end{aligned}$$

It is clear that for any rational number α the series for $\mathcal{E}(\alpha)$ and $\mathcal{E}^*(\alpha)$ become finite sums and therefore belong to $\mathbb{Q}$. In the case of quadratic irrational numbers α we have $\mathcal{E}(\alpha) \in \mathbb{Q}(\alpha)$ and $\mathcal{E}^*(\alpha) \in \mathbb{Q}(\alpha)$ ([2, Theorem 3]). But for quadratic irrationals $\mathcal{E}(\alpha) \in \mathbb{Q}(\alpha) \setminus \mathbb{Q}$ does not hold in general. For example, $\mathcal{E}((3-\sqrt{5})/2) = 1$ (see [3, Lemma 8]). On the other hand $\mathcal{E}(\alpha) \in \mathbb{Q}(\alpha)$ is not true for all real numbers α. For $\alpha = e = \exp(1)$ we have $\mathcal{E}(e) \notin \mathbb{Q}(e)$, since e and $\int_0^1 \exp(-t^2)\,dt$ are algebraically independent over $\mathbb{Q}$. This follows from a remark on page 193 in [8]. Similarly, one can show that $\mathcal{E}^*(e) \notin \mathbb{Q}(e)$.

The authors [3] studied the value distribution of the error sum functions in more detail. They constructed two algorithms which prove that the set of values of $\mathcal{E}$ is dense in the interval $I_{\mathcal{E}} = [0,\rho]$, and that the set of values of $\mathcal{E}^*$ is dense in the interval $I_{\mathcal{E}^*} = [0,1]$ (see [3, Theorems 1, 2]). But, given any uniformly modulo one distributed sequence $(\alpha_\nu)_{\nu\ge 1}$ of real numbers, the sequences $(\mathcal{E}(\alpha_\nu))_{\nu\ge 1}$ and $(\mathcal{E}^*(\alpha_\nu))_{\nu\ge 1}$ are not uniformly distributed in $I_{\mathcal{E}}$ and $I_{\mathcal{E}^*}$, respectively (see [3, Theorems 3, 4]). In this paper we show that any dense subset of $(0,1)$ is mapped by $\mathcal{E}(\alpha)$ and $\mathcal{E}^*(\alpha)$ into a set which is dense in $I_{\mathcal{E}}$ and $I_{\mathcal{E}^*}$, respectively. Then, we continue to study the analytic properties of the error sum functions. The function $\mathcal{E}^*$ has already been investigated by Ridley and Petruska [7]. Among other things they showed that $\mathcal{E}^*(\alpha)$ is continuous at every irrational point α, and discontinuous when α is rational. Moreover, they computed the integral $\int_0^1 \mathcal{E}^*(\alpha)\,d\alpha$ by applying the functional equation

$$\mathcal{E}^*(\alpha) + \mathcal{E}^*(1-\alpha) = \max\{\alpha, 1-\alpha\} \quad \text{except at} \quad \alpha = 0 \quad \text{and} \quad \alpha = \frac{1}{2}.$$

Inspired by the work of Ridley and Petruska, we prove similar results for the error sum function $\mathcal{E}$. We compute the integral $\int_0^1 \mathcal{E}(\alpha)\,d\alpha$ by using a multiple sum, which expresses the integral in terms of denominators of convergents. Unfortunately, the functional equation

$$\mathcal{E}(\alpha) - \mathcal{E}(1-\alpha) = \begin{cases} \alpha - 1, & \text{if} \quad 0 < \alpha < 1/2; \\ \alpha, & \text{if} \quad 1/2 < \alpha < 1; \end{cases}$$

cannot be used to evaluate the integral $\int_0^1 \mathcal{E}(\alpha)\,d\alpha$.

The main results of this paper are given by the following theorems.

Theorem 1. *Let $(\alpha_n)_{n\geq 1}$ be a sequence of real numbers forming a dense set $\{\alpha_n : n \in \mathbb{N}\}$ in $(0,1)$. Then the set $\{\mathcal{E}(\alpha_n) : n \in \mathbb{N}\}$ is dense in $(0,\rho)$, and the set $\{\mathcal{E}^*(\alpha_n) : n \in \mathbb{N}\}$ is dense in $(0,1)$.*

Theorem 2. *The function $\mathcal{E}(\alpha)$ is discontinuous at every rational point α, and it is continuous at every irrational point α.*

Example 3. Let n,k be integers with $n,k \geq 3$. For $x = 1/n$ we have

$$\begin{aligned}
\mathcal{E}\Big(\frac{1}{n} + \frac{1}{n^k}\Big) &= \frac{2}{n} + \frac{3}{n^k} \quad \rightarrow \quad \frac{2}{n} \quad (k \to \infty)\,, \\
\mathcal{E}\Big(\frac{1}{n} - \frac{1}{n^k}\Big) &= \frac{1}{n} - \frac{1}{n^k} + \frac{2}{n^{k-1}} \quad \rightarrow \quad \frac{1}{n} \quad (k \to \infty)\,, \\
\mathcal{E}^*\Big(\frac{1}{n} + \frac{1}{n^k}\Big) &= \frac{2}{n^{k-1}} - \frac{1}{n^k} \quad \rightarrow \quad 0 \quad (k \to \infty)\,, \\
\mathcal{E}^*\Big(\frac{1}{n} - \frac{1}{n^k}\Big) &= \frac{1}{n} - \frac{3}{n^k} \quad \rightarrow \quad \frac{1}{n} \quad (k \to \infty)\,.
\end{aligned}$$

These expressions are obtained by using the identities

$$\begin{aligned}
\frac{1}{n} + \frac{1}{n^k} &= [\,0; n-1, 1, n^{k-2}-1, n\,]\,, \\
\frac{1}{n} - \frac{1}{n^k} &= [\,0; n, n^{k-2}-1, 1, n-1\,]\,.
\end{aligned}$$

□

Let $m \geq 1$, and let $a_1, \ldots, a_m$ be positive integers. Set

$$\frac{p_m}{q_m} = [\,0; a_1, \ldots, a_m\,]\,,$$

where p_m and q_m with $q_m > 0$ are coprime integers.

Theorem 4. *We have*

$$\int_0^1 \mathcal{E}(\alpha)\,d\alpha = \frac{1}{2} + \frac{1}{2}\sum_{m=1}^{\infty}\sum_{a_1=1}^{\infty}\cdots\sum_{a_m=1}^{\infty}\frac{1}{q_m(q_m+q_{m-1})^2}\,,$$

and

$$\int_0^1 \mathcal{E}^*(\alpha)\,d\alpha = \frac{1}{2} + \frac{1}{2}\sum_{m=1}^{\infty}\sum_{a_1=1}^{\infty}\cdots\sum_{a_m=1}^{\infty}\frac{(-1)^m}{q_m(q_m+q_{m-1})^2}\,.$$

With the first identity from the preceding theorem, we compute the mean value of the function $\mathcal{E}$.

Theorem 5. *We have*

$$\int_0^1 \mathcal{E}(\alpha)\, d\alpha = -\frac{5}{8} + \frac{3\zeta(2)\log 2}{2\zeta(3)} = 0.79778798\ldots,$$

where $\zeta(s)$ denotes the Riemann zeta function.

Remark 6. Ridley and Petruska [7] proved that

$$\int_0^1 \mathcal{E}^*(\alpha)\, d\alpha = \frac{3}{8}\,.$$

We point out that by Theorem 5 and Remark 6 the mean values of $\mathcal{E}$ and $\mathcal{E}^*$ are less than half of the maximum value of $\mathcal{E}$ and $\mathcal{E}^*$, respectively.

In Section 5 we generalize the error sum function $\mathcal{E}$ to the approximation with algebraic numbers of bounded degree. Here, the Mahler function $w_n(H,\alpha)$ will be involved.

2 Proof of Theorem 1

We will only prove the statement concerning the values of the function $\mathcal{E}$, since there are no additional arguments for the function $\mathcal{E}^*$.

It is shown in the proof of Theorem 1 in [3] that the set $\{\mathcal{E}(\alpha) : \alpha \in \mathbb{Q} \cap (0,1)\}$ is dense in $(0,\rho)$. Hence, for any real number $\eta \in (0,\rho)$ and for any $\delta > 0$ there is a rational number $r \in (0,1)$ satisfying

$$\big|\eta - \mathcal{E}(r)\big| \,<\, \frac{\delta}{3}\,. \tag{1}$$

By

$$r \,=\, [0; a_1, a_2, \ldots, a_t] \,=\, \frac{p_t}{q_t}$$

we denote the continued fraction expansion of r. Without loss of generality we may assume that t satisfies

$$\frac{1+\sqrt{2}}{(\sqrt{2})^{t-1}} \,<\, \frac{\delta}{3}\,. \tag{2}$$

This can be seen by the following argument: For any number $r' = [0; a_1, \ldots, a_{t'}]$ satisfying $|\eta - \mathcal{E}(r')| < \delta/3$ and $t' < t$ we construct a number $r = [0; a_1, a_2, \ldots, a_t]$ with $a_{t'+1} = \cdots = a_t = b$, such that t satisfies (2) and b is sufficiently large (see [3, Lemma 1]). Namely, for r_k

defined by $r_k := [0; a_1, \ldots, a_{t'}, \underbrace{b, \ldots, b}_{k}]$ we have

$$\begin{aligned}
\left|\mathcal{E}(r) - \mathcal{E}(r')\right| &= \left|\mathcal{E}(r_{t-t'}) - \mathcal{E}(r_0)\right| = \Big| \sum_{k=0}^{t-t'-1} \mathcal{E}(r_{k+1}) - \mathcal{E}(r_k) \Big| \\
&\le \sum_{k=0}^{t-t'-1} \big| \mathcal{E}(r_{k+1}) - \mathcal{E}(r_k) \big| < \sum_{k=0}^{t-t'-1} \frac{1}{b} = \frac{t-t'}{b} \\
&< \frac{t}{b} \to 0 \qquad (b \to \infty)\,.
\end{aligned}$$

Since the set $\{\alpha_n : n \in \mathbb{N}\}$ is dense in $(0,1)$ by the assumption in the theorem, there is a positive integer m satisfying

$$\alpha_m = [0; a_1, a_2, \ldots, a_t, a_{t+1}, \ldots]$$

and

$$\left|r - \alpha_m\right| < \frac{\delta}{3(t+1)q_t}\,. \tag{3}$$

Let p_ν/q_ν be the convergents of α_m. Then, by applying the inequalities (1), (3) and (2) we have

$$\begin{aligned}
\left|\eta - \mathcal{E}(\alpha_m)\right| &= \left|\eta - \mathcal{E}(r) + \mathcal{E}(r) - \mathcal{E}(\alpha_m)\right| \le \left|\eta - \mathcal{E}(r)\right| + \left|\mathcal{E}(r) - \mathcal{E}(\alpha_m)\right| \\
&< \frac{\delta}{3} + \Big| \sum_{\nu=0}^{t} |q_\nu r - p_\nu| - \sum_{\nu \ge 0} |q_\nu \alpha_m - p_\nu| \Big| \\
&\le \frac{\delta}{3} + \sum_{\nu=0}^{t} |r - \alpha_m| q_t + \sum_{\nu \ge t+1} |q_\nu \alpha_m - p_\nu| \\
&\le \frac{\delta}{3} + \sum_{\nu=0}^{t} \frac{\delta}{3(t+1)} + \sum_{\nu \ge t+1} \frac{1}{q_\nu} \\
&\le \frac{2\delta}{3} + \sum_{\nu \ge t} \frac{1}{(\sqrt{2})^\nu} \\
&= \frac{2\delta}{3} + \frac{1+\sqrt{2}}{(\sqrt{2})^{t-1}} \\
&< \delta\,,
\end{aligned}$$

which completes the proof of Theorem 1. $\square$

3 Proof of Theorem 2

Since the function $\mathcal{E}$ is periodic of period one, it suffices to prove Theorem 2 for $\alpha \in [0,1)$. We will prove the statement on continuity first. Let $\eta \in [0,1)$ be a real irrational number, say

$$\eta = [0; a_1, a_2, \ldots] ,$$

and let $(\xi_n)_{n\geq 1}$ be a sequence of real numbers converging to η. By $I_m = I_m(a_1, \ldots, a_m)$ we denote the interval defined uniquely by

$$[0; b_1, b_2, \ldots] \in I_m \quad \iff \quad (b_1 = a_1 \wedge \cdots \wedge b_m = a_m) . \tag{4}$$

The boundary points of I_m are rational numbers, and therefore the irrational number η lies in the interior of I_m for any $m \geq 1$. With $\lim_{n\to\infty} \xi_n = \eta$ we conclude on

$$\xi_n \in I_m \quad (n \geq n_0)$$

for some positive integer $n_0 = n_0(m)$. Hence, by (4), we have

$$\xi_n = [0; a_1, \ldots, a_m, \ldots]. \tag{5}$$

Let p_ν/q_ν for $\nu \geq 0$ be the convergents of η and let $p_\nu^{(n)}/q_\nu^{(n)}$ be the convergents of ξ_n. Then, from (5), it follows that

$$\frac{p_\nu}{q_\nu} = \frac{p_\nu^{(n)}}{q_\nu^{(n)}} \quad (0 \leq \nu \leq m) .$$

For a fixed positive integer m and any $n \geq n_0$ we estimate

$$\begin{aligned}
|\mathcal{E}(\eta) - \mathcal{E}(\xi_n)| &= \Big| \sum_{\nu\geq 0} |q_\nu \eta - p_\nu| - \sum_{\nu\geq 0} |q_\nu^{(n)} \xi_n - p_\nu^{(n)}| \Big| \\
&\leq \Big| \sum_{\nu=0}^{m} (-1)^\nu q_\nu (\eta - \xi_n) \Big| + \sum_{\nu\geq m+1} |q_\nu \eta - p_\nu| + \sum_{\nu\geq m+1} |q_\nu^{(n)} \xi_n - p_\nu^{(n)}| \\
&\leq \Big| \sum_{\nu=0}^{m} (-1)^\nu q_\nu (\eta - \xi_n) \Big| + \sum_{\nu\geq m+1} \frac{1}{q_\nu} + \sum_{\nu\geq m+1} \frac{1}{q_\nu^{(n)}} \\
&\leq \Big| \sum_{\nu=0}^{m} (-1)^\nu q_\nu (\eta - \xi_n) \Big| + \sum_{\nu\geq m+1} \frac{1}{2^{(\nu-1)/2}} + \sum_{\nu\geq m+1} \frac{1}{2^{(\nu-1)/2}} \\
&= \Big| \sum_{\nu=0}^{m} (-1)^\nu q_\nu (\eta - \xi_n) \Big| + \frac{2\sqrt{2}}{\sqrt{2}-1} \cdot \frac{1}{(\sqrt{2})^m} .
\end{aligned}$$

Since m can be chosen arbitrary large and ξ_n tends to η for increasing n, we conclude on

$$\lim_{n\to\infty} \mathcal{E}(\xi_n) = \mathcal{E}(\eta) .$$

This proves that the function $\mathcal{E}(\alpha)$ is continuous at every irrational point α.

To prove the statement on discontinuity we shall at first discuss the case when η is a rational number in $(0, 1)$. Let

$$\eta = [0; a_1, a_2, \ldots, a_m]$$

for some integers $m \geq 1$ and $a_m > 1$. Moreover, let $(\xi_n^{(1)})_{n\geq 2}$ and $(\xi_n^{(2)})_{n\geq 2}$ be two sequences of rationals defined by

$$\xi_n^{(1)} = [0; a_1, \ldots, a_m, n] \quad \text{and} \quad \xi_n^{(2)} = [0; a_1, \ldots, a_m - 1, 1, n] \quad (n \geq 2)\,.$$

Obviously we have

$$\lim_{n\to\infty} \xi_n^{(1)} = \eta = \lim_{n\to\infty} \xi_n^{(2)}\,. \tag{6}$$

Let $p_\nu^{(1)}/q_\nu^{(1)}$ for $\nu = 0, \ldots, m+1$ be the convergents of $\xi_n^{(1)}$. By $p_\nu^{(2)}/q_\nu^{(2)}$ for $\nu = 0, \ldots, m+2$ we denote the convergents of $\xi_n^{(2)}$. Then we have

$$\frac{p_\nu^{(1)}}{q_\nu^{(1)}} = \frac{p_\nu^{(2)}}{q_\nu^{(2)}} \quad (0 \leq \nu \leq m-1)\,.$$

Therefore we may set $p_\nu := p_\nu^{(1)} = p_\nu^{(2)}$ and $q_\nu := q_\nu^{(1)} = q_\nu^{(2)}$ for $\nu = 0, \ldots, m-1$. We compute

$$\begin{aligned}
\mathcal{E}(\xi_n^{(2)}) - \mathcal{E}(\xi_n^{(1)}) - &\sum_{\nu=0}^{m-1} (-1)^\nu (\xi_n^{(2)} - \xi_n^{(1)}) q_\nu \\
&= (-1)^m((a_m-1)q_{m-1} + q_{m-2})\xi_n^{(2)} - (-1)^m((a_m-1)p_{m-1} + p_{m-2}) \\
&\quad + (-1)^{m+1}(a_m q_{m-1} + q_{m-2})\xi_n^{(2)} - (-1)^{m+1}(a_m p_{m-1} + p_{m-2}) \\
&\quad - (-1)^m(a_m q_{m-1} + q_{m-2})\xi_n^{(1)} + (-1)^m(a_m p_{m-1} + p_{m-2}) \\
&= (-1)^m(\xi_n^{(2)} - \xi_n^{(1)})(a_m q_{m-1} + q_{m-2}) + (-1)^m(p_{m-1} - q_{m-1}\xi_n^{(2)}) \\
&\quad + (-1)^{m+1}(a_m q_{m-1} + q_{m-2})\xi_n^{(2)} - (-1)^{m+1}(a_m p_{m-1} + p_{m-2})\,.
\end{aligned}$$

For $n \to \infty$, by (6) and with $\eta = p_m^{(1)}/q_m^{(1)}$ we obtain the limit

$$\begin{aligned}
\lim_{n\to\infty} \left(\mathcal{E}(\xi_n^{(2)}) - \mathcal{E}(\xi_n^{(1)})\right) &= (-1)^m \left[(p_{m-1} - q_{m-1}\eta) + (p_m^{(1)} - q_m^{(1)}\eta)\right] \\
&= (-1)^m \frac{p_{m-1}^{(1)} q_m^{(1)} - p_m^{(1)} q_{m-1}^{(1)}}{q_m^{(1)}} = \frac{1}{q_m^{(1)}}\,.
\end{aligned}$$

In particular, by $1/q_m^{(1)} \neq 0$, this proves that the function $\mathcal{E}$ is discontinuous at η.

It remains to prove that $\mathcal{E}$ is discontinuous at $\eta = 0$. Let $\xi_n^{(1)} := [0; n]$ and $\xi_n^{(2)} := [-1; 1, n]$. Then both sequences $(\xi_n^{(1)})_{n\geq 1}$ and $(\xi_n^{(2)})_{n\geq 1}$ tend to 0 for increasing n, but

$$\mathcal{E}(\xi_n^{(1)}) = \frac{1}{n} \to 0 \quad (n \to \infty),$$

wheras $\mathcal{E}(\xi_n^{(2)}) = 1$ holds for every positive integer n. Hence, Theorem 2 is proven. □

4 Proofs of Theorem 4 and Theorem 5

Proof of Theorem 4. Let m and $a_1, \ldots, a_m$ be positive integers. Set

$$\xi_1 = [0; a_1, \ldots, a_{m-1}, a_m]\,, \qquad \xi_2 = [0; a_1, \ldots, a_{m-1}, a_m + 1]\,.$$

Then we have $\xi_1 < \xi_2$ for even m and $\xi_2 < \xi_1$ otherwise. We define $I_m := (\xi_1, \xi_2)$ for even m and $I_m := (\xi_2, \xi_1)$ for odd m, which depend on $a_1, \ldots, a_m$. The intervals I_m are disjoint for different m-tuples $(a_1, \ldots, a_m)$. For any fixed m the union of all closed intervals $\overline{I}_m$ gives the interval $[0, 1]$. With this decomposition of $[0, 1]$ we obtain

$$\begin{aligned}
\int_0^1 \mathcal{E}(\alpha)\, d\alpha &= \int_0^1 \sum_{m=0}^{\infty} (-1)^m (q_m\alpha - p_m)\, d\alpha \\
&= \sum_{m=0}^{\infty} (-1)^m \int_0^1 (q_m\alpha - p_m)\, d\alpha \\
&= \frac{1}{2} + \sum_{m=1}^{\infty} (-1)^m \sum_{a_1=1}^{\infty} \cdots \sum_{a_m=1}^{\infty} \int_{I_m} (q_m\alpha - p_m)\, d\alpha \\
&= \frac{1}{2} + \sum_{m=1}^{\infty} \sum_{a_1=1}^{\infty} \cdots \sum_{a_m=1}^{\infty} \int_{\xi_1}^{\xi_2} (q_m\alpha - p_m)\, d\alpha
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
\int_0^1 \mathcal{E}^*(\alpha)\, d\alpha &= \int_0^1 \sum_{m=0}^{\infty} (q_m\alpha - p_m)\, d\alpha \\
&= \frac{1}{2} + \sum_{m=1}^{\infty} \sum_{a_1=1}^{\infty} \cdots \sum_{a_m=1}^{\infty} \int_{I_m} (q_m\alpha - p_m)\, d\alpha \\
&= \frac{1}{2} + \sum_{m=1}^{\infty} (-1)^m \sum_{a_1=1}^{\infty} \cdots \sum_{a_m=1}^{\infty} \int_{\xi_1}^{\xi_2} (q_m\alpha - p_m)\, d\alpha
\end{aligned} \tag{8}$$

Every point $\alpha \in I_m$ satisfies $\alpha = [0; a_1, \ldots, a_{m-1}, a_m, \ldots]$, hence the convergents p_ν/q_ν for $\nu \leq m$ depend on I_m, but not on $\alpha \in I_m$. Therefore, we derive

$$\int_{\xi_1}^{\xi_2} (q_m\alpha - p_m)\, d\alpha = (\xi_2 - \xi_1)\, \frac{(\xi_2 + \xi_1)q_m - 2p_m}{2}\,.$$

Using

$$\xi_1 = \frac{p_m}{q_m} \quad \text{and} \quad \xi_2 = \frac{(a_m + 1)p_{m-1} + p_{m-2}}{(a_m + 1)q_{m-1} + q_{m-2}}$$

we compute the expressions

$$\xi_2 - \xi_1 = \frac{(-1)^m}{(q_m + q_{m-1})q_m}$$

and

$$\xi_2 + \xi_1 = \frac{p_{m-1}q_m + q_{m-1}p_m + 2p_mq_m}{(q_m + q_{m-1})q_m}\,,$$

which give

$$\int_{\xi_1}^{\xi_2} (q_m\alpha - p_m)\, d\alpha = \frac{1}{2q_m(q_m + q_{m-1})^2}\,.$$

Substituting this integral into (7) and (8), we finally get the formulas stated in the theorem. □

Proof of Theorem 5. First we show that

$$\frac{1}{2}+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{a_1=1}^{\infty}\cdots\sum_{a_m=1}^{\infty}\frac{1}{q_m(q_m+q_{m-1})^2}=-\frac{3}{8}+\sum_{a=1}^{\infty}\frac{1}{a}\sum_{\substack{b=0\\ \gcd(a,b)=1}}^{a-1}\frac{1}{(a+b)^2}\,. \tag{9}$$

For the denominators of two subsequent convergents of the continued fraction expansion of $\alpha=[0;a_1,\ldots,a_m,\ldots]$ it is well-known that $\gcd(q_m,q_{m-1})=1$. For fixed $q_m=a$ we count the solutions of $q_{m-1}=b$ with $\gcd(a,b)=1$ and $0\le b\le a-1$ in the multiple sum on the left-hand side of (9). It is necessary to distinguish the cases $m\ge 2$ and $m=1$.
Case 1: $m\ge 2$. First let $a_1=1$. Then,

$$\frac{q_{m-1}}{q_m}=[0;a_m,\ldots,a_2,1]=[0;a_m,\ldots,a_2+1]\,.$$

For $a_1\ge 2$ we have

$$\frac{q_{m-1}}{q_m}=[0;a_m,\ldots,a_2,a_1]=[0;a_m,\ldots,a_2,a_1-1,1]\,.$$

Case 2: $m=1$. For $a_1=1$ we have a unique representation of the fraction

$$\frac{q_{m-1}}{q_m}=\frac{q_0}{q_1}=\frac{1}{a_1}=\frac{1}{1}=[0;1]\,,$$

since the integer part $a_0=0$ must not be changed. For $a_1\ge 2$ there are again two representations:

$$\frac{q_{m-1}}{q_m}=\frac{q_0}{q_1}=\frac{1}{a_1}=[0;a_1]=[0;a_1-1,1]\,.$$

Therefore it is clear that for fixed $q_m=a$ every b with $\gcd(a,b)=1$ and $0\le b\le a-1$ occurs exactly two times in the multiple sum on the left-hand side of (9), except for $m=1$ and $a_1=1$. For this exceptional case we separate the term

$$\frac{1}{2q_1(q_1+q_0)^2}=\frac{1}{8}$$

from the multiple sum. Then we obtain

$$
\begin{aligned}
&\frac{1}{2}+\frac{1}{2}\sum_{m=1}^{\infty}\sum_{a_1=1}^{\infty}\cdots\sum_{a_m=1}^{\infty}\frac{1}{q_m(q_m+q_{m-1})^2}\\
&\qquad=\frac{1}{2}+\frac{1}{2}\sum_{m=2}^{\infty}\sum_{a_1=1}^{\infty}\cdots\sum_{a_m=1}^{\infty}\frac{1}{q_m(q_m+q_{m-1})^2}+\frac{1}{2}\sum_{a_1=2}^{\infty}\frac{1}{q_1(q_1+1)^2}+\frac{1}{8}\\
&\qquad=\frac{1}{2}+\sum_{a=1}^{\infty}\frac{1}{a}\sum_{\substack{b=1\\ \gcd(a,b)=1}}^{a-1}\frac{1}{(a+b)^2}+\frac{1}{8}\\
&\qquad=\frac{1}{2}+\sum_{a=1}^{\infty}\frac{1}{a}\sum_{\substack{b=0\\ \gcd(a,b)=1}}^{a-1}\frac{1}{(a+b)^2}-1+\frac{1}{8}\\
&\qquad=-\frac{3}{8}+\sum_{a=1}^{\infty}\frac{1}{a}\sum_{\substack{b=0\\ \gcd(a,b)=1}}^{a-1}\frac{1}{(a+b)^2}\,,
\end{aligned}
$$

which proves the identity in (9).

Next we treat the double sum on the right-hand side of (9). Let μ denote the Möbius function. Then we derive

$$
\begin{aligned}
\sum_{a=1}^{\infty}\frac{1}{a}\sum_{\substack{b=0\\ \gcd(a,b)=1}}^{a-1}\frac{1}{(a+b)^2}&=\sum_{a=1}^{\infty}\frac{1}{a}\sum_{b=0}^{a-1}\sum_{\substack{d>0\\ d\mid\gcd(a,b)}}\frac{\mu(d)}{(a+b)^2}=\sum_{a=1}^{\infty}\sum_{b=0}^{a-1}\sum_{\substack{d>0\\ d\mid a\wedge d\mid b}}\frac{\mu(d)}{a(a+b)^2}\\
&=\sum_{d=1}^{\infty}\sum_{\substack{a=1\\ d\mid a}}^{\infty}\sum_{\substack{b=0\\ d\mid b}}^{a-1}\frac{\mu(d)}{a(a+b)^2}=\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{n-1/d}\frac{\mu(d)}{nd(nd+md)^2}
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^3}\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{1}{n(n+m)^2}=\frac{1}{\zeta(3)}\sum_{a=1}^{\infty}\sum_{b=0}^{a-1}\frac{1}{a(a+b)^2}\\
&=\frac{1}{\zeta(3)}\sum_{a=1}^{\infty}\frac{1}{a}\sum_{c=a}^{2a-1}\frac{1}{c^2}=\frac{1}{\zeta(3)}\sum_{c=1}^{\infty}\frac{1}{c^2}\sum_{a=\lfloor c/2\rfloor+1}^{c}\frac{1}{a}\\
&=\frac{1}{\zeta(3)}\sum_{c=1}^{\infty}\frac{1}{c^2}\sum_{a=1}^{c}\frac{(-1)^{a+1}}{a}\\
&=\frac{1}{\zeta(3)}\sum_{c=1}^{\infty}\frac{1}{c^2}\sum_{a=1}^{c-1}\frac{(-1)^{a+1}}{a}+\frac{1}{\zeta(3)}\sum_{c=1}^{\infty}\frac{(-1)^{c+1}}{c^3}\\
&=-\frac{\zeta(2,-1)}{\zeta(3)}+\frac{3}{4}\,,
\end{aligned}
$$

where

$$\zeta(2,-1)=\sum_{c=1}^{\infty}\frac{1}{c^2}\sum_{a=1}^{c-1}\frac{(-1)^a}{a}=\sum_{c>a>0}\frac{(-1)^a}{ac^2}$$

is a special case of the multivariate zeta function (see [1, Section 2.6]), satisfying

$$\zeta(2,-1)=\zeta(3)-\frac{3}{2}\zeta(2)\log 2\,.$$

Collecting together we obtain from (9) that

$$\int_0^1 \mathcal{E}(\alpha)\,d\alpha=-\frac{3}{8}-1+\frac{3}{4}+\frac{3}{2}\frac{\zeta(2)\log 2}{\zeta(3)}=-\frac{5}{8}+\frac{3\zeta(2)\log 2}{2\zeta(3)}\,,$$

which completes the proof of the theorem.

□

Remark 7. Let n be a positive integer. We consider a modified error sum function given by

$$\sum_{m\geq 0}|\alpha q_m-p_m|^n \qquad (0<\alpha<1)\,.$$

By similar methods as used to deduce Theorems 4 and 5 we obtain the following identities:

$$\begin{aligned}\int_0^1\sum_{m\geq 0}|\alpha q_m-p_m|^n\,d\alpha &= \frac{1}{n+1}+\frac{1}{n+1}\sum_{m=1}^{\infty}\sum_{a_1=1}^{\infty}\cdots\sum_{a_m=1}^{\infty}\frac{1}{q_m(q_m+q_{m-1})^{n+1}}\\ &=\frac{1}{n+1}\left(1-\frac{1}{2^{n+1}}-\frac{2\zeta(n+1,-1)}{\zeta(n+2)}\right)\end{aligned}$$

with the multivariate zeta function $\zeta(n+1,-1)$ defined by

$$\zeta(n+1,-1)=\sum_{m_2>m_1>0}\frac{(-1)^{m_1}}{m_1m_2^{n+1}}\,.$$

This yields an asymptotic expansion, namely

$$\int_0^1\sum_{m\geq 0}|\alpha q_m-p_m|^n\,d\alpha=\frac{1}{n+1}+\mathcal{O}\left(\frac{1}{(n+1)2^n}\right) \qquad (n\to\infty)\,.$$

5 Generalization of the error sum function $\mathcal{E}$

In this section we show that the error sum function $\mathcal{E}$ is the special case of a more general concept involving the theory of approximation with algebraic numbers of bounded degree. We need some notations to recall the definition of the Mahler functions $w_n(H,\alpha)$ and $w_n(\alpha)$. For more details on this function we refer to [5].
For any polynomial $P(x)\in\mathbb{Z}[x]$ we denote by $H(P)$ the *height* of the polynomial P, which is given by the maximum value of the modulus of the coefficients. Let n,H be positive integers

and $\alpha \in \mathbb{C}$ with $|\alpha| \le 1/2$ and $\deg \alpha > n$. For α being transcendental we define $\deg \alpha = \infty$. Set

$$\begin{aligned} w_n(H,\alpha) &:= \min_{\substack{P \in \mathbb{Z}[x] \setminus \{0\} \\ \deg P \le n \\ H(P) \le H}} \big|P(\alpha)\big|, \\ w_n(\alpha) &:= \limsup_{H \to \infty} \frac{-\log w_n(H,\alpha)}{\log H}. \end{aligned}$$

$w_n(\alpha)$ is the largest positive real number such that for every $\varepsilon > 0$ there are infinitely many polynomials P from $\mathbb{Z}[x]$ of degree at most n satisfying

$$\big|P(\alpha)\big| < \big(H(P)\big)^{-w_n(\alpha)+\varepsilon}.$$

So the function $w_n(H,\alpha)$ is needed to define the important Mahler function $w_n(\alpha)$. From the definition of $w_n(H,\alpha)$ it follows immediately that $w_1(H,\alpha) \ge w_2(H,\alpha) \ge \cdots \ge w_n(H,\alpha)$ holds for all integers $n = 1, 2, \ldots$.
Given α and some positive integer n with $\deg \alpha > n$, there is a unique sequence $(H_m)_{m \ge 0}$ of positive integers satisfying the following conditions:

$$\begin{array}{rl} (i) & 1 = H_0 < H_1 < \cdots < H_m < \ldots \\ (ii) & w_n(H_0,\alpha) > w_n(H_1,\alpha) > \cdots > w_n(H_m,\alpha) > \ldots \\ (iii) & w_n(H_m,\alpha) = w_n(H_{m+1}-1,\alpha) \qquad (m = 0, 1, \ldots) \end{array}$$

We define the generalized error sum function

$$\mathcal{E}_n(\alpha) := \sum_{m=0}^{\infty} w_n(H_m,\alpha).$$

Note that $\mathcal{E}_n(\alpha) = \mathcal{E}_n(-\alpha)$ holds, since the same is obviously true for the Mahler function: $w_n(H,\alpha) = w_n(H,-\alpha)$. For $n = 1$ and $\alpha \in (-1/2, 1/2) \setminus \mathbb{Q}$ we have $p_0/q_0 \in \{-1/1, 0/1\}$ and $p_1/q_1 = 1/a_1$, where $a_1 = 1$ holds if and only if $-1/2 < \alpha < 0$. This implies that

$$w_1(H_m,\alpha) = \begin{cases} |q_m\alpha - p_m|, & \text{if } 0 < \alpha < 1/2; , \\ |q_{m+1}\alpha - p_{m+1}|, & \text{if } -1/2 < \alpha < 0; \end{cases} \qquad (m = 0, 1, \ldots).$$

Therefore,

$$\mathcal{E}_1(\alpha) = \begin{cases} \mathcal{E}(\alpha), & \text{if } 0 < \alpha < 1/2; , \\ \mathcal{E}(\alpha) - \alpha - 1, & \text{if } -1/2 < \alpha < 0; , \end{cases}$$

where $\alpha + 1$ equals $q_0\alpha - p_0$ in the second case. Let

$$\mathcal{E}_n := \sup\big\{\mathcal{E}_n(\alpha) : \alpha \in (-1/2, 1/2) \wedge \deg \alpha > n\big\} \qquad (n = 1, 2, \ldots).$$

Then it is clear that for $n = 1, 2, \ldots$

$$\begin{aligned} \mathcal{E}_n &= \sup\Big\{\sum_{m=0}^{\infty} w_n(H_m,\alpha) : \alpha \in (-1/2, 1/2) \wedge \deg \alpha > n\Big\} \\ &\le \sup\Big\{\sum_{m=0}^{\infty} w_1(H_m,\alpha) : \alpha \in (-1/2, 1/2) \wedge \deg \alpha > n\Big\} \\ &\le \mathcal{E}_1 = \sup\big\{\mathcal{E}_1(\alpha) : \alpha \in (-1/2, 1/2) \wedge \deg \alpha > 1\big\} \\ &\le \rho. \end{aligned}$$

This bound can be improved by applying two inequalities based on Siegel's Lemma. Let $\alpha \in \mathbb{C}$ with $|\alpha| < 1/2$. For real α and any positive integers n, H we have

$$w_n(H,\alpha) < (n+1)H^{-n}\,. \tag{10}$$

For $\alpha \notin \mathbb{R}$ and any positive integers n, H we have

$$w_n(H,\alpha) < \sqrt{2}(n+1)H^{-(n-1)/2}\,. \tag{11}$$

These inequalities can be found on page 69 in [5], where the constants C_1 and C_2 are given by [5, Hilfssatz 27, Hilfssatz 28]. In what follows we distinguish whether α is real or not.

Case 1: $\alpha \in \mathbb{R}$. By using

$$w_n(1,\alpha) \le \max_{-1/2 \le x \le 1/2} |x^n| = \frac{1}{2^n}$$

we obtain with (10) and the Riemann zeta function for $n \ge 2$

$$\begin{aligned}
\mathcal{E}_n(\alpha) &= \sum_{m=0}^{\infty} w_n(H_m,\alpha) = w_n(1,\alpha) + \sum_{m=1}^{\infty} w_n(H_m,\alpha) \\
&\le \frac{1}{2^n} + \sum_{m=1}^{\infty} w_n(m+1,\alpha) \le \frac{1}{2^n} + \sum_{m=1}^{\infty} \frac{n+1}{(m+1)^n} \\
&= \frac{1}{2^n} + (n+1)\big(\zeta(n)-1\big) \quad \to 0 \qquad (\text{for } n\to\infty)\,.
\end{aligned}$$

Case 2: $\alpha \notin \mathbb{R}$. Here we consider the polynomial z^n for $|z| \le 1/2$. Then,

$$w_n(1,\alpha) \le \max_{|z|\le 1/2} |z^n| = \frac{1}{2^n}\,.$$

With (11) we repeat the arguments from Case 1 for $n \ge 4$:

$$\begin{aligned}
\mathcal{E}_n(\alpha) &\le w_n(1,\alpha) + \sum_{m=1}^{\infty} \frac{\sqrt{2}(n+1)}{(m+1)^{(n-1)/2}} \\
&\le \frac{1}{2^n} + \sqrt{2}(n+1)\left(\zeta\left(\frac{n-1}{2}\right) - 1\right) \quad \to 0 \qquad (\text{for } n\to\infty)\,.
\end{aligned}$$

Note that the inequality

$$\frac{1}{2^n} + (n+1)\big(\zeta(n)-1\big) < \rho$$

holds for $n \ge 3$, whereas

$$\frac{1}{2^n} + \sqrt{2}(n+1)\left(\zeta\left(\frac{n-1}{2}\right) - 1\right) < \rho$$

is true for $n \ge 5$.

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