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A Note on Lacunary Lonely Runners

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Abstract

We give a simple argument proving the lonely runner conjecture in the case where the speed of the runners forms a certain lacunary sequence. This improves an earlier result by Pandey, and is then used to derive that for each number of runners the lonely runner conjecture is true for a set of nonzero measure in a natural probability space.

1 Introduction

Consider n people running at a circular track of length 1. The i-th runner runs with speed $v_i > 0$ and no two runners run with the same speed. Assume furthermore that all runners start at $t = 0$ at the same point. Is it then the case that each runner gets lonely at some point $t > 0$, i.e., has a circular distance of at least $1/(n+1)$ to the next runner? This appealing problem is equivalent (with $n \leftrightarrow n-1$, the interpretation as a race is due to Goddyn) to the following conjecture due to Wills [10] and (independently) Cusick [5]. Using the notation $\{x\} := x - \lfloor x \rfloor$ for $x \in \mathbb{R}$ to denote the fractional part, we can state the conjecture as follows.

Conjecture 1 (Wills 1967, Cusick 1974)**.** For any $v_1, v_2, \ldots, v_n \in \mathbb{R}_+$ there exists a $t > 0$ with

$$\{v_k t\} \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right] \qquad \text{for all } \ 1 \leq k \leq n.$$

Betke and Wills [2] and (independently) Cusick [6] proved the statement for $n = 3$, the case $n = 4$ was dealt with by Cusick and Pomerance [7] — a simplified proof for $n = 4$ is due to Bienia, Goddyn, Gvozdjak, Sebő and Tarsi [3]. Bohman, Holzman and Kleitman [4] proved $n = 5$ (later simplified by Renault [9]) and recently $n = 6$ was shown by Barajas and Serra [1].

The conjecture is rather remarkable insofar as it is easily stated, rather natural and quite impenetrable. Proofs of the special cases are not just ad hoc, but quite involved and rely on various alternative formulations: Betke and Wills saw it as a problem regarding diophantine approximation, Cusick saw it as a view obstruction problem, and it has been connected with flows on graphs and matroids since.

2 The results

This short note aims to focus on a slightly different viewpoint; indeed, it could be quite interesting to gain a perspective on things that is independent of the number of runners and rather captures some inherent behavior instead. An example of such a result was recently shown by Pandey [8].

Theorem 2 (Pandey)**.** *Assume the nonnegative reals $v_1, v_2, \dots, v_n$ satisfy*

$$v_{k+1} \geq 2\frac{n+1}{n-1}v_k \qquad \text{for all } 1 \leq k \leq n-1,$$

then there exists $0 < t < 1/v_1$ with

$$\{v_k t\} \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right] \qquad \text{for all} \quad 1 \leq k \leq n.$$

The intuition behind a result of this type is clear, and it is not at all surprising that a theorem of this kind should hold. The speed condition implies that by changing the time only very slightly, we can effectively keep the first k runners in place, while the last $n-k$ runners move very quickly and assume all kinds of constellations - among those constellations also some we like. A drawback of this result is that, since $v_n/v_1 \geq 2^{n-1}$, the velocities of the runners need to vary widely in size. We improve this result by making a minor modification in Pandey's construction to remove the factor 2.

Theorem 3. *Assume the nonnegative reals $v_1, v_2, \dots, v_n$ satisfy*

$$v_{k+1} \geq \frac{n+1}{n-1}v_k \qquad \text{for all } 1 \leq k \leq n-1,$$

then there exists $0 < t < 1/v_1$ with

$$\{v_k t\} \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right] \qquad \text{for all} \quad 1 \leq k \leq n.$$

Proof. We define a descending chain of intervals

$$\left[\frac{1}{v_1}\frac{1}{n+1}, \frac{1}{v_1}\frac{n}{n+1}\right] =: I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n,$$

where each interval has at least a certain minimal length (λ denotes the Lebesgue measure)

$$\lambda(I_k) \geq \frac{n-1}{n+1}\frac{1}{v_k}$$

and, finally, for all $t \in I_k$, we have

$$\{v_k t\} \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right].$$

So far, this is equivalent to Pandey's construction. The new idea consists of the following observation. Assume $I_1, \dots, I_k$ are already constructed in a manner satisfying the above constraints. Then, by the lacunarity condition, the range of the linear function

$$\begin{aligned} f : I_k &\rightarrow \mathbb{R}_+ \\ t &\rightarrow v_{k+1} t \end{aligned}$$

is again an interval J of length at least

$$\lambda(J) = v_{k+1}\lambda(I) \geq \frac{n-1}{n+1}\frac{v_{k+1}}{v_k} \geq 1$$

and therefore $\{v_{k+1}t\}$ with $t \in I_k$ covers the entire unit interval. Therefore, if we take from the set of reals

$$\left\{t \in I_k : \{v_{k+1}t\} \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right]\right\}$$

one of the connected components of maximal length and call it I_{k+1}, then

$$\lambda(I_{k+1}) \geq \frac{n-1}{n+1}\frac{1}{v_{k+1}}$$

and this interval satisfies all desired properties. Since it follows from the lower bound on the measure that I_n is not empty, any $t \in I_n$ proves the result. $\square$

This has rather dramatic consequences, since it effectively allows us to even deal with scenarios, where

$$v_n \leq e^2 v_1,$$

independent of the actual number of runners n (though, of course, the lacunarity condition is still a very strong one).

Since the lonely runner conjecture has proven to be a highly difficult problem indeed, it could also be interesting to prove the conjecture 'in measure', i.e., to prove it for a large subset of a suitable probability space. We can use the above result to at least establish its truth for a small subset. The scaling invariance of the problem, i.e., the conjecture holds for

$$(v_1, \dots, v_n) \qquad \text{if and only if it holds for } (\lambda v_1, \dots, \lambda v_n)$$

for every $\lambda > 0$, does suggest $[0,1]^n$ as a somewhat natural probability space from which to choose the speed of the runners (though there are many other possibilities that spring to mind, the part of the surface of the unit sphere with only nonzero coordinates being one of them).

A theorem due to Kronecker (see also [4]) implies that the lonely runner conjecture holds if $1, v_1, v_2, \dots, v_n$ are linearly independent over $\mathbb{Q}$ (actually, much stronger statements hold in this case). This implies that

$$\lambda\left(\left\{(v_1, \dots, v_n) \in [0,1]^n \middle| \text{ the conjecture fails } \right\}\right) = 0,$$

where λ is the usual Lebesgue measure. This result is somewhat nice but tells us very little about the set of critical speeds (there are even everywhere dense sets with Lebesgue measure 0). It would hence be interesting, to establish the existence of large sets for which the lonely runner conjecture is true and where the sets look less fractal than $\mathbb{R} \setminus \mathbb{Q}$ and more continuous (like the union of many small intervals). The suitable measure for this task is the Jordan measure J (for example, the inner Jordan measure of the complement of an everywhere dense set is 0). It is conceivable that the lonely runner conjecture is false and the resulting exceptional set is not Jordan measurable (i.e., inner and outer Jordan measure do not coincide), we therefore give a lower for the inner Jordan measure.

Corollary 4. *Let $\Omega_n = [0,1]^n$ and let J be the $n-$dimensional Jordan measure. Then we have*

$$J\left(\left\{(v_1, \dots, v_n) \in \Omega_n \middle| \exists\, t > 0 : \{v_k t\} \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right] \text{ for all } 1 \le k \le n\right\}\right) \gtrsim \sqrt{n}(2e^{-3})^n.$$

Proof. Define $0 < a_1 < a_2 < \dots < a_{2n} = 1$ via

$$a_1 = \left(\frac{n-1}{n+1}\right)^{2n-1} \qquad \text{and} \qquad a_{k+1} = \frac{n+1}{n-1} a_k.$$

Then the intervals $I_1, \dots, I_n$ given by $I_k = (a_{2k-1}, a_{2k})$ fulfill

$$\forall t \in I_k : a_{2k} < \frac{n+1}{n-1} x < a_{2k+1}.$$

Hence, if we have $v_1, v_2, \dots, v_n$ and each v_i lies in some interval I_j and conversely every interval I_i contains some point v_j, then the previous result implies that the lonely runner conjecture holds for these values. Hence

$$J\left(\{\boldsymbol{v} \in \Omega_n : \text{ the conjecture holds}\}\right) \ge J\left(\bigcup_{\pi \in S_n} I_{\pi(1)} \times I_{\pi(2)} \times \dots \times I_{\pi(n)}\right) = n! \prod_{k=1}^{n} J(I_k),$$

where the union runs over all permutations $\pi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. Inserting the definition now yields

$$\prod_{k=1}^{n} J(I_k) = \prod_{k=1}^{n} (a_{2k} - a_{2k-1}) = \left(\frac{2}{n-1}\right)^n \prod_{k=1}^{n} a_{2k-1}$$

However, this product can be exactly evaluated

$$\prod_{k=1}^{n} a_{2k-1} = a_1^n \prod_{k=1}^{n} \left(\frac{n+1}{n-1}\right)^{2k-2} = \left(\frac{n-1}{n+1}\right)^{(2n-1)n} \left(\frac{n+1}{n-1}\right)^{(n-1)n} = \left(\frac{n-1}{n+1}\right)^{n^2}$$

and

$$\left(\frac{n-1}{n+1}\right)^{n^2} = \left(\left(1 - \frac{2}{n+1}\right)^{n+1}\right)^{n-1} \left(1 - \frac{2}{n+1}\right) \gtrsim e^{-2n}.$$

Thus, altogether, by Stirling's formula

$$n! \left(\frac{2}{n-1}\right)^n \prod_{k=1}^{n} a_{2k-1} \gtrsim \sqrt{n} \left(2e^{-3}\right)^n.$$

□

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