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Riordan Arrays, Orthogonal Polynomials as Moments, and Hankel Transforms

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Abstract

Taking the examples of Legendre and Hermite orthogonal polynomials, we show how to interpret the fact that these orthogonal polynomials are moments of other orthogonal polynomials in terms of their associated Riordan arrays. We use these means to calculate the Hankel transforms of the associated polynomial sequences.

1 Introduction

In this note, we shall re-interpret some of the results of Ismail and Stanton [15, 16] in terms of Riordan arrays. These authors give functionals [15] whose moments are the Hermite, Laguerre, and various Meixner families of polynomials. In this note, we shall confine ourselves to Legendre and Hermite polynomials. Indeed, the types of orthogonal polynomials representable with Riordan arrays is very limited (see below), but it is nevertheless instructive to show that a number of them can be exhibited as moments, again using (parameterized) Riordan arrays.

The essence of the paper is to show that a Riordan array L (either ordinary or exponential) defines a family of orthogonal polynomials (via its inverse L^{-1}) if and only if its production matrix [8, 9, 10] is tri-diagonal. The sequence of moments μ_n associated to the family of orthogonal polynomials then appears as the elements of the first column of L. In terms of generating functions, this means that if $L = (g, f)$ (or $L = [g, f]$), then $g(x)$ is the generating function of the moment sequence. By defining suitable parameterized Riordan arrays, we can exhibit the Legendre and Hermite polynomials as such moment sequences.

While partly expository in nature, the note assumes a certain familiarity with integer sequences, generating functions, orthogonal polynomials [5, 12, 30], Riordan arrays [25, 29], production matrices [10, 23], and the Hankel transform of sequences [2, 7, 20]. We provide background material in this note to give a hopefully coherent narrative. Many interesting examples of sequences and Riordan arrays can be found in Neil Sloane's On-Line Encyclopedia of Integer Sequences (OEIS), [27, 28]. Sequences are frequently referred to by their OEIS number. For instance, the binomial matrix $\mathbf{B}$ ("Pascal's triangle") is A007318.

The plan of the paper is as follows:

1. This Introduction
2. Preliminaries on integer sequences and (ordinary) Riordan arrays
3. Orthogonal polynomials and Riordan arrays
4. Exponential Riordan arrays and orthogonal polynomials
5. The Hankel transform of an integer sequence
6. Legendre polynomials
7. Legendre polynomials as moments
8. Hermite polynomials
9. Hermite polynomials as moments
10. Acknowledgements
11. Appendix - The Stieltjes transform of a measure

2 Preliminaries on integer sequences and Riordan arrays

For an integer sequence a_n, that is, an element of $\mathbb{Z}^{\mathbb{N}}$, the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is called the *ordinary generating function* or g.f. of the sequence. a_n is thus the coefficient of x^n in this series. We denote this by $a_n = [x^n]f(x)$. For instance, $F_n = [x^n]\frac{x}{1-x-x^2}$ is the n-th Fibonacci number A000045, while $C_n = [x^n]\frac{1-\sqrt{1-4x}}{2x}$ is the n-th Catalan number A000108. The article [21] gives examples of the use of the operator $[x^n]$. We use the notation $0^n = [x^n]1$ for the sequence $1, 0, 0, 0, \ldots$, A000007. Thus $0^n = [n = 0] = \delta_{n,0} = \binom{0}{n}$. Here, we have used the Iverson bracket notation [13], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false.

For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with $f(0) = 0$ we define the reversion or compositional inverse of f to be the power series $\bar{f}(x)$ such that $f(\bar{f}(x)) = x$. We shall sometimes write this as $\bar{f} = \mathrm{Rev} f$.

For a lower triangular matrix $(a_{n,k})_{n,k\geq 0}$ the row sums give the sequence with general term $\sum_{k=0}^{n} a_{n,k}$ while the diagonal sums form the sequence with general term

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-k,k}.$$

The *Riordan group* [25, 29], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \cdots$ and $f(x) = f_1 x + f_2 x^2 + \cdots$ where $f_1 \neq 0$ [29]. We assume in addition that $f_1 = 1$ in what follows. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g, f)(h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence a_n, is called the *sequence array* of the sequence a_n. Its general term is a_{n-k} (or more precisely, $[k \leq n]a_{n-k}$). Such arrays are also called *Appell* arrays as they form the elements of the so-called Appell subgroup.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence with ordinary generating function $\mathcal{A}\ (x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbb{Z}^{\mathbb{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbb{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \mapsto (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 1. The so-called *binomial matrix* $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$, and hence as an array coincides with Pascal's triangle. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

Example 2. If a_n has generating function $g(x)$, then the generating function of the sequence

$$b_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-2k}$$

is equal to

$$\frac{g(x)}{1-x^2} = \left(\frac{1}{1-x^2}, x\right) \cdot g(x),$$

while the generating function of the sequence

$$d_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} a_{n-2k}$$

is equal to

$$\frac{1}{1-x^2} g\left(\frac{x}{1-x^2}\right) = \left(\frac{1}{1-x^2}, \frac{x}{1-x^2}\right) \cdot g(x).$$

The row sums of the matrix (g, f) have generating function

$$(g, f) \cdot \frac{1}{1-x} = \frac{g(x)}{1-f(x)}$$

while the diagonal sums of (g, f) (sums of left-to-right diagonals in the North East direction) have generating function $g(x)/(1-xf(x))$. These coincide with the row sums of the "generalized" Riordan array (g, xf):

$$(g, xf) \cdot \frac{1}{1-x} = \frac{g(x)}{1-xf(x)}.$$

For instance the Fibonacci numbers F_{n+1} are the diagonal sums of the binomial matrix $\mathbf{B}$ given by $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 3 & 1 & 0 & 0 & \ldots \\ 1 & 4 & 6 & 4 & 1 & 0 & \ldots \\ 1 & 5 & 10 & 10 & 5 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while they are the row sums of the "generalized" or "stretched" [6] Riordan array $\left(\frac{1}{1-x}, \frac{x^2}{1-x}\right)$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 4 & 3 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Each Riordan array $(g(x), f(x))$ has bi-variate generating function given by

$$\frac{g(x)}{1-yf(x)}.$$

For instance, the binomial matrix $\mathbf{B}$ has generating function

$$\frac{\frac{1}{1-x}}{1-y\frac{x}{1-x}} = \frac{1}{1-x(1+y)}.$$

For a sequence $a_0, a_1, a_2, \ldots$ with g.f. $g(x)$, the "aeration" of the sequence is the sequence $a_0, 0, a_1, 0, a_2, \ldots$ with interpolated zeros. Its g.f. is $g(x^2)$.

The aeration of a (lower-triangular) matrix $\mathbf{M}$ with general term $m_{i,j}$ is the matrix whose general term is given by

$$m^r_{\frac{i+j}{2},\frac{i-j}{2}} \frac{1+(-1)^{i-j}}{2},$$

where $m^r_{i,j}$ is the i,j-th element of the reversal of $\mathbf{M}$:

$$m^r_{i,j} = m_{i,i-j}.$$

In the case of a Riordan array (or indeed any lower triangular array), the row sums of the aeration are equal to the diagonal sums of the reversal of the original matrix.

Example 3. The Riordan array $(c(x^2), xc(x^2))$ is the aeration of $(c(x), xc(x))$ A033184. Here

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}$$

is the g.f. of the Catalan numbers. Indeed, the reversal of $(c(x), xc(x))$ is the matrix with general element

$$[k \le n+1]\binom{n+k}{k}\frac{n-k+1}{n+1},$$

which begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 2 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 5 & 5 & 0 & 0 & \ldots \\ 1 & 4 & 9 & 14 & 14 & 0 & \ldots \\ 1 & 5 & 14 & 28 & 42 & 42 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This is A009766. Then $(c(x^2), xc(x^2))$ has general element

$$\binom{n+1}{\frac{n-k}{2}}\frac{k+1}{n+1}\frac{1+(-1)^{n-k}}{2},$$

and begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots \\ 2 & 0 & 3 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & 4 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is A053121. Note that

$$(c(x^2), xc(x^2)) = \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right)^{-1}.$$

We observe that the diagonal sums of the reverse of $(c(x), xc(x))$ coincide with the row sums of $(c(x^2), xc(x^2))$, and are equal to the central binomial coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ A001405.

An important feature of Riordan arrays is that they have a number of sequence characterizations [4, 14]. The simplest of these is as follows.

Proposition 4. *[14, Theorem 2.1, Theorem 2.2] Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then D is a Riordan array if and only if there exist two sequences $A = [a_0, a_1, a_2, \dots]$ and $Z = [z_0, z_1, z_2, \dots]$ with $a_0 \neq 0$, $z_0 \neq 0$ such that*

- $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}, \quad (k, n = 0, 1, \dots)$
- $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}, \quad (n = 0, 1, \dots)$.

The coefficients $a_0, a_1, a_2, \dots$ and $z_0, z_1, z_2, \dots$ are called the A-sequence and the Z-sequence of the Riordan array $D = (g(x), f(x))$, respectively. Letting $A(x)$ be the generating function of the A-sequence and $Z(x)$ be the generating function of the Z-sequence, we have

$$A(x) = \frac{x}{\bar{f}(x)}, \quad Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right). \tag{1}$$

3 Orthogonal polynomials and Riordan arrays

By an *orthogonal polynomial sequence* $(p_n(x))_{n \geq 0}$ we shall understand [5, 12] an infinite sequence of polynomials $p_n(x)$, $n \geq 0$, of degree n, with real coefficients (often integer coefficients) that are mutually orthogonal on an interval $[x_0, x_1]$ (where $x_0 = -\infty$ is allowed, as well as $x_1 = \infty$), with respect to a weight function $w : [x_0, x_1] \to \mathbb{R}$:

$$\int_{x_0}^{x_1} p_n(x) p_m(x) w(x) dx = \delta_{nm} \sqrt{h_n h_m},$$

where

$$\int_{x_0}^{x_1} p_n^2(x) w(x) dx = h_n.$$

We assume that w is strictly positive on the interval (x_0, x_1). Every such sequence obeys a so-called "three-term recurrence" :

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x)$$

for coefficients a_n, b_n and c_n that depend on n but not x. We note that if

$$p_j(x) = k_j x^j + k'_j x^{j-1} + \dots \qquad j = 0, 1, \dots$$

then

$$a_n = \frac{k_{n+1}}{k_n}, \qquad b_n = a_n \left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n} \right), \qquad c_n = a_n \left(\frac{k_{n-1} h_n}{k_n h_{n-1}} \right).$$

Since the degree of $p_n(x)$ is n, the coefficient array of the polynomials is a lower triangular (infinite) matrix. In the case of monic orthogonal polynomials (where $k_n = 1$ for all n) the diagonal elements of this array will all be 1. In this case, we can write the three-term recurrence as

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0.$$

The *moments* associated to the orthogonal polynomial sequence are the numbers

$$\mu_n = \int_{x_0}^{x_1} x^n w(x) dx.$$

We can find $p_n(x)$, α_n and β_n (and in the right circumstances, $w(x)$ - see the Appendix) from a knowledge of these moments. To do this, we let Δ_n be the Hankel determinant $|\mu_{i+j}|_{i,j\geq 0}^n$ and $\Delta_{n,x}$ be the same determinant, but with the last row equal to $1, x, x^2, \dots$. Then

$$p_n(x) = \frac{\Delta_{n,x}}{\Delta_{n-1}}.$$

More generally, we let $H \begin{pmatrix} u_1 & \dots & u_k \\ v_1 & \dots & v_k \end{pmatrix}$ be the determinant of Hankel type with (i,j)-th term $\mu_{u_i + v_j}$. Let

$$\Delta_n = H \begin{pmatrix} 0 & 1 & \dots & n \\ 0 & 1 & \dots & n \end{pmatrix}, \qquad \Delta'_n = H_n \begin{pmatrix} 0 & 1 & \dots & n-1 & n \\ 0 & 1 & \dots & n-1 & n+1 \end{pmatrix}.$$

Then we have

$$\alpha_n = \frac{\Delta'_n}{\Delta_n} - \frac{\Delta'_{n-1}}{\Delta_{n-1}}, \qquad \beta_n = \frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}.$$

We shall say that a family of polynomials $\{p_n(x)\}_{n\geq 0}$ is [31] *formally orthogonal*, if there exists a linear functional $\mathcal{L}$ on polynomials such that

1. $p_n(x)$ is a polynomial of degree n,

2. $\mathcal{L}(p_n(x)p_m(x)) = 0$ for $m \neq n$,

3. $\mathcal{L}(p_n^2(x)) \neq 0$.

Consequences of this definition include [31] that

$$\mathcal{L}(x^m p_n(x)) = \kappa_n \delta_{mn}, \quad 0 \leq m \leq n, \quad \kappa_n \neq 0,$$

and if $q(x) = \sum_{k=0}^n a_k p_k(x)$, then $a_k = \mathcal{L}(qp_k)/\mathcal{L}(p_k^2)$.

The sequence of numbers $\mu_n = \mathcal{L}(x^n)$ is called the sequence of moments of the family of orthogonal polynomials defined by $\mathcal{L}$. Note that where a suitable weight function $w(x)$ exists, then we can realize the functional $\mathcal{L}$ as

$$\mathcal{L}(p(x)) = \int_{\mathbb{R}} p(x)w(x)\,dx.$$

If the family $p_n(x)$ is an orthogonal family for the functional $\mathcal{L}$, then it is also orthogonal for $c\mathcal{L}$, where $c \neq 0$. In the sequel, we shall always assume that $\mu_0 = \mathcal{L}(p_0(x)) = 1$.

The following well-known results (the first is the well-known "Favard's Theorem"), which we essentially reproduce from [18], specify the links between orthogonal polynomials, three term recurrences, and the recurrence coefficients and the g.f. of the moment sequence of the orthogonal polynomials.

Theorem 5. *[18] (Cf. [31], Théorème 9 on p.I-4, or [32], Theorem 50.1). Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n = 0, 1, \ldots$ Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ with $\beta_n \neq 0$ for all $n \geq 1$, such that the three-term recurrence*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \textit{for} \quad n \geq 1,$$

holds, with initial conditions $p_0(x) = 1$ and $p_1(x) = x - \alpha_0$.

Theorem 6. *[18] (Cf. [31], Proposition 1, (7), on p. V-5, or [32], Theorem 51.1). Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, which is orthogonal with respect to some functional $\mathcal{L}$. Let*

$$p_{n+1} = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad \textit{for} \quad n \geq 1,$$

be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the generating function

$$g(x) = \sum_{k=0}^{\infty} \mu_k x^k$$

for the moments $\mu_k = \mathcal{L}(x^k)$ satisfies

$$g(x) = \cfrac{\mu_0}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

Given a family of monic orthogonal polynomials

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \qquad p_0(x) = 1, \qquad p_1(x) = x - \alpha_0,$$

we can write

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k.$$

Then we have

$$\sum_{k=0}^{n+1} a_{n+1,k} x^k = (x - \alpha_n) \sum_{k=0}^{n} a_{n,k} x^k - \beta_n \sum_{k=0}^{n-1} a_{n-1,k} x^k$$

from which we deduce

$$a_{n+1,0} = -\alpha_n a_{n,0} - \beta_n a_{n-1,0} \tag{2}$$

and

$$a_{n+1,k} = a_{n,k-1} - \alpha_n a_{n,k} - \beta_n a_{n-1,k} \tag{3}$$

We note that if α_n and β_n are constant, equal to α and β, respectively, then the sequence $(1, -\alpha, -\beta, 0, 0, \ldots)$ forms an A-sequence for the coefficient array. The question immediately arises as to the conditions under which a Riordan array (g, f) can be the coefficient array of a family of orthogonal polynomials. A partial answer is given by the following proposition.

Proposition 7. *Every Riordan array of the form*

$$\left(\frac{1}{1 + rx + sx^2}, \frac{x}{1 + rx + sx^2}\right)$$

is the coefficient array of a family of monic orthogonal polynomials.

Proof. The array $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ [17] has a C-sequence $C(x) = \sum_{n \geq 0} c_n x^n$ given by

$$\frac{x}{1 + rx + sx^2} = \frac{x}{1 - xC(x)},$$

and thus

$$C(x) = -r - sx.$$

This means that the Riordan array $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ is determined by the fact that

$$a_{n+1,k} = a_{n,k-1} + \sum_{i \geq 0} c_i a_{n-i,k} \quad \text{for } n, k = 0, 1, 2, \ldots$$

where $a_{n,-1} = 0$. In the case of $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ we have

$$a_{n+1,k} = a_{n,k-1} - r a_{n,k} - s a_{n-1,k}.$$

Working backwards, this now ensures that

$$p_{n+1}(x) = (x - r)p_n(x) - s p_{n-1}(x),$$

where $p_n(x) = \sum_{k=0}^{n} a_{n,k} x^n$. The result now follows from Theorem 5. $\square$

We note that in this case the three-term recurrence coefficients α_n and β_n are constants. We have in fact the following proposition (see the next section for information on the Chebyshev polynomials).

Proposition 8. *The Riordan array* $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)$ *is the coefficient array of the modified Chebyshev polynomials of the second kind given by*

$$P_n(x) = (\sqrt{s})^n U_n\left(\frac{x-r}{2\sqrt{s}}\right), \quad n = 0, 1, 2, \ldots$$

Proof. The production array of $\left(\frac{1}{1+rx+sx^2}, \frac{x}{1+rx+sx^2}\right)^{-1}$ is given by

$$\left(\begin{array}{ccccccc} r & 1 & 0 & 0 & 0 & 0 & \ldots \\ s & r & 1 & 0 & 0 & 0 & \ldots \\ 0 & s & r & 1 & 0 & 0 & \ldots \\ 0 & 0 & s & r & 1 & 0 & \ldots \\ 0 & 0 & 0 & s & r & 1 & \ldots \\ 0 & 0 & 0 & 0 & s & r & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The result is now a consequence of the article [11] by Elouafi, for instance. □

The complete answer can be found by considering the associated production matrix of a Riordan arrray, in the following sense.

The concept of a *production matrix* [8, 9, 10] is a general one, but for this work we find it convenient to review it in the context of Riordan arrays. Thus let P be an infinite matrix (most often it will have integer entries). Letting $\mathbf{r}_0$ be the row vector

$$\mathbf{r}_0 = (1, 0, 0, 0, \ldots),$$

we define $\mathbf{r}_i = \mathbf{r}_{i-1}P$, $i \geq 1$. Stacking these rows leads to another infinite matrix which we denote by A_P. Then P is said to be the *production matrix* for A_P.
If we let

$$u^T = (1, 0, 0, 0, \ldots, 0, \ldots)$$

then we have

$$A_P = \left(\begin{array}{c} u^T \\ u^T P \\ u^T P^2 \\ \vdots \end{array}\right)$$

and

$$\bar{I} A_P = A_P P$$

where $\bar{I} = (\delta_{i+1,j})_{i,j\geq 0}$ (where δ is the usual Kronecker symbol):

$$\bar{I} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have

$$P = A_P^{-1} \bar{I} A_P. \tag{4}$$

Writing $\overline{A_P} = \bar{I} A_P$, we can write this equation as

$$P = A_P^{-1} \overline{A_P}. \tag{5}$$

Note that $\overline{A_P}$ is a "beheaded" version of A_P; that is, it is A_P with the first row removed.

The production matrix P is sometimes [23, 26] called the Stieltjes matrix S_{A_P} associated to A_P. Other examples of the use of production matrices can be found in [1], for instance. The sequence formed by the row sums of A_P often has combinatorial significance and is called the sequence associated to P. Its general term a_n is given by $a_n = u^T P^n e$ where

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

In the context of Riordan arrays, the production matrix associated to a proper Riordan array takes on a special form:

Proposition 9. *[10, Proposition 3.1] Let P be an infinite production matrix and let A_P be the matrix induced by P. Then A_P is an (ordinary) Riordan matrix if and only if P is of the form*

$$P = \begin{pmatrix} \xi_0 & \alpha_0 & 0 & 0 & 0 & 0 & \dots \\ \xi_1 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \dots \\ \xi_2 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \dots \\ \xi_3 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \dots \\ \xi_4 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \dots \\ \xi_5 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\xi_0 \neq 0$, $\alpha_0 \neq 0$. Moreover, columns 0 and 1 of the matrix P are the Z- and A-sequences, respectively, of the Riordan array A_P.

We recall that we have

$$A(x) = \frac{x}{\bar{f}(x)}, \quad Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right).$$

Example 10. We consider the Riordan array $\mathbf{L}$ where

$$L^{-1}=\left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right).$$

The production matrix (Stieltjes matrix) of

$$L=\left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}$$

is given by

$$P=S_L=\begin{pmatrix} a+\lambda & 1 & 0 & 0 & 0 & 0 & \ldots \\ b+\mu & a & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We note that since

$$\begin{aligned} L^{-1} &= \left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right) \\ &= (1-\lambda x-\mu x^2, x)\cdot\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right), \end{aligned}$$

we have

$$L=\left(\frac{1-\lambda x-\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}=\left(\frac{1}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)^{-1}\cdot\left(\frac{1}{1-\lambda x-\mu x^2}, x\right).$$

If we now let

$$L_1=\left(\frac{1}{1+ax}, \frac{x}{1+ax}\right)\cdot L,$$

then [22] we obtain that the Stieltjes matrix for L_1 is given by

$$S_{L_1}=\begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & \ldots \\ b+\mu & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have in fact the following general result [22]:

Proposition 11. *If $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is tridiagonal, then necessarily*

$$P = S_L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ b_1 & a & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$f(x) = Rev\frac{x}{1+ax+bx^2} \qquad and \qquad g(x) = \frac{1}{1-a_1x-b_1xf},$$

and vice-versa.

This leads to the important corollary

Corollary 12. *If $L = (g(x), f(x))$ is a Riordan array and $P = S_L$ is tridiagonal, with*

$$P = S_L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ b_1 & a & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6}$$

then L^{-1} is the coefficient array of the family of orthogonal polynomials $p_n(x)$ where $p_0(x) = 1$, $p_1(x) = x - a_1$, and

$$p_{n+1}(x) = (x-a)p_n(x) - b_n p_{n-1}(x), \qquad n \geq 2,$$

where b_n is the sequence $0, b_1, b, b, b, \ldots$.

Proof. By Favard's theorem, it suffices to show that L^{-1} defines a family of polynomials $\{p_n(x)\}$ that obey the above three-term recurrence. Now L is lower-triangular and so L^{-1} is the coefficient array of a family of polynomials $p_n(x)$ (with the degree of $p_n(x)$ being n), where

$$L^{-1}\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix} = \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix}.$$

We have

$$S_L \cdot L^{-1} = L^{-1} \cdot \bar{L} \cdot L^{-1} = L^{-1} \cdot \bar{I} \cdot L \cdot L^{-1} = L^{-1} \cdot \bar{I}.$$

Thus

$$S_L \cdot L^{-1} \cdot (1, x, x^2, \dots)^T = L^{-1} \cdot \bar{I} \cdot (1, x, x^2, \dots)^T = L^{-1} \cdot (x, x^2, x^3, \dots)^T.$$

We therefore obtain

$$\begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \dots \\ b_1 & a & 1 & 0 & 0 & 0 & \dots \\ 0 & b & a & 1 & 0 & 0 & \dots \\ 0 & 0 & b & a & 1 & 0 & \dots \\ 0 & 0 & 0 & b & a & 1 & \dots \\ 0 & 0 & 0 & 0 & b & a & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} xp_0(x) \\ xp_1(x) \\ xp_2(x) \\ xp_3(x) \\ \vdots \end{pmatrix},$$

from which we infer that

$$p_1(x) = x - a_1,$$

and

$$p_{n+1}(x) + ap_n(x) + b_n p_{n-1}(x) = xp_n(x), \quad n \geq 1,$$

or

$$p_{n+1}(x) = (x - a)p_n(x) - b_n p_{n-1}(x), \quad n \geq 1.$$

□

If we now start with a family of orthogonal polynomials $\{p_n(x)\}$, $p_0(x) = 1$, $p_1(x) = x - a_1$, that for $n \geq 1$ obey a three-term recurrence

$$p_{n+1}(x) = (x - a)p_n(x) - b_n p_{n-1}(x),$$

where b_n is the sequence $0, b_1, b, b, b, \dots$, then we can define [23] an associated Riordan array $L = (g(x), f(x))$ by

$$f(x) = \text{Rev}\frac{x}{1 + ax + bx^2} \qquad \text{and} \qquad g(x) = \frac{1}{1 - a_1 x - b_1 x f}.$$

Clearly, L^{-1} is then the coefficient array of the family of polynomials $\{p_n(x)\}$. Combining these results, we have

Theorem 13. *A Riordan array $L = (g(x), f(x))$ is the inverse of the coefficient array of a family of orthogonal polynomials if and only if its production matrix $P = S_L$ is tri-diagonal.*

Proof. If L has a tri-diagonal production matrix, then by Corollary (12), L^{-1} is the coefficient array of orthogonal polynomials. It conversely L^{-1} is the coefficient array of a family of orthogonal polynomials, then using the fact they these polynomials obey a three-term recurrence and the uniqueness of the Z- and A-sequences, we see using equations (2) and (3) along with Proposition **9**, that the production matrix is tri-diagonal. □

Proposition 14. *Let $L = (g(x), f(x))$ be a Riordan array with tri-diagonal production matrix S_L. Then*

$$[x^n]g(x) = \mathcal{L}(x^n),$$

where $\mathcal{L}$ is the linear functional that defines the associated family of orthogonal polynomials.

Proof. Let $L = (l_{i,j})_{i,j\geq 0}$. We have [31]

$$x^n = \sum_{i=0}^{n} l_{n,i} p_i(x).$$

Applying $\mathcal{L}$, we get

$$\mathcal{L}(x^n) = \mathcal{L}\left(\sum_{i=0}^{n} l_{n,i} p_i(x)\right) = \sum_{i=0}^{n} l_{n,i} \mathcal{L}(p_i(x)) = \sum_{i=0}^{n} l_{n,i} \delta_{i,0} = l_{n,0} = [x^n]g(x).$$

$\square$

Thus under the conditions of the proposition, by Theorem 6, $g(x)$ is the g.f. of the moment sequence $\mu_n = \mathcal{L}(x^n)$. Hence $g(x)$ has the continued fraction expansion

$$g(x) = \cfrac{1}{1 - a_1 x - \cfrac{b_1 x^2}{1 - ax - \cfrac{bx^2}{1 - ax - \cfrac{bx^2}{1 - ax - \cdots}}}}.$$

This can also be established directly. To see this, we use the

Lemma 15. *Let*

$$f(x) = Rev \frac{x}{1 + ax + bx^2}.$$

Then

$$\frac{f}{x} = \frac{1}{1 - ax - bx^2(f/x)}.$$

Proof. By definition, $f(x)$ is the solution $u(x)$, with $u(0) = 0$, of

$$\frac{u}{1 + au + bu^2} = x.$$

We find

$$u(x) = \frac{1 - ax - \sqrt{1 - 2ax + (a^2 - 4b)x^2}}{2bx}.$$

Solving the equation

$$v(x) = \frac{1}{1 - av - bx^2 v},$$

we obtain

$$v(x) = \frac{1 - ax - \sqrt{1 - 2ax + (a^2 - 4b)x^2}}{2bx^2} = \frac{f(x)}{x}.$$

$\square$

Thus we have

$$\frac{f(x)}{x}=\frac{1}{x}\mathrm{Rev}\frac{x}{1+ax+bx^2}=\cfrac{1}{1-ax-\cfrac{bx^2}{1-ax-\cfrac{bx^2}{1-\cdots}}}.$$

Now

$$g(x)=\frac{1}{1-a_1x-b_1xf}=\frac{1}{1-a_1x-b_1x^2(f/x)}$$

immediately implies by the above lemma that

$$g(x)=\cfrac{1}{1-a_1x-\cfrac{b_1x^2}{1-ax-\cfrac{bx^2}{1-ax-\cfrac{bx^2}{1-\ldots}}}}.$$

We note that the elements of the rows of L^{-1} can be identified with the coefficients of the characteristic polynomials of the successive principal sub-matrices of P.

Example 16. We consider the Riordan array

$$\left(\frac{1}{1+ax+bx^2},\frac{x}{1+ax+bx^2}\right).$$

Then the production matrix (Stieltjes matrix) of the inverse Riordan array $\left(\frac{1}{1+ax+bx^2},\frac{x}{1+ax+bx^2}\right)^{-1}$ left-multiplied by the k-th binomial array

$$\left(\frac{1}{1-kx},\frac{x}{1-kx}\right)=\left(\frac{1}{1-x},\frac{x}{1-x}\right)^k$$

is given by

$$P=\begin{pmatrix} a+k & 1 & 0 & 0 & 0 & 0 & \ldots\\ b & a+k & 1 & 0 & 0 & 0 & \ldots\\ 0 & b & a+k & 1 & 0 & 0 & \ldots\\ 0 & 0 & b & a+k & 1 & 0 & \ldots\\ 0 & 0 & 0 & b & a+k & 1 & \ldots\\ 0 & 0 & 0 & 0 & b & a+k & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and vice-versa. This follows since

$$\left(\frac{1}{1+ax+bx^2},\frac{x}{1+ax+bx^2}\right)\cdot\left(\frac{1}{1+kx},\frac{x}{1+kx}\right)=\left(\frac{1}{1+(a+k)x+bx^2},\frac{x}{1+(a+k)x+bx^2}\right).$$

In fact we have the more general result:

$$\left(\frac{1+\lambda x+\mu x^2}{1+ax+bx^2}, \frac{x}{1+ax+bx^2}\right)\cdot\left(\frac{1}{1+kx}, \frac{x}{1+kx}\right) = \left(\frac{1+\lambda x+\mu x^2}{1+(a+k)x+bx^2}, \frac{x}{1+(a+k)x+bx^2}\right).$$

The inverse of this last matrix therefore has production array

$$\left(\begin{array}{ccccccc} a+k-\lambda & 1 & 0 & 0 & 0 & 0 & \ldots \\ b-\mu & a+k & 1 & 0 & 0 & 0 & \ldots \\ 0 & b & a+k & 1 & 0 & 0 & \ldots \\ 0 & 0 & b & a+k & 1 & 0 & \ldots \\ 0 & 0 & 0 & b & a+k & 1 & \ldots \\ 0 & 0 & 0 & 0 & b & a+k & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

4 Exponential Riordan arrays

The *exponential Riordan group* [3, 8, 10], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = g_0 + g_1x + g_2x^2 + \cdots$ and $f(x) = f_1x + f_2x^2 + \cdots$ where $g_0 \neq 0$ and $f_1 \neq 0$. In what follows, we shall assume

$$g_0 = f_1 = 1.$$

The associated matrix is the matrix whose i-th column has exponential generating function $g(x)f(x)^i/i!$ (the first column being indexed by 0). The matrix corresponding to the pair f, g is denoted by $[g, f]$. The group law is given by

$$[g, f] \cdot [h, l] = [g(h \circ f), l \circ f].$$

The identity for this law is $I = [1, x]$ and the inverse of $[g, f]$ is $[g, f]^{-1} = [1/(g \circ \bar{f}), \bar{f}]$ where $\bar{f}$ is the compositional inverse of f. We use the notation $e\mathcal{R}$ to denote this group.

If $\mathbf{M}$ is the matrix $[g, f]$, and $\mathbf{u} = (u_n)_{n\geq 0}$ is an integer sequence with exponential generating function $\mathcal{U}\ (x)$, then the sequence $\mathbf{Mu}$ has exponential generating function $g(x)\mathcal{U}(f(x))$. Thus the row sums of the array $[g, f]$ have exponential generating function given by $g(x)e^{f(x)}$ since the sequence $1, 1, 1, \ldots$ has exponential generating function e^x.

As an element of the group of exponential Riordan arrays, the Binomial matrix $\mathbf{B}$ is given by $\mathbf{B} = [e^x, x]$. By the above, the exponential generating function of its row sums is given by $e^xe^x = e^{2x}$, as expected (e^{2x} is the e.g.f. of 2^n).

Example 17. We consider the exponential Riordan array $[\frac{1}{1-x}, x]$, A094587. This array has elements

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 6 & 6 & 3 & 1 & 0 & 0 & \dots \\ 24 & 24 & 12 & 4 & 1 & 0 & \dots \\ 120 & 120 & 60 & 20 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and general term $[k \le n]\frac{n!}{k!}$, and inverse

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -3 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & -4 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which is the array $[1-x, x]$. In particular, we note that the row sums of the inverse, which begin $1, 0, -1, -2, -3, \dots$ (that is, $1-n$), have e.g.f. $(1-x)\exp(x)$. This sequence is thus the binomial transform of the sequence with e.g.f. $(1-x)$ (which is the sequence starting $1, -1, 0, 0, 0, \dots$).

Example 18. We consider the exponential Riordan array $L = [1, \frac{x}{1-x}]$. The general term of this matrix may be calculated as follows

$$\begin{aligned} T_{n,k} &= \frac{n!}{k!}[x^n]\frac{x^k}{(1-x)^k} \\ &= \frac{n!}{k!}[x^{n-k}](1-x)^{-k} \\ &= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{-k}{j}(-1)^j x^j \\ &= \frac{n!}{k!}[x^{n-k}]\sum_{j=0}^{\infty}\binom{k+j-1}{j}x^j \\ &= \frac{n!}{k!}\binom{k+n-k-1}{n-k} \\ &= \frac{n!}{k!}\binom{n-1}{n-k}. \end{aligned}$$

Thus its row sums, which have e.g.f. $\exp\left(\frac{x}{1-x}\right)$, have general term $\sum_{k=0}^{n}\frac{n!}{k!}\binom{n-1}{n-k}$. This is A000262, the 'number of "sets of lists": the number of partitions of $\{1, .., n\}$ into any number of lists, where a list means an ordered subset'. Its general term is equal to $(n-1)! L_{n-1}(1, -1)$.

We will use the following [8, 10],important result concerning matrices that are production matrices for exponential Riordan arrays.

Proposition 19. *Let* $A = (a_{n,k})_{n,k\geq 0} = [g(x), f(x)]$ *be an exponential Riordan array and let*

$$c(y) = c_0 + c_1 y + c_2 y^2 + \dots, \qquad r(y) = r_0 + r_1 y + r_2 y^2 + \dots \tag{7}$$

be two formal power series that that

$$r(f(x)) = f'(x) \tag{8}$$

$$c(f(x)) = \frac{g'(x)}{g(x)}. \tag{9}$$

Then

$$(i) \qquad a_{n+1,0} = \sum_i i! c_i a_{n,i} \tag{10}$$

$$(ii) \qquad a_{n+1,k} = r_0 a_{n,k-1} + \frac{1}{k!}\sum_{i\geq k} i!(c_{i-k} + k r_{i-k+1}) a_{n,i} \tag{11}$$

or, assuming $c_k = 0$ *for* $k < 0$ *and* $r_k = 0$ *for* $k < 0$*,*

$$a_{n+1,k} = \frac{1}{k!}\sum_{i\geq k-1} i!(c_{i-k} + k r_{i-k+1}) a_{n,i}. \tag{12}$$

Conversely, starting from the sequences defined by (7), the infinite array $(a_{n,k})_{n,k\geq 0}$ *defined by (12) is an exponential Riordan array.*

A consequence of this proposition is that the production matrix $P = (p_{i,j})_{i,j\geq 0}$ for an exponential Riordan array obtained as in the proposition satisfies [10]

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{i-j+1}) \qquad (c_{-1} = 0).$$

Furthermore, the bivariate exponential function

$$\phi_P(t,z) = \sum_{n,k} p_{n,k} t^k \frac{z^n}{n!}$$

of the matrix P is given by

$$\phi_P(t,z) = e^{tz}(c(z) + t r(z)).$$

Note in particular that we have

$$r(x) = f'(\bar{f}(x)), \tag{13}$$

and

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))}. \tag{14}$$

Example 20. The production matrix of $\left[1, \frac{x}{1+x}\right]$ A111596 is given by

$$\left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -2 & 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & -4 & 1 & 0 & 0 & \dots \\ 0 & 0 & 6 & -6 & 1 & 0 & \dots \\ 0 & 0 & 0 & 12 & -8 & 1 & \dots \\ 0 & 0 & 0 & 0 & 20 & -10 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The row sums of L^{-1} have e.g.f. $\exp\left(\frac{x}{1+x}\right)$, and start $1, 1, -1, 1, 1, -19, 151, \ldots$. This is A111884. This follows since we have $g(x) = 1$ and so $g'(x) = 0$, implying that $c(x) = 0$, and $f(x) = \frac{x}{1+x}$ which gives us $\bar{f}(x) = \frac{x}{1-x}$ and $f'(x) = \frac{1}{(1+x)^2}$. Thus $f'(\bar{f}(x)) = c(x) = (1-x)^2$. Hence the bivariate generating function of P is $e^{xy}(1-x)^2 y$, as required.

Example 21. The exponential Riordan array $\mathbf{A} = \left[\frac{1}{1-x}, \frac{x}{1-x}\right]$, or

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 4 & 1 & 0 & 0 & 0 & \dots \\ 6 & 18 & 9 & 1 & 0 & 0 & \dots \\ 24 & 96 & 72 & 16 & 1 & 0 & \dots \\ 120 & 600 & 600 & 200 & 25 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

has general term

$$T_{n,k} = \frac{n!}{k!}\binom{n}{k}.$$

It is closely related to the Laguerre polynomials. Its inverse $\mathbf{A}^{-1}$ is the exponential Riordan array $\left[\frac{1}{1+x}, \frac{x}{1+x}\right]$ with general term $(-1)^{n-k}\frac{n!}{k!}\binom{n}{k}$. This is A021009, the triangle of coefficients of the Laguerre polynomials $L_n(x)$.

The production matrix of the matrix $\mathbf{A}^{-1} = \left[\frac{1}{1+x}, \frac{x}{1+x}\right]$ is given by

$$\left(\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & 0 & 0 & \dots \\ 0 & 4 & 5 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9 & 7 & 1 & 0 & \dots \\ 0 & 0 & 0 & 16 & 9 & 1 & \dots \\ 0 & 0 & 0 & 0 & 25 & 11 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

This follows since we have $g(x) = \frac{1}{1-x}$, and so $g'(x) = \frac{1}{(1-x)^2}$, and $f(x) = \frac{x}{1-x}$ which yields $\bar{f}(x) = \frac{x}{1+x}$ and $f'(x) = \frac{1}{(1-x)^2}$. Then

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = 1 + x$$

while

$$r(x) = f'(\bar{f}(x)) = (1+x)^2.$$

Thus the bivariate generating function of P is given by

$$e^{xy}(1+x+(1+x)^2y).$$

We note that

$$\mathbf{A} = \exp(\mathbf{S}),$$

where

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 4 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 9 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 16 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 25 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 22. The exponential Riordan array $\left[e^x, \ln\left(\frac{1}{1-x}\right)\right]$, or

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 8 & 6 & 1 & 0 & 0 & \ldots \\ 1 & 24 & 29 & 10 & 1 & 0 & \ldots \\ 1 & 89 & 145 & 75 & 15 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the coefficient array for the polynomials

$${}_2F_0(-n, x; -1)$$

which are an unsigned version of the Charlier polynomials (of order 0) [12, 24, 30]. This is A094816. It is equal to

$$[e^x, x] \cdot \left[1, \ln\left(\frac{1}{1-x}\right)\right],$$

or the product of the binomial array $\mathbf{B}$ and the array of (unsigned) Stirling numbers of the first kind. The production matrix of the inverse of this matrix is given by

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & -3 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 3 & -4 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 4 & -5 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 5 & -6 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which indicates the orthogonal nature of these polynomials. We can prove this as follows. We have

$$\left[e^x, \ln\left(\frac{1}{1-x}\right)\right]^{-1} = \left[e^{-(1-e^{-x})}, 1-e^{-x}\right].$$

Hence $g(x) = e^{-(1-e^{-x})}$ and $f(x) = 1-e^{-x}$. We are thus led to the equations

$$\begin{aligned} r(1-e^{-x}) &= e^{-x}, \\ c(1-e^{-x}) &= -e^{-x}, \end{aligned}$$

with solutions $r(x) = 1-x$, $c(x) = x-1$. Thus the bi-variate generating function for the production matrix of the inverse array is

$$e^{tz}(z-1+t(1-z)),$$

which is what is required.

We can infer the following result from the article [23] by Peart and Woan.

Proposition 23. *If $L = [g(x), f(x)]$ is an exponential Riordan array and $P = S_L$ is tridiagonal, then necessarily*

$$P = S_L = \begin{pmatrix} \alpha_0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \beta_1 & \alpha_1 & 1 & 0 & 0 & 0 & \dots \\ 0 & \beta_2 & \alpha_2 & 1 & 0 & 0 & \dots \\ 0 & 0 & \beta_3 & \alpha_3 & 1 & 0 & \dots \\ 0 & 0 & 0 & \beta_4 & \alpha_4 & 1 & \dots \\ 0 & 0 & 0 & 0 & \beta_5 & \alpha_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\{\alpha_i\}_{i\geq 0}$ is an arithmetic sequence with common difference α, $\left\{\frac{\beta_i}{i}\right\}_{i\geq 1}$ is an arithmetic sequence with common difference β, and

$$\ln(g) = \int (\alpha_0 + \beta_1 f)\, dx, \quad g(0) = 1,$$

where f is given by

$$f' = 1 + \alpha f + \beta f^2, \quad f(0) = 0,$$

and vice-versa.

In the above, we note that $\alpha_0 = a_1$ where $g(x)$ is the g.f. of $a_0 = 1, a_1, a_2, \ldots$. We have the important

Corollary 24. *If $L = [g(x), f(x)]$ is an exponential Riordan array and $P = S_L$ is tridiagonal, with*

$$P = S_L = \begin{pmatrix} \alpha_0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \beta_1 & \alpha_1 & 1 & 0 & 0 & 0 & \dots \\ 0 & \beta_2 & \alpha_2 & 1 & 0 & 0 & \dots \\ 0 & 0 & \beta_3 & \alpha_3 & 1 & 0 & \dots \\ 0 & 0 & 0 & \beta_4 & \alpha_4 & 1 & \dots \\ 0 & 0 & 0 & 0 & \beta_5 & \alpha_5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then L^{-1} is the coefficient array of the family of monic orthogonal polynomials $p_n(x)$ where $p_0(x) = 1$, $p_1(x) = x - a_1 = x - \alpha_0$, and

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad n \geq 0.$$

Proof. By Favard's theorem, it suffices to show that L^{-1} defines a family of polynomials $\{p_n(x)\}$ that obey the above three-term recurrence. Now L is lower-triangular and so L^{-1} is the coefficient array of a family of polynomials $p_n(x)$, where

$$L^{-1}\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{pmatrix} = \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix}.$$

We have

$$S_L \cdot L^{-1} = L^{-1} \cdot \bar{L} \cdot L^{-1} = L^{-1} \cdot \bar{I} \cdot L \cdot L^{-1} = L^{-1} \cdot \bar{I}.$$

Thus

$$S_L \cdot L^{-1} \cdot (1, x, x^2, \ldots)^T = L^{-1} \cdot \bar{I} \cdot (1, x, x^2, \ldots)^T = L^{-1} \cdot (x, x^2, x^3, \ldots)^T.$$

We therefore obtain

$$\begin{pmatrix} \alpha_0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ \beta_1 & \alpha_1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & \beta_2 & \alpha_2 & 1 & 0 & 0 & \ldots \\ 0 & 0 & \beta_3 & \alpha_3 & 1 & 0 & \ldots \\ 0 & 0 & 0 & \beta_4 & \alpha_4 & 1 & \ldots \\ 0 & 0 & 0 & 0 & \beta_5 & \alpha_5 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} xp_0(x) \\ xp_1(x) \\ xp_2(x) \\ xp_3(x) \\ \vdots \end{pmatrix},$$

from which we infer

$$p_1(x) = x - \alpha_0,$$

and

$$p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x) = xp_n(x), \quad n \geq 1,$$

or

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x) \quad n \geq 1.$$

□

If we now start with a family of orthogonal polynomials $\{p_n(x)\}$ that obeys a three-term recurrence

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x), \quad n \geq 1,$$

with $p_0(x) = 1$, $p_1(x) = x - \alpha_0$, where $\{\alpha_i\}_{i\geq 0}$ is an arithmetic sequence with common difference α and $\left\{\frac{\beta_i}{i}\right\}_{i\geq 1}$ is an arithmetic sequence with common difference β, then we can define [23] an associated exponential Riordan array $L = [g(x), f(x)]$ by

$$f' = 1 + \alpha f + \beta f^2, \quad f(0) = 0,$$

and

$$\ln(g) = \int (\alpha_0 + \beta_1 f)\, dx, \quad g(0) = 1.$$

Clearly, L^{-1} is then the coefficient array of the family of polynomials $\{p_n(x)\}$. Gathering these results, we have

Theorem 25. *An exponential Riordan array $L = [g(x), f(x)]$ is the inverse of the coefficient array of a family of orthogonal polynomials if and only if its production matrix $P = S_L$ is tri-diagonal.*

Proposition 26. *Let $L = [g(x), f(x)]$ be an exponential Riordan array with tri-diagonal production matrix S_L. Then*

$$n![x^n]g(x) = \mathcal{L}(x^n) = \mu_n,$$

where $\mathcal{L}$ is the linear functional that defines the associated family of orthogonal polynomials.

Proof. Let $L = (l_{i,j})_{i,j\geq 0}$. We have

$$x^n = \sum_{i=0}^{n} l_{n,i} p_i(x).$$

Applying $\mathcal{L}$, we get

$$\mathcal{L}(x^n) = \mathcal{L}\left(\sum_{i=0}^{n} l_{n,i} p_i(x)\right) = \sum_{i=0}^{n} l_{n,i} \mathcal{L}(p_i(x)) = \sum_{i=0}^{n} l_{n,i}\delta_{i,0} = l_{n,0} = n![x^n]g(x).$$

□

Corollary 27. *Let $L = [g(x), f(x)]$ be an exponential Riordan array with tri-diagonal production matrix S_L. Then the moments μ_n of the associated family of orthogonal polynomials are given by the terms of the first column of L.*

Thus under the conditions of the proposition, $g(x)$ is the g.f. of the moment sequence $\mathcal{L}(x^n)$. Hence $g(x)$ has the continued fraction expansion

$$g(x) = \cfrac{1}{1 - \alpha_0 x - \cfrac{\beta_1 x^2}{1 - \alpha_1 x - \cfrac{\beta_2 x^2}{1 - \alpha_2 x - \cfrac{\beta_3 x^2}{1 - \alpha_3 x - \cdots}}}}.$$

When we come to study Hermite polynomials, we shall be working with elements of the exponential Appell subgroup. By the *exponential Appell subgroup* $\mathcal{A}e\mathcal{R}$ of $e\mathcal{R}$ we understand the set of arrays of the form $[f(x), x]$.

Let $\mathbf{A} \in \mathcal{A}e\mathcal{R}$ correspond to the sequence $(a_n)_{n\geq 0}$, with e.g.f. $f(x)$. Let $\mathbf{B} \in \mathcal{A}e\mathcal{R}$ correspond to the sequence (b_n), with e.g.f. $g(x)$. Then we have

1. The row sums of $\mathbf{A}$ are the binomial transform of (a_n).
2. The inverse of $\mathbf{A}$ is the sequence array for the sequence with e.g.f. $\frac{1}{f(x)}$.
3. The product $\mathbf{AB}$ is the sequence array for the exponential convolution $a * b(n) = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$ with e.g.f. $f(x)g(x)$.

For instance, the row sums of $\mathbf{A} = [f(x), x]$ will have e.g.f. given by

$$[f(x), x] \cdot e^x = f(x)e^x = e^x f(x) = [e^x, x] \cdot f(x),$$

which is the e.g.f. of the binomial transform of (a_n).

Example 28. We consider the matrix $[\cosh(x), x]$, A119467, with elements

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 6 & 0 & 1 & 0 & \dots \\ 0 & 5 & 0 & 10 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row sums of this matrix have e.g.f. $\cosh(x)\exp(x)$, which is the e.g.f. of the sequence $1, 1, 2, 4, 8, 16, \ldots$. The inverse matrix is $[\operatorname{sech}(x), x]$, A119879, with entries

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & -3 & 0 & 1 & 0 & 0 & \dots \\ 5 & 0 & -6 & 0 & 1 & 0 & \dots \\ 0 & 25 & 0 & -10 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The row sums of this matrix have e.g.f. $\operatorname{sech}(x)\exp(x)$. This is A155585.

5 The Hankel transform of an integer sequence

The *Hankel transform* of a given sequence $A = \{a_0, a_1, a_2, ...\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \ldots\}$ where $h_n = |a_{i+j}|_{i,j=0}^{n}$, i.e

$$A = \{a_n\}_{n \in \mathbb{N}_0} \quad \rightarrow \quad h = \{h_n\}_{n \in \mathbb{N}_0} : \quad h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix}. \tag{15}$$

The Hankel transform of a sequence a_n and its binomial transform are equal.

In the case that a_n has g.f. $g(x)$ expressible in the form

$$g(x)=\cfrac{a_0}{1-\alpha_0 x-\cfrac{\beta_1 x^2}{1-\alpha_1 x-\cfrac{\beta_2 x^2}{1-\alpha_2 x-\cfrac{\beta_3 x^2}{1-\alpha_3 x-\cdots}}}}$$

then we have [18]

$$h_n=a_0^{n+1}\beta_1^n\beta_2^{n-1}\cdots\beta_{n-1}^2\beta_n=a_0^{n+1}\prod_{k=1}^n \beta_k^{n+1-k}. \tag{16}$$

Note that this independent from α_n.

We note that α_n and β_n are in general not integers. Now let $H\begin{pmatrix} u_1 & \dots & u_k \\ v_1 & \dots & v_k\end{pmatrix}$ be the determinant of Hankel type with (i,j)-th term $\mu_{u_i+v_j}$. Let

$$\Delta_n=H\begin{pmatrix} 0 & 1 & \dots & n \\ 0 & 1 & \dots & n\end{pmatrix},\qquad \Delta'_n=H_n\begin{pmatrix} 0 & 1 & \dots & n-1 & n \\ 0 & 1 & \dots & n-1 & n+1\end{pmatrix}.$$

Then we have

$$\alpha_n=\frac{\Delta'_n}{\Delta_n}-\frac{\Delta'_{n-1}}{\Delta_{n-1}},\qquad \beta_n=\frac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}. \tag{17}$$

6 Legendre polynomials

We recall that the Legendre polynomials $P_n(x)$ can be defined by

$$P_n(x)=\sum_{k=0}^n (-1)^k\binom{n}{k}^2\left(\frac{1+x}{2}\right)^{n-k}\left(\frac{1-x}{2}\right)^k.$$

Their generating function is given by

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{n=0}^\infty P_n(x)t^n.$$

We note that the production matrix of the inverse of the coefficient array of these polynomials is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 & \dots \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 & \dots \\ 0 & 0 & \frac{3}{7} & 0 & \frac{4}{7} & 0 & \dots \\ 0 & 0 & 0 & \frac{4}{9} & 0 & \frac{5}{9} & \dots \\ 0 & 0 & 0 & 0 & \frac{5}{11} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which corresponds to the fact that the $P_n(x)$ satisfy the following three-term recurrence

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

The shifted Legendre polynomials $\tilde{P}_n(x)$ are defined by

$$\tilde{P}_n(x) = P_n(2x-1).$$

They satisfy

$$\tilde{P}_n(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-x)^k = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{2k}\binom{2k}{k}x^k.$$

Their coefficient array begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 2 & 0 & 0 & 0 & 0 & \dots \\ 1 & -6 & 6 & 0 & 0 & 0 & \dots \\ -1 & 12 & -30 & 20 & 0 & 0 & \dots \\ 1 & -20 & 90 & -140 & 70 & 0 & \dots \\ -1 & 30 & -210 & 560 & -630 & 252 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

and so the first few terms begin

$$1, 2x-1, 6x^2-6x+1, 20x^3-30x^2+12x-1, \dots$$

We clearly have

$$\frac{1}{\sqrt{1-2(2x-1)t+t^2}} = \sum_{n=0}^{\infty} \tilde{P}_n(x)t^n.$$

7 Legendre polynomials as moments

Our goal in this section is to represent the Legendre polynomials as the first column of a Riordan array whose production matrix is tri-diagonal. We first of all consider the so-called shifted Legendre polynomials. We have

Proposition 29. *The inverse* $\mathbf{L}$ *of the Riordan array*

$$\left(\frac{1+r(1-r)x^2}{1+(2r-1)x+r(r-1)x^2}, \frac{x}{1+(2r-1)x+r(r-1)x^2}\right)$$

has as its first column the shifted Legendre polynomials $\tilde{P}_n(r)$. *The production matrix of* $\mathbf{L}$ *is tri-diagonal.*

Proof. Indeed, standard Riordan array techniques show that we have

$$\begin{aligned}\mathbf{L} &= \left(\frac{1+r(1-r)x^2}{1+(2r-1)x+r(r-1)x^2}, \frac{x}{1+(2r-1)x+r(r-1)x^2}\right)^{-1} \\ &= \left(\frac{1}{\sqrt{1-2(2r-1)x+x^2}}, \frac{1-(2r-1)x-\sqrt{1-2(2r-1)x+x^2}}{2r(r-1)x}\right).\end{aligned}$$

This establishes the first part. Now using equations (1), with

$$f(x) = \frac{1-(2r-1)x-\sqrt{1-2(2r-1)x+x^2}}{2r(r-1)x}, \quad \bar{f}(x) = \frac{x}{1+(2r-1)x+r(r-1)x^2},$$

and

$$g(x) = \frac{1}{\sqrt{1-2(2r-1)x+x^2}},$$

we find that

$$Z(x) = (2r-1)+2rx(r-1), \quad A(x) = 1+x(2r-1)+x^2r(r-1).$$

Hence by Proposition 9 the production matrix $P = S_L$ of $\mathbf{L}$ is given by

$$\left(\begin{array}{ccccccc} 2r-1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2r(r-1) & 2r-1 & 1 & 0 & 0 & 0 & \dots \\ 0 & r(r-1) & 2r-1 & 1 & 0 & 0 & \dots \\ 0 & 0 & r(r-1) & 2r-1 & 1 & 0 & \dots \\ 0 & 0 & 0 & r(r-1) & 2r-1 & 1 & \dots \\ 0 & 0 & 0 & 0 & r(r-1) & 2r-1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

□

Corollary 30. *The shifted Legendre polynomials are moments of the family of orthogonal polynomials whose coefficient array is given by*

$$\mathbf{L}^{-1} = \left(\frac{1+r(1-r)x^2}{1+(2r-1)x+r(r-1)x^2}, \frac{x}{1+(2r-1)x+r(r-1)x^2}\right).$$

Proof. This follows from the above result and Proposition 14. □

Proposition 31. *The Hankel transform of the sequence $\tilde{P}_n(r)$ is given by $2^n(r(r-1))^{\binom{n+1}{2}}$.*

Proof. From the above, the g.f. of $\tilde{P}_n(r)$ is given by

$$\cfrac{1}{1-(2r-1)x-\cfrac{2r(r-1)x^2}{1-(2r-1)x-\cfrac{r(r-1)x^2}{1-(2r-1)x-\cfrac{r(r-1)x^2}{1-\cdots}}}}.$$

The result now follows from Equation (16). □

We note that

$$\tilde{P}_n(r)=\frac{1}{\pi}\int_{-2\sqrt{r(r-1)}+2r-1}^{2\sqrt{r(r-1)}+2r-1}\frac{x^n}{\sqrt{-x^2+2(2r-1)x-1}}\,dx$$

gives an explicit moment representation for $\tilde{P}_n(r)$.

Turning now to the Legendre polynomials $P_n(x)$, we have the following result.

Proposition 32. *The inverse* $\mathbf{L}$ *of the Riordan array*

$$\left(\frac{1+\frac{1-r^2}{4}x^2}{1+rx+\frac{r^2-1}{2}x^2},\frac{x}{1+rx+\frac{r^2-1}{2}x^2}\right)$$

has as its first column the Legendre polynomials $P_n(r)$*. The production matrix of* $\mathbf{L}$ *is tri-diagonal.*

Proof. We have

$$\begin{aligned}\mathbf{L} &= \left(\frac{1+\frac{1-r^2}{4}x^2}{1+rx+\frac{r^2-1}{2}x^2},\frac{x}{1+rx+\frac{r^2-1}{2}x^2}\right)^{-1}\\ &= \left(\frac{1}{\sqrt{1-2rx+x^2}},\frac{2(1-rx-\sqrt{1-2rx+x^2})}{x(r^2-1)}\right).\end{aligned}$$

This proves the first assertion. Using equations (9) again, we obtain the following tri-diagonal matrix as the production matrix of $\mathbf{L}$:

$$\left(\begin{array}{ccccccc} r & 1 & 0 & 0 & 0 & 0 & \ldots\\ \frac{r^2-1}{2} & r & 1 & 0 & 0 & 0 & \ldots\\ 0 & \frac{r^2-1}{4} & r & 1 & 0 & 0 & \ldots\\ 0 & 0 & \frac{r^2-1}{4} & r & 1 & 0 & \ldots\\ 0 & 0 & 0 & \frac{r^2-1}{4} & r & 1 & \ldots\\ 0 & 0 & 0 & 0 & \frac{r^2-1}{4} & r & \ldots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{array}\right).$$

□

Corollary 33.

$$\mathbf{L}^{-1}=\left(\frac{1+\frac{1-r^2}{4}x^2}{1+rx+\frac{r^2-1}{2}x^2},\frac{x}{1+rx+\frac{r^2-1}{2}x^2}\right)$$

is the coefficient array of a set of orthogonal polynomials for which the Legendre polynomials are moments.

Proposition 34. *The Hankel transform of* $P_n(r)$ *is given by*

$$\frac{(r^2-1)^{\binom{n+1}{2}}}{2^{n^2}}.$$

Proof. From the above, we obtain that the g.f. of $P_n(r)$ can be expressed as

$$\cfrac{1}{1-rx-\cfrac{\frac{r^2-1}{2}x^2}{1-rx-\cfrac{\frac{r^2-1}{4}x^2}{1-rx-\cfrac{\frac{r^2-1}{4}x^2}{1-\cdots}}}}.$$

The result now follows from Equation (16). ☐

We end this section by noting that

$$P_n(r) = \frac{1}{\pi}\int_{r-\sqrt{r^2-1}}^{r+\sqrt{r^2-1}} \frac{x^n}{\sqrt{-x^2+2rx-1}}\,dx$$

gives an explicit moment representation for $P_n(r)$.

8 Hermite polynomials

The Hermite polynomials may be defined as

$$H_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}.$$

The generating function for $H_n(x)$ is given by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

The unitary Hermite polynomials (also called normalized Hermite polynomials) are given by

$$He_n(x) = 2^{-\frac{n}{2}} H_n(\sqrt{2}x) = \sum_{k=0}^{n} \frac{n!}{(-2)^{\frac{n-k}{2}} k! \left(\frac{n-k}{2}\right)!} \frac{1+(-1)^{n-k}}{2} x^k.$$

Their generating function is given by

$$e^{xt-\frac{t^2}{2}} = \sum_{n=0}^{\infty} He_n(x)\frac{t^n}{n!}.$$

We note that the coefficient array of He_n is a proper exponential Riordan array, equal to

$$\left[e^{-\frac{x^2}{2}}, x\right].$$

This array A066325 begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -3 & 0 & 1 & 0 & 0 & \cdots \\ 3 & 0 & -6 & 0 & 1 & 0 & \cdots \\ 0 & 15 & 0 & -10 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is the aeration of the alternating sign version of the Bessel coefficient array A001497. The inverse of this matrix has production matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which corresponds to the fact that we have the following three-term recurrence for He_n:

$$He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x).$$

9 Hermite polynomials as moments

Proposition 35. *The proper exponential Riordan array*

$$\mathbf{L} = \left[e^{rx-\frac{x^2}{2}}, x\right]$$

has as first column the unitary Hermite polynomials $He_n(r)$. This array has a tri-diagonal production array.

Proof. The first column of $\mathbf{L}$ has generating function $e^{rx-\frac{x^2}{2}}$, from which the first assertion follows. We now use equations (13) and (14) to calculate the production matrix $P = S_L$. We have $f(x) = x$, so that $\bar{f}(x) = x$ and $f'(\bar{f}(x)) = 1 = r(x)$. Also, $g(x) = e^{rx-\frac{x^2}{2}}$ implies that $g'(x) = g(x)(r - x)$, and hence

$$c(x) = \frac{g'(\bar{f}(x))}{g(\bar{f}(x))} = r - x.$$

Thus the production array of $\mathbf{L}$ is indeed tri-diagonal, beginning

$$\begin{pmatrix} r & 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & r & 1 & 0 & 0 & 0 & \ldots \\ 0 & -2 & r & 1 & 0 & 0 & \ldots \\ 0 & 0 & -3 & r & 1 & 0 & \ldots \\ 0 & 0 & 0 & -4 & r & 1 & \ldots \\ 0 & 0 & 0 & 0 & -5 & r & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

□

We note that $\mathbf{L}$ starts

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ r & 1 & 0 & 0 & 0 & 0 & \ldots \\ r^2-1 & 2r & 1 & 0 & 0 & 0 & \ldots \\ r(r^2-3) & 3(r^2-1) & 3r & 1 & 0 & 0 & \ldots \\ r^4-6r^2+3 & 4r(r^2-3) & 6(r^2-1) & 4r & 1 & 0 & \ldots \\ r(r^4-10r^2+15) & 5(r^4-6r^2+3) & 10r(r^2-3) & 10(r^2-1) & 5r & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus

$$\mathbf{L}^{-1} = \left[e^{-rx+\frac{x^2}{2}}, x\right]$$

is the coefficient array of a set of orthogonal polynomials which have as moments the unitary Hermite polynomials. These new orthogonal polynomials satisfy the three-term recurrence

$$\mathfrak{H}_{n+1}(x) = (x-r)\mathfrak{H}_n(x) + n\mathfrak{H}_{n-1}(x),$$

with $\mathfrak{H}_0 = 1$, $\mathfrak{H}_1 = x - r$.

Proposition 36. *The Hankel transform of the sequence $He_n(r)$ is given by $(-1)^{\binom{n+1}{2}} \prod_{k=0}^n k!$*

Proof. By the above, the g.f. of $H_n(r)$ is given by

$$\cfrac{1}{1-rx+\cfrac{2x^2}{1-rx+\cfrac{3x^2}{1-rx+\cfrac{4x^2}{1-\ldots}}}}.$$

The result now follows from Equation (16). □

Turning now to the Hermite polynomials $H_n(x)$, we have the following result.

Proposition 37. *The proper exponential Riordan array*

$$\mathbf{L} = \left[e^{2rx-x^2}, x\right]$$

has as first column the Hermite polynomials $H_n(r)$. This array has a tri-diagonal production array.

Proof. The first column of $\mathbf{L}$ has generating function e^{2rx-x^2}, from which the first assertion follows. Using equations (13) and (14) we find that the production array of $\mathbf{L}$ is indeed tri-diagonal, beginning

$$\left(\begin{array}{ccccccc} 2r & 1 & 0 & 0 & 0 & 0 & \dots \\ -2 & 2r & 1 & 0 & 0 & 0 & \dots \\ 0 & -4 & 2r & 1 & 0 & 0 & \dots \\ 0 & 0 & -6 & 2r & 1 & 0 & \dots \\ 0 & 0 & 0 & -8 & 2r & 1 & \dots \\ 0 & 0 & 0 & 0 & -10 & 2r & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

$\square$

We note that $\mathbf{L}$ starts

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2r & 1 & 0 & 0 & 0 & 0 & \dots \\ 2(2r^2-1) & 4r & 1 & 0 & 0 & 0 & \dots \\ 4r(2r^2-3) & 6(2r^2-1) & 6r & 1 & 0 & 0 & \dots \\ 4(4r^3-12r^2+3) & 16r(2r^2-3) & 12(2r^2-1) & 8r & 1 & 0 & \dots \\ 8r(4r^4-20r^2+15) & 20(4r^4-12r^2+3) & 40r(2r^2-3) & 20(2r^2-1) & 10r & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

Thus

$$\mathbf{L}^{-1} = \left[e^{-2rx+x^2}, x\right]$$

is the coefficient array of a set of orthogonal polynomials which have as moments the Hermite polynomials. These new orthogonal polynomials satisfy the three-term recurrence

$$\mathfrak{H}_{n+1}(x) = (x-2r)\mathfrak{H}_n(x) + 2n\mathfrak{H}_{n-1}(x),$$

with $\mathfrak{H}_0 = 1$, $\mathfrak{H}_1 = x - 2r$.

Proposition 38. *The Hankel transform of the sequence $H_n(r)$ is given by $(-1)^{\binom{n+1}{2}}\prod_{k=0}^n 2^k k!$*

Proof. By the above, the g.f. of $H_n(r)$ is given by

$$\cfrac{1}{1-2rx+\cfrac{2x^2}{1-2rx+\cfrac{4x^2}{1-2rx+\cfrac{6x^2}{1-\dots}}}}.$$

The result now follows from Equation (16). $\square$

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11 Appendix - The Stieltjes transform of a measure

The *Stieltjes transform* of a measure μ on $\mathbb{R}$ is a function G_μ defined on $\mathbb{C} \setminus \mathbb{R}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\mu(t).$$

If f is a bounded continuous function on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} f(x)\mu(x) = -\lim_{y\to 0^+} \int_{\mathbb{R}} f(x)\Im G_\mu(x+iy)dx.$$

If μ has compact support, then G_μ is holomorphic at infinity and for large z,

$$G_\mu(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}},$$

where $a_n = \int_{\mathbb{R}} t^n \mu(t)$ are the moments of the measure. If $\mu(t) = d\psi(t) = \psi'(t)dt$ then (Stieltjes-Perron)

$$\psi(t) - \psi(t_0) = -\frac{1}{\pi} \lim_{y\to 0^+} \int_{t_0}^{t} \Im G_\mu(x+iy)dx.$$

If now $g(x)$ is the generating function of a sequence a_n, with $g(x) = \sum_{n=0}^{\infty} a_n x^n$, then we can define

$$G(z) = \frac{1}{z} g\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}.$$

By this means, under the right circumstances we can retrieve the density function for the measure that defines the elements a_n as moments.

Example 39. We let $g(z) = \frac{1-\sqrt{1-4z}}{2z}$ be the g.f. of the Catalan numbers. Then

$$G(z) = \frac{1}{z} g\left(\frac{1}{z}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{x-4}{x}}\right).$$

Then

$$\Im G_\mu(x+iy) = -\frac{\sqrt{2}\sqrt{\sqrt{x^2+y^2}\sqrt{x^2-8x+y^2+16} - x^2+4x-y^2}}{4\sqrt{x^2+y^2}},$$

and so we obtain

$$\begin{aligned} \psi'(x) &= -\frac{1}{\pi} \lim_{y\to 0^+} \left\{ -\frac{\sqrt{2}\sqrt{\sqrt{x^2+y^2}\sqrt{x^2-8x+y^2+16} - x^2+4x-y^2}}{4\sqrt{x^2+y^2}} \right\} \\ &= \frac{1}{2\pi} \frac{\sqrt{x(4-x)}}{x}. \end{aligned}$$

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