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Functorial Orbit Counting

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Abstract

We study the functorial and growth properties of closed orbits for maps. By viewing an arbitrary sequence as the orbit-counting function for a map, iterates and Cartesian products of maps define new transformations between integer sequences. An orbit monoid is associated to any integer sequence, giving a dynamical interpretation of the Euler transform.

1 Introduction

Many combinatorial or dynamical questions involve counting the number of closed orbits or the periodic points under iteration of a map. Here we consider functorial properties of orbit-counting in the following sense. Associated to a map $T : X \to X$ with the property that T has only finitely many orbits of each length are combinatorial data (counts of fixed points and periodic orbits), analytic data (a zeta function and a Dirichlet series) and algebraic data (the orbit monoid). On the other hand, the collection of such maps is closed under disjoint unions, direct products, iteration, and other operations. Our starting point is to ask how the associated data behaves under those operations. A feature of this work is that these natural operations applied to maps with simple orbit structures give novel constructions of sequences with combinatorial or arithmetic interest. Routine calculations are suppressed here for brevity; complete details, related results, and further applications will appear in the thesis of the first author [16].

We define the following categories:

- MAPS $\mathfrak{M}$, comprising all pairs (X, T) where T is a map $X \to X$ with $\mathcal{F}_T(n) = |\{x \in X \mid T^n(x) = x\}| < \infty$ for all $n \geqslant 1$;
- ORBITS $\mathfrak{O} = \mathbb{N}_0^{\mathbb{N}}$, comprising all sequences $(a_n)_{n\geqslant 1}$ with $a_n \geqslant 0$ for all $n \geqslant 1$; and

- FIXED POINTS $\mathfrak{F} \subseteq \mathfrak{O}$, comprising any sequence $a = (a_n)$ with the property that there is some $(X, T) \in \mathfrak{M}$ with $a_n = \mathcal{F}_T(n)$ for all $n \geqslant 1$.

For $(X, T) \in \mathfrak{M}$, a closed orbit of length n under T is any set of the form $\tau = \{x, Tx, \dots, T^n x = x\}$ with cardinality $|\tau| = n$, and we write $\mathcal{O}_T(n)$ for the number of closed orbits of length n. Clearly

$$\mathcal{F}_T(n) = \sum_{d|n} d\, \mathcal{O}_T(d), \tag{1}$$

so

$$\mathcal{O}_T(n) = \frac{1}{n} \sum_{d|n} \mu(n/d)\, \mathcal{F}_T(d), \tag{2}$$

and this defines a bijection between $\mathfrak{F}$ and $\mathfrak{O}$ (see Everest, van der Poorten, Puri and the second author [7, 17] for more on the combinatorial applications of this bijection, and for non-trivial examples of sequences in $\mathfrak{F}$; see Baake and Neumärker [3] for more on spectral properties of the operators on $\mathfrak{O}$). Since the space X will not concern us, we will fix it to be some countable set and refer to an element of $\mathfrak{M}$ as a map T.

Recall from Knopfmacher [12] that an *additive arithmetic semigroup* is a free abelian monoid G equipped with a non-empty set of generators P, and a weight function

$$\partial : G \to \mathbb{N} \cup \{0\}$$

with $\partial(a + b) = \partial(a) + \partial(b)$ for all $a, b \in G$, satisfying the finiteness property

$$P(n) = |\{p \in P \mid \partial(p) = n\}| < \infty$$

for all $n \geqslant 1$. Given $T \in \mathfrak{M}$, we define the orbit monoid $\mathcal{G}_T$ associated to T to be the free abelian monoid generated by the closed orbits of T, equipped with the weight function

$$\partial(a_1\tau_1 + \cdots + a_r\tau_r) = a_1|\tau_1| + \cdots + a_r|\tau_r|,$$

and write $\mathcal{G}_T(n)$ for the number of elements of weight n. Finally, define ORBIT MONOIDS $\mathfrak{G}$ to be the category of all such monoids associated to maps in $\mathfrak{M}$. Notice that every additive arithmetic semigroup in the sense of Knopfmacher [12] is an orbit monoid, since for any sequence (a_n) there is a map T with $\mathcal{O}_T(n) = a_n$ for all $n \geqslant 1$ (indeed, Windsor [23] shows that the map T may be chosen to be a C^∞ diffeomorphism of a torus). We will write $\mathcal{G}_T$ for the sequence $(\mathcal{G}_T(n))$, since the sequence determines the monoid up to isomorphism.

As usual, we write $\zeta = (1, 1, 1, \dots)$ and $\mu = (1, -1, -1, 0, \dots)$ for the zeta and Möbius functions viewed as sequences, and use the same symbols to denote their Dirichlet series.

There are natural generating functions associated to an element $T \in \mathfrak{M}$. If $\mathcal{F}_T(n)$ is exponentially bounded then the *dynamical zeta function*

$$\zeta_T(s) = \exp \sum_{n \geqslant 1} \frac{s^n}{n} \mathcal{F}_T(n)$$

converges in some complex disk (see Artin and Mazur [1]). If $\mathcal{O}_T(n)$ is polynomially bounded then the *orbit Dirichlet series*

$$\mathsf{d}_T(s) = \sum_{n\geqslant 1} \frac{\mathcal{O}_T(n)}{n^s},$$

converges in some half-plane. The basic relation (1) is expressed in terms of these generating functions by the two identities

$$\zeta_T(s) = \prod_{n\geqslant 1} (1-s)^{-\mathcal{O}_T(n)} = \prod_{\tau} \left(1 - s^{|\tau|}\right)^{-1},$$

where the product is taken over all closed orbits of T, and

$$\mathsf{d}_T(s)\zeta(s+1) = \sum_{n\geqslant 1} \frac{\mathcal{F}_T(n)}{n^{s+1}}.$$

Finally, a natural measure of the rate of growth in $\mathcal{O}_T$ is the number

$$\pi_T(N) = |\{\tau \mid |\tau| \leqslant N\}|$$

of closed orbits of length no more than N. Asymptotics for π_T are analogous to the prime number theorem. In the case of exponential growth, that is under the assumption that $\limsup_{n\to\infty} \frac{1}{n} \log \mathcal{O}_T(n) = h > 0$, a more smoothly averaged measure of orbit growth is given by

$$\mathcal{M}_T(N) = \sum_{|\tau|\leqslant N} \frac{1}{\mathrm{e}^{h|\tau|}},$$

and asymptotics for $\mathcal{M}_T$ are analogous to Mertens' theorem.

2 Functorial properties

Most functorial properties are immediate, so we simply record them here. Write $T_1 \times T_2$ for the Cartesian product of two maps, $T_1 \sqcup T_2$ for the disjoint union, defined by

$$(T_1 \sqcup T_2)(x) = \begin{cases} T_1(x), & \text{if } x \in X_1; \\ T_2(x), & \text{if } x \in X_2; \end{cases}$$

and write T^k with $k \geqslant 1$ for the kth iterate of T. Then

1. $\mathcal{F}_{T_1\times T_2} = \mathcal{F}_{T_1}\mathcal{F}_{T_2}$ (pointwise product);
2. $\mathsf{d}_{T_1\sqcup T_2} = \mathsf{d}_{T_1} + \mathsf{d}_{T_2}$;
3. $\zeta_{T_1\sqcup T_2} = \zeta_{T_1}\zeta_{T_2}$;
4. $\mathcal{F}_{T^k}(n) = \mathcal{F}_T(kn)$ for all $k \geqslant 1$ and $n \geqslant 1$.

In contrast to the first of these, it is clear that computing the number of closed orbits under the Cartesian product of two maps is more involved.

Lemma 1. $\mathcal{O}_{T_1\times T_2}(n) = \sum_{\substack{d_1,d_2\in\mathbb{N},\\ \operatorname{lcm}(d_1,d_2)=n}} \mathcal{O}_{T_1}(d_1)\,\mathcal{O}_{T_2}(d_2)\gcd(d_1,d_2)$.

Proof. If (x_1,x_2) lies on a $T_1\times T_2$-orbit of length n, then, in particular,

$$T_1^n(x_1)=x_1$$

and

$$T_2^n(x_2)=x_2,$$

so x_i lies on a T_i-orbit of length d_i for some d_i dividing n, for $i=1,2$. On the other hand, if x_i lies on a T_i-orbit of length d_i for $i=1,2$ then the $T_1\times T_2$-orbit of (x_1,x_2) has cardinality $\operatorname{lcm}(d_1,d_2)$. On the other hand, if τ_1,τ_2 are orbits of length d_1,d_2 with $\operatorname{lcm}(d_1,d_2)=n$, then there are d_1d_2 points in the set $\tau_1\times\tau_2$, so this must split up into $d_1d_2/n=\gcd(d_1,d_2)$ orbits of length n under $T_1\times T_2$. $\square$

Example 2. Let T be a map with one orbit of each length, so $\mathsf{d}_T(s)=\zeta(s)$. Then, by Lemma 1 and a calculation,

$$\mathcal{O}_{T\times T}(n) = \sum_{\substack{d_1,d_2\in\mathbb{N},\\ \operatorname{lcm}(d_1,d_2)=n}} \gcd(d_1,d_2) = \sum_{d\mid n}\sigma(d)\mu(n/d)^2,$$

so

$$\mathsf{d}_{T\times T}(s) = \frac{\zeta(s)^2\zeta(s-1)}{\zeta(2s)} = 1+\frac{4}{2^s}+\frac{5}{3^s}+\frac{10}{4^s}+\frac{7}{5^s}+\frac{20}{6^s}+\frac{9}{7^s}+\frac{22}{8^s}+\cdots. \tag{3}$$

By identifying an orbit of length n under T with a cyclic group C_n of order n, we see from the proof of Lemma 1 that $\mathcal{O}_{T\times T}(n)$ is the number of cyclic subgroups of $C_n\times C_n$, so $\mathcal{O}_{T\times T}$ is A060648.

Example 2 is generalized in Example 16, where it corresponds to the case $P=\varnothing$.

Example 3. Let p be a prime, and assume that $\mathcal{O}_T(n)=p^n$ for all $n\geqslant 1$, so $\zeta_T(s)=\frac{1}{1-ps}$. Then

$$\mathcal{O}_{T\times T}(n) = \frac{1}{n}\sum_{d\mid n}\left(\mu(n/d)\sum_{d_1\mid d} d_1p^{d_1}\sum_{d_2\mid d} d_2p^{d_2}\right) = \frac{1}{n}\sum_{d\mid n}\mu(d)p^{2n/d},$$

which is the number of irreducible polynomials of degree $n+1$ over $\mathbb{F}_{p^2}$. In the case $p=2$ this gives the sequence A027377, and in the case $p=3$ this gives A027381.

The behavior of orbits under iteration is more involved. To motivate the rather dense formula below, consider the orbits of length n under T^p for some prime p. Points on an m-orbit under T lie on an orbit of length $m/\gcd(m,p)$ under T^p. If $m\neq n, np$ then $m/\gcd(m,p)\neq n$, so the only points that can contribute to $\mathcal{O}_{T^p}(n)$ are points lying on n-orbits or on np-orbits under T. Each np-orbit under T splits into p orbits of length n under T^p. An n-orbit under T defines an n-orbit under T^p only if $p \nmid n$. It follows that

$$\mathcal{O}_{T^p}(n) = \begin{cases} p\,\mathcal{O}_T(pn) + \mathcal{O}_T(n), & \text{if } p \nmid n; \\ p\,\mathcal{O}_T(pn), & \text{if } p | n. \end{cases} \tag{4}$$

In order to state the general case, fix the power m and write $m = \boldsymbol{p}^{\boldsymbol{a}} = p_1^{a_1} \cdots p_r^{a_r}$ for the decomposition into primes of $m \in \mathbb{N}$; for any set $J \subseteq I = \{p_1, \dots, p_r\}$ write $\boldsymbol{p}_J^{\boldsymbol{a}_J} = \prod_{p_j \in J} p_j^{a_j}$. Finally, write $\mathcal{D}(n)$ for the set of prime divisors of n.

Theorem 4. *Let $m = \boldsymbol{p}^{\boldsymbol{a}}$ and $J = J(n) = \mathcal{D}(m) \setminus \mathcal{D}(n)$. Then*

$$\mathcal{O}_{T^m}(n) = \sum_{d | \boldsymbol{p}_J^{\boldsymbol{a}_J}} \tfrac{m}{d}\,\mathcal{O}_T(\tfrac{mn}{d}). \tag{5}$$

Proof. Notice that J depends on n, so the formula (5) involves a splitting into cases depending on the primes dividing n, just as in the case of a single prime discussed above. We argue by induction on the length $\sum_{i=1}^r a_i$ of m. If the length of m is 1, then m is a prime and (5) reduces to (4). Assume now that (5) holds for $\sum_{i=1}^r a_i \leqslant k$, and let m have length k; write $I = \mathcal{D}(m)$ and $J = \mathcal{D}(m) \setminus \mathcal{D}(n)$. We consider the effect of multiplying m by a prime q on the formula (5), and write $I' = \mathcal{D}(mq)$, $J' = \mathcal{D}(mq) \setminus \mathcal{D}(n)$.

If $q \in \mathcal{D}(n) \setminus I$ then by (4) we have

$$\mathcal{O}_{T^{mq}}(n) = q\,\mathcal{O}_{T^m}(qn) = q \sum_{d | \boldsymbol{p}_J^{\boldsymbol{a}_J}} \tfrac{m}{d}\,\mathcal{O}_T(\tfrac{mnq}{d}) = q \sum_{d | \boldsymbol{p}_{J'}^{\boldsymbol{a}_{J'}}} \tfrac{m}{d}\,\mathcal{O}_T(\tfrac{mnq}{d}),$$

in accordance with (5).

If $q \in I \cap \mathcal{D}(n)$, then $I' = I$ and $J' = J$, so

$$\mathcal{O}_{T^{mq}}(n) = q\,\mathcal{O}_{T^m}(qn) = q \sum_{d | \boldsymbol{p}_{J'}^{\boldsymbol{a}_{J'}}} \tfrac{m}{d}\,\mathcal{O}_T(\tfrac{mnq}{d})$$

as required.

If $q \notin I \cup \mathcal{D}(n)$ then

$$\begin{aligned} \mathcal{O}_{T^{mq}}(n) &= q\,\mathcal{O}_{T^m}(qn) + \mathcal{O}_{T^m}(n) \\ &= q \sum_{d | \boldsymbol{p}_J^{\boldsymbol{a}_J}} \tfrac{m}{d}\,\mathcal{O}_T(\tfrac{mqn}{d}) + \sum_{d | \boldsymbol{p}_J^{\boldsymbol{a}_J}} \tfrac{m}{d}\,\mathcal{O}_T(\tfrac{mn}{d}) \\ &= \sum_{d | \boldsymbol{p}_J^{\boldsymbol{a}_J} q} \tfrac{qm}{d}\,\mathcal{O}_T(\tfrac{mqn}{d}) \end{aligned}$$

as required.

Finally, if $q \in I \setminus \mathcal{D}(n)$ then

$$\begin{aligned}
\mathcal{O}_{T^{mq}}(n) &= q\,\mathcal{O}_{T^m}(qn) + \mathcal{O}_{T^m}(n) \\
&= q \sum_{d \mid \boldsymbol{p}_{J\setminus\{q\}}^{\boldsymbol{a}_{J\setminus\{q\}}}} \tfrac{m}{d}\,\mathcal{O}_T(\tfrac{mqn}{d}) + \sum_{d \mid \boldsymbol{p}_J^{\boldsymbol{a}_J}} \tfrac{m}{d}\,\mathcal{O}_T(\tfrac{mn}{d}) \\
&= \sum_{\substack{d \mid \boldsymbol{p}_J^{\boldsymbol{a}_J} q, \\ q \nmid d}} \tfrac{mq}{d}\,\mathcal{O}_T\,\tfrac{mqn}{d} + \sum_{\substack{d \mid \boldsymbol{p}_J^{\boldsymbol{a}_J} q, \\ q \mid d}} \tfrac{m}{d/q}\,\mathcal{O}_T(\tfrac{mn/d}{d/q}) \\
&= \sum_{d \mid \boldsymbol{p}_J^{\boldsymbol{a}_J} q} \frac{qm}{d}\,\mathcal{O}_T(\tfrac{mqn}{d}),
\end{aligned}$$

completing the proof. □

This defines a family of transformations on sequences, taking $\mathcal{O}_T$ to $\mathcal{O}_{T^k}$ for each $k \geqslant 1$.

Example 5. Let $T \in \mathfrak{M}$ have $\mathcal{O}_T(n) = n$ for all $n \geqslant 1$, so $\mathsf{d}_T(s) = \zeta(s-1)$. Then by (2) we have

$$\mathcal{O}_{T^2}(n) = \frac{1}{n} \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \mathcal{F}_{T^2}(d) = \frac{1}{n} \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \sigma_2(2d)$$

since $\mathcal{F}_{T^2}(d) = \mathcal{F}_T(2d) = \sum_{e \mid 2d} e\,\mathcal{O}_T(e) = \sum_{e \mid 2d} e^2$, so

$$\mathsf{d}_{T^2}(s) = \left(5 - \frac{2}{2^s}\right) \zeta(s-1) = 5 + \frac{8}{2^s} + \frac{15}{3^s} + \frac{16}{4^s} + \frac{25}{5^s} + \frac{24}{6^s} + \frac{35}{7^s} + \frac{32}{8^s} + \cdots .$$

Thus $\mathcal{O}_{T^2}$ is A091574 (up to an offset).

Example 6. More generally, if $\mathsf{d}_T(s) = \zeta(s-1)$ and p is a prime, then

$$\mathcal{O}_{T^p}(n) = \frac{1}{n} \sum_{d \mid n} \mu\left(\frac{n}{d}\right) \sigma_2(pd) = \begin{cases} (p^2+1)n, & \text{if } p \nmid n; \\ p^2 n & \text{if } p \mid n; \end{cases}$$

so $\mathsf{d}_{T^p}(s) = \left(p^2 + 1 - \dfrac{p}{p^s}\right) \zeta(s-1)$. Composite powers are more involved; full details are in [16]. For example,

$$\mathcal{O}_{T^4}(n) = \begin{cases} 16n, & \text{if } n \text{ is even;} \\ 21n & \text{if } n \text{ is odd;} \end{cases}$$

so $\mathsf{d}_{T^4}(s) = \left(2 - \frac{10}{2^s}\right) \zeta(s-1)$.

An important family of dynamical systems – those of *finite combinatorial rank* – have been studied by Everest, Miles, Stevens and Ward [6]. These have the property that their orbit Dirichlet series is "Dirichlet–rational", that is there is a finite set $C \subseteq \mathbb{Z}$ with the property that $\mathsf{d}_T(s)$ is a rational function in the variables $\{c^{-s} \mid c \in C\}$. An easy consequence of Theorem 4 is that this property is preserved under iteration.

Corollary 7. *If maps S and T have Dirichlet–rational orbit Dirichlet series, then so do $S\times T$ and T^k for any $k \geqslant 1$.*

Example 8. The quadratic map $T : x \mapsto 1-cx^2$ on the interval $[-1,1]$ at the Feigenbaum value $c = 1.401155\cdots$ (see Feigenbaum's lecture notes [8]; this is at the end of a period-doubling cascade) gives a particularly simple example of a Dirichlet–rational Dirichlet series. This map has

$$\mathcal{O}_T(n) = \begin{cases} 1, & \text{if } n = 2^k \text{ for some } k \geqslant 0; \\ 0 & \text{if not;} \end{cases}$$

so $\mathsf{d}_T(s) = \frac{1}{1-2^{-s}}$ and $\mathcal{O}_T$ is (up to an offset) the Fredholm-Rueppel sequence A036987. By (1) we have $\mathcal{F}_T(n) = 2\lfloor n \rfloor_2 - 1$, where $\lfloor n \rfloor_2 = |n|_2^{-1}$ denotes the 2-part of n, so $\mathcal{F}_T$ is A038712. Using this we see that

$$\mathcal{O}_{T^2}(n) = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right)(2\lfloor 2d \rfloor_2 - 1) = \begin{cases} 3, & \text{if } n = 1; \\ 2, & \text{if } n = 2^k \text{ for some } k \geqslant 1; \\ 0, & \text{if not;} \end{cases}$$

so

$$\mathsf{d}_{T^2}(s) = \frac{3-2^{-s}}{1-2^{-s}}.$$

More generally, the formula for $\mathcal{F}_T$ shows that

$$\mathsf{d}_{T^k}(s) = \lfloor k \rfloor_2 - 1 + \lfloor k \rfloor_2 \mathsf{d}_T(s)$$

for any $k \geqslant 1$.

Example 9. With T as in Example 8, a similar calculation using Lemma 1 shows that

$$\mathsf{d}_{T\times T}(s) = \frac{3}{1-2^{-(s-1)}} - \frac{2}{1-2^{-s}}.$$

If $S \in \mathfrak{M}$ has

$$\mathcal{O}_S(n) = \begin{cases} 1, & \text{if } n = 3^k \text{ for some } k \geqslant 0; \\ 0, & \text{if not;} \end{cases}$$

then $O_{T\times S}$ is A065333, the characteristic function of the 3-smooth numbers, so

$$\mathsf{d}_{T\times S}(s) = \frac{1}{(1-2^{-s})(1-3^{-s})},$$

$$\mathsf{d}_{T\times T\times S}(s) = \frac{3}{(1-2^{-(s-1)})(1-3^{-s})} - \frac{2}{(1-2^{-s})(1-3^{-s})},$$

and so on.

The thesis of the first author [16] characterizes the existence of "roots": that is, given a sequence $a \in \mathfrak{O}$ and $k \geqslant 1$ to determine if there is some $T \in \mathfrak{M}$ with $\mathcal{O}_{T^k} = a$. Instances of no roots, unique roots, and uncountably many roots occur.

3 Multiplicative sequences

Multiplicative sequences in $\mathfrak{O}$ are particularly easy to work with, and in this section we describe simple examples of such sequences, and some properties of their product systems. In particular, we show how simple orbit sequences may factorize (that is, be the orbit sequence of the product of two maps) in many different ways. Since $\mathcal{F}_T$ and $\mathcal{O}_T$ are related by convolution with μ and multiplication by n, it is clear that $\mathcal{F}_T$ is multiplicative if and only if $\mathcal{O}_T$ is multiplicative. The next lemma is equally straightforward; we include the proof to illustrate how the correspondence between $\mathcal{F}_T$ and $\mathcal{O}_T$ may be exploited.

Lemma 10. *If any two of $\mathcal{O}_T$, $\mathcal{O}_S$ and $\mathcal{O}_{T\times S}$ are multiplicative, then so is the third.*

Proof. Assume the first two are multiplicative and $\gcd(m,n)=1$. Then

$$\begin{aligned}
\mathcal{O}_{T\times S}(mn) &= \tfrac{1}{mn}\sum_{d|mn}\mu(\tfrac{mn}{d})\,\mathcal{F}_{T\times S}(d) = \tfrac{1}{mn}\sum_{d|m}\sum_{d'|n}\mu(\tfrac{m}{d})\mu(\tfrac{n}{d'})\,\mathcal{F}_T(dd')\,\mathcal{F}_S(dd')\\
&= \tfrac{1}{mn}\sum_{d|m}\sum_{d'|n}\mu(\tfrac{m}{d})\mu(\tfrac{n}{d'})\,\mathcal{F}_T(d)\,\mathcal{F}_T(d')\,\mathcal{F}_S(d)\,\mathcal{F}_{S'}(d')\\
&= \tfrac{1}{mn}\sum_{d|m}\sum_{d'|n}\mu(\tfrac{m}{d})\mu(\tfrac{n}{d'})\,\mathcal{F}_{T\times S}(d)\,\mathcal{F}_{T\times S}(d') = \mathcal{O}_{T\times S}(m)\,\mathcal{O}_{T\times S}(n).
\end{aligned}$$

Now assume that $\mathcal{O}_S$ is not multiplicative while $\mathcal{O}_T$ is, and choose m,n of minimal product with the property that $\gcd(m,n)=1$ and $\mathcal{O}_S(mn)\neq\mathcal{O}_S(m)\,\mathcal{O}_S(n)$. Then, by construction, if $ab<mn$ and $\gcd(a,b)=1$ we have $\mathcal{O}_S(ab)=\mathcal{O}_S(a)\,\mathcal{O}_S(b)$, so we must have $\mathcal{F}_S(mn)\neq\mathcal{F}_S(m)\,\mathcal{F}_S(n)$. If $mn=1$ then

$$\mathcal{O}_{T\times S}(1)=\mathcal{F}_{T\times S}(1)=\mathcal{F}_T(1)\,\mathcal{F}_S(1)=\mathcal{O}_T(1)\,\mathcal{O}_S(1)=\mathcal{O}_S(1)\neq 1,$$

so $\mathcal{O}_{T\times S}$ is not multiplicative. If $mn>1$ then a calculation gives

$$\begin{aligned}
\mathcal{O}_{T\times S}(mn) &= \tfrac{1}{mn}\sum_{\substack{d|m,d'|n,\\ dd'<mn}}\mu(\tfrac{mn}{dd'})\,\mathcal{F}_{T\times S}(dd')+\tfrac{1}{mn}\,\mathcal{F}_{T\times S}(mn)\\
&= \tfrac{1}{m}\sum_{d|m}\mu(\tfrac{m}{d})\,\mathcal{F}_{T\times S}(d)\cdot\tfrac{1}{n}\sum_{d'|n}\mu(\tfrac{n}{d'})\,\mathcal{F}_{T\times S}(d')\\
&\qquad\qquad -\tfrac{1}{mn}\,\mathcal{F}_{T\times S}(m)\,\mathcal{F}_{T\times S}(n)+\tfrac{1}{mn}\,\mathcal{F}_{T\times S}(mn)\\
&= \mathcal{O}_{T\times S}(m)\,\mathcal{O}_{T\times S}(n)-\tfrac{1}{mn}\,\mathcal{F}_T(mn)\underbrace{\mathcal{F}_S(m)\,\mathcal{F}_S(n)}_{\neq\mathcal{F}_S(mn)}\\
&\neq \mathcal{O}_{T\times S}(m)\,\mathcal{O}_{T\times S}(n)
\end{aligned}$$

since $\mathcal{F}_T(mn)\geqslant\mathcal{F}_T(1)=1$, so $\mathcal{O}_{T\times S}$ is not multiplicative. □

This gives a bijective proof of (3) as follows. Write $\ell(n)$ for the number of primitive lattices of index n in $\mathbb{Z}^2$, so that (3) is equivalent to the statement

$$\mathcal{O}_{T\times T}(n)=\sum_{d|n}\ell(d).$$

Both sides of this equation are multiplicative, so it is enough to prove this for $n = p^r$ a prime power. The primitive lattices of index p^j in $\mathbb{Z}^2$ are in one-to-one correspondence with

$$\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid ac = p^j,\ a, c \geqslant 1,\ 0 \leqslant b < p^j,\ \gcd(a, b, c) = 1 \right\}.$$

It follows that $\sum_{d|p^r} \ell(d) = p^r + 2\sum_{j=0}^{r-1} p^j$, in agreement with the formula for $\mathcal{O}_{T\times T}(p^r)$.

Write $\mathbb{P}$ for the set of all prime numbers, and for a subset $P \subseteq \mathbb{P}$ write $P^c = \mathbb{P} \setminus P$.

Example 11. For any set P of primes, define $s_P \in \mathfrak{O}$ by

$$s_P(n) = \begin{cases} 0, & \text{if } p|n \text{ for some } p \in P; \\ 1, & \text{if not.} \end{cases}$$

Lemma 12. *If T is a map with $\mathcal{O}_T = s_P$ and $k \geqslant 1$, then*

$$\mathcal{O}_n(T^k) = \begin{cases} \displaystyle\prod_{p\in Q, p|n} p^{a_p} \cdot \prod_{p\in Q, p\nmid n} \sigma(p^{a_p}) & \textit{if } p \nmid n \textit{ for all } p \in P; \\ \qquad\qquad 0 & \textit{if } p|n \textit{ for some } p \in P, \end{cases}$$

where $k = \displaystyle\prod_{p\in\mathcal{P}} p^{a_p} \cdot \prod_{p\in\mathcal{Q}} p^{a_p}$ with $P \cap Q = \varnothing$ is the prime decomposition of k.

Proof. Let $J \subseteq P \cup Q$ and $I \subseteq Q$. Then

$$\begin{aligned} \mathcal{O}_{T^k}(n) &= \sum_{d|\boldsymbol{p}_J{}^{\boldsymbol{a}_J}} (k/d)\, \mathcal{O}_T(kn/d) \\ &\qquad\qquad \text{(where } p \nmid n \text{ for } p \in J, p|n \text{ for } p \in (P \cup Q) \setminus J) \\ &= \sum_{d|\prod_{p\nmid n} p^{a_p}} \prod_{p\in Q} (p^{a_p}/d)\, \mathcal{O}_T(p^{a_p} n/d), \end{aligned}$$

showing the second case. If $p \nmid n$ for $p \in I$, $p|n$ for $p \in Q \setminus I$ and $p \nmid n$ for any $p \in P$

then $\mathcal{O}_T(p^{a_p}n/d) = 1$, so

$$\begin{aligned}
\mathcal{O}_{T^k}(n) &= \sum_{d \mid \prod_{p\nmid n} p^{a_p}} \prod_{p\in Q} (p^{a_p}/d) \\
&= \left(\prod_{p\in Q} p^{a_p}\right) \sum_{d \mid \prod_{p\nmid n} p^{a_p}} 1/d \\
&= \prod_{p\in Q} p^{a_p} \left(\sum_{d \mid \prod_{p\nmid n} p^{a_p}} d/\textstyle\prod_{p\nmid n} p^{a_p} \right) \\
&= \sum_{d \mid \prod_{p\nmid n} p^{a_p}} \left(d/\textstyle\prod_{p\nmid n} p^{a_p}\right) \displaystyle\prod_{p\nmid n} p^{a_p} \prod_{p\mid n} p^{a_p} \\
&= \prod_{p\mid n} p^{a_p} \left(\prod_{p\nmid n} \sum_{d\mid p^{a_p}} d \right) \\
&= \prod_{\substack{p\in Q \\ p\mid n}} p^{a_p} \cdot \prod_{\substack{p\in Q \\ p\nmid n}} \sigma(p^{a_p}),
\end{aligned}$$

showing the first case. □

It is clear from Lemma 1 that if $\mathcal{O}_S = s_P$ and $\mathcal{O}_T = s_{P^c}$ then $\mathcal{O}_{T\times S} = \zeta$, so the sequence ζ factorizes in uncountably many ways into the orbit count of two combinatorially distinct systems. Indeed, these sequences provide the *only* combinatorial factorization of ζ into the orbit count of the product of two systems.

Proposition 13. *If S and T are maps with $\mathcal{O}_{S\times T} = \zeta$, then there is a set $P \subseteq \mathbb{P}$ for which $\mathcal{O}_T = s_p$ and $\mathcal{O}_S = s_{P^c}$.*

Proof. It is clear from Lemma 1 that $\mathcal{O}_S$ and $\mathcal{O}_T$ take values in $\{0, 1\}$, and moreover that $\{\mathcal{O}_S(p), \mathcal{O}_T(p)\} = \{0, 1\}$ for $p \in \mathbb{P}$. Fix a pair of maps satisfying the hypothesis, and let $P = \{p \in \mathbb{P} \mid \mathcal{O}_T(p) = 0\}$, so that $P^c = \{p \in \mathbb{P} \mid \mathcal{O}_S(p) = 0\}$.

Assume that $p|n$ for some $p \in P$, so that $\mathcal{O}_T(p) = 0$ and $\mathcal{O}_S(p) = 1$. If $\mathcal{O}_T(n) = 1$, then

$$1 = \sum_{\operatorname{lcm}(d,d')=n} \gcd(d, d')\, \mathcal{O}_T(d)\, \mathcal{O}_S(d') \geqslant p,$$

which is impossible, so $\mathcal{O}_T(n) = 0$. By symmetry, if $p|n$ for some $p \in P^c$, then $\mathcal{O}_S(n) = 0$.

Now if $n \neq 1$ is not divisible by any $p \in P$, then

$$\begin{aligned}
1 &= \sum_{\operatorname{lcm}(d,d')=n} \gcd(d, d')\, \mathcal{O}_T(d)\, \mathcal{O}_S(d') \\
&= \sum_{\substack{\operatorname{lcm}(d,d')=n, \\ \mathcal{D}(d)\subseteq P^c, \mathcal{D}(d')\subseteq P}} \gcd(d, d')\, \mathcal{O}_T(d)\, \mathcal{O}_S(d') \\
&= \mathcal{O}_T(n) \sum_{d'\mid n} d'\, \mathcal{O}_S(d'),
\end{aligned}$$

so $\mathcal{O}_T(n) = 1$. It follows that $\mathcal{O}_T = s_P$, and by symmetry $\mathcal{O}_S = s_{P^c}$ as required. $\square$

Products of the systems in Example 11 enjoy remarkable combinatorial properties, illustrated in the examples below. The calculations in the examples all follow from the next lemma. Let $S \subseteq \mathbb{P}$ be a set of primes. Write $\lfloor n \rfloor_S$ for the S-part of n, that is

$$\lfloor n \rfloor_S = \prod_{p \in S} |n|_p^{-1}.$$

Write $\gcd(n, S)$ as shorthand for $\gcd(n, \prod_{p \in S} p)$.

Lemma 14. *For any set $S \subseteq \mathbb{P}$,*

$$\sum_{n \geqslant 1} \frac{\lfloor n \rfloor_S}{n^s} = \zeta(s) \prod_{p \in S} \left(\frac{p^s - 1}{p^s - p} \right). \tag{6}$$

Proof. Recall that

$$\sum_{n \geqslant 1, \gcd(n,S)=1} \frac{1}{n^s} = \prod_{p \in S} \left(1 - p^{-s}\right) \zeta(s). \tag{7}$$

Write $S = \{p_1, \dots\}$. Then, writing $n = p_1^{a_1} \cdots p_r^{a_r} m$ with $\gcd(m, S) = 1$,

$$\begin{aligned} \sum_{n \geqslant 1} \frac{\lfloor n \rfloor_S}{n^s} &= \sum_{\gcd(m,S)=1} \sum_{a_1 \geqslant 0} \cdots \sum_{a_r \geqslant 0} \frac{p_1^{a_1} \cdots p_r^{a_r}}{(p_1^{a_1} \cdots p_r^{a_r})^s m^s} \\ &= \sum_{\gcd(m,S)=1} \frac{1}{m^s} \sum_{a_2 \geqslant 0} \cdots \sum_{a_r \geqslant 0} \frac{p_2^{a_2} \cdots p_r^{a_r}}{(p_2^{a_2} \cdots p_r^{a_r})^s} \sum_{a_1 \geqslant 0} \frac{1}{(p_1^{a_1})^{s-1}} \\ &= \sum_{\gcd(m,S)=1} \frac{1}{m^s} \sum_{a_2 \geqslant 0} \cdots \sum_{a_r \geqslant 0} \frac{p_2^{a_2} \cdots p_r^{a_r}}{(p_2^{a_2} \cdots p_r^{a_r})^s} \left(\frac{1}{1 - p_1^{-(s-1)}} \right), \end{aligned}$$

so by induction we have

$$\begin{aligned} \sum_{n \geqslant 1} \frac{\lfloor n \rfloor_S}{n^s} &= \sum_{\gcd(m,S)=1} \frac{1}{m^s} \prod_{p \in S} \left(\frac{1}{1 - p^{-(s-1)}} \right) \\ &= \zeta(s) \prod_{p \in S} \left(1 - p^{-s}\right) \frac{1}{1 - p^{-(s-1)}} \end{aligned}$$

since I can write $\sum_{\gcd(m,S)=1}$ as $(1 - p_1^{-s}) \sum_{\gcd(m, S \setminus \{p_1\})=1}$ as in (7), as required. $\square$

Notice that (6) interpolates between $\zeta(s)$ (when $S = \varnothing$) and $\zeta(s-1)$ (when $S = \mathbb{P}$). Of course how the abscissa of convergence moves from 1 at $S = \varnothing$ to 2 at $S = \mathbb{P}$ is rather subtle. A similar argument gives the following.

Lemma 15. *Let S be a set of primes. Then*

$$a_{S,n} = \prod_{p \in S} \left(\frac{1}{p-1} \right) \left(\frac{p+1}{|n|_p} - 2 \right)$$

for all $n \geqslant 1$ *if and only if*

$$\sum_{n\geqslant 1} \frac{a_{S,n}}{n^s} = \zeta(s) \prod_{p\in S} \frac{p^s+1}{p^s-p}.$$

Proof. First, if $q \notin S$ and $\gcd(m,q)=1$, then

$$a_{S,mq^k} = a_{S,m} \tag{8}$$

for all $k \geqslant 0$, since $|q|_p = 1$ for all $p \in S$. Second, we have an extension of the identity (7): if $q \notin S$, then

$$\sum_{m\geqslant 1,\gcd(m,q)=1} \frac{a_{S,m}}{m^s} = \left(1-q^{-s}\right) \sum_{n\geqslant 1} \frac{a_{S,n}}{n^s} \tag{9}$$

by the usual argument and (8). We now prove the lemma by induction on the cardinality of S. Assume we have the lemma for some set S, and assume that $q \notin S$. Write $S' = S \cup \{q\}$, and notice that

$$\begin{aligned}
\sum_{n\geqslant 1} \frac{a_{S',n}}{n^s} &= \sum_{n\geqslant 1} \frac{\frac{1}{q-1}\left(\frac{q+1}{|n|_q}-2\right) a_{S,n}}{n^s} \\
&= \left(\frac{q+1}{q-1}\right) \sum_{n\geqslant 1} \frac{a_{S,n}/|n|_q}{n^s} - \frac{2}{q-1}\sum_{n\geqslant 1} \frac{a_{S,n}}{n^s} \\
&= \left(\frac{q+1}{q-1}\right) \sum_{m\geqslant 1,\gcd(m,q)=1} \sum_{k\geqslant 0} \frac{q^k a_{S,mq^k}}{(q^k)^s m^s} - \frac{2}{q-1}\sum_{n\geqslant 1} \frac{a_{S,n}}{n^s} \\
&= \left(\frac{q+1}{q-1}\right) \sum_{m\geqslant 1,\gcd(m,q)=1} \frac{a_{S,m}}{m^s} \sum_{k\geqslant 0} \frac{1}{(q^k)^{s-1}} - \frac{2}{q-1}\sum_{n\geqslant 1} \frac{a_{S,n}}{n^s} \\
&\qquad \text{by (8)} \\
&= \frac{q+1}{q-1}\left(1-q^{-s}\right) \frac{1}{1-q^{1-s}} \sum_{n\geqslant 1} \frac{a_{S,n}}{n^s} - \frac{2}{q-1}\sum_{n\geqslant 1} \frac{a_{S,n}}{n^s} \\
&\qquad \text{by (9).}
\end{aligned}$$

So if we write $\phi(s) = \sum_{n\geqslant 1} \frac{a_{S,n}}{n^s}$, then

$$\begin{aligned}
\sum_{n\geqslant 1} \frac{a_{S',n}}{n^s} &= \phi(s)\left(\left(\frac{q+1}{q-1}\right)\left(1-q^{-s}\right)\left(\frac{1}{1-q^{1-s}}\right) - \frac{2}{q-1}\right) \\
&= \phi(s)\left(\frac{q^s+1}{q^s-q}\right),
\end{aligned}$$

showing the lemma for the set S'. All that remains is to check the case of a singleton $S = \{p\}$, which is easy. $\square$

Example 16. If $\mathcal{O}_S = s_P$ and $\mathcal{O}_T = \zeta$ then, by Lemmas 1 and 15,

$$\mathsf{d}_{S\times T}(s) = \prod_{p\in P}\left(\tfrac{1-p^{1-s}}{1+p^{-s}}\right) \tfrac{\zeta^2(s)\zeta(s-1)}{\zeta(2s)} = \zeta(s)\prod_{p\notin P} \tfrac{1+p^{-s}}{1-p^{1-s}}.$$

Example 17. Taking $P = \{2\}$, $\mathcal{O}_S = s_P$, $\mathcal{O}_T = \zeta$ again, we have

$$\mathsf{d}_{S\times T}(s) = \left(\tfrac{1-2^{1-s}}{1+2^{-s}}\right)\tfrac{\zeta^2(s)\zeta(s-1)}{\zeta(2s)}.$$

The sequence $\mathcal{O}_{S\times T} = (1, 1, 5, 1, 7, 5, 9, 1, 17, \ldots)$ is A035109, which arises in work of Baake and Moody [2, Eq. (5.10)], where it is shown to count the elements of $\mathbb{Z}^3$ with m distinct colours so that one colour occupies a similarity sublattice of index m while the other colours code the cosets.

Example 18. Let $\mathsf{d}_S(s) = \zeta(s-a)$ and $\mathsf{d}_T(s) = \zeta(s-b)$. Then a calculation shows that

$$\mathcal{O}_{S\times T}(n) = \frac{1}{n}\sum_{d|n}\mu(n/d)\sigma_{a+1}(d)\sigma_{b+1}(d)$$

(the details are in [16]), so Ramanujan's formula gives

$$\mathsf{d}_{S\times T}(s) = \frac{\zeta(s-a)\zeta(s-b)\zeta(s-a-b-1)}{\zeta(2s-a-b)}.$$

These examples give an indication of how analytic properties of d_S and d_T relate to those of d_{T^k} and $\mathsf{d}_{S\times T}$, and this is pursued in [16].

4 Counting in orbit monoids

Counting in $\mathcal{G}_T$ involves counting additive partitions, with two changes: some parts may be missing (that is, a "restricted" additive partition) and some parts may come in several versions. Thus the sequence $\mathcal{G}_T$ is the *Euler transform* of the sequence $\mathcal{O}_T$ (see Sloane and Plouffe [18, pp. 20–22]).

Lemma 19. *For any map $T \in \mathfrak{M}$,*

$$1 + \sum_{n=1}^{\infty}\mathcal{G}_T(n)s^n = \prod_{i=1}^{\infty}\left(1-s^i\right)^{-\mathcal{O}_T(i)} = \zeta_T(s) \tag{10}$$

and

$$n\,\mathcal{G}_T(n) - \mathcal{F}_T(n) - \sum_{k=1}^{n-1}\mathcal{F}_T(k)\,\mathcal{G}_T(n-k) = 0 \tag{11}$$

for all $n \geqslant 1$.

Proof. The first equality in (10) is clear, since the coefficient of s^n in the right-hand side counts partitions of n into parts i with multiplicity $\mathcal{O}_T(i)$; the second equality is the usual Euler product expansion of the dynamical zeta function. The recurrence relation (11) may be seen by expanding the zeta function as

$$1 + \sum_{n=1}^{\infty}\mathcal{G}_T(n)s^n = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=1}^{\infty}\mathcal{F}_T(n)\frac{s^n}{n}\right)^k$$

and verifying that (11) satisfies this relation. □

Thus the categories $\mathfrak{F}$, $\mathfrak{O}$ and $\mathfrak{G}$ are related as follows,

$$\begin{tikzcd} & \mathfrak{F} \arrow[dl, dotted, leftrightarrow, "\text{generating function}"'] \arrow[dr, dotted, leftrightarrow, "\text{Möbius convolution}"] & \\ \mathfrak{G} \arrow[rr, dotted, leftrightarrow, "\text{Euler}"'] & & \mathfrak{O} \end{tikzcd}$$

and we indicate in this section how various growth properties of any one sequence relate to growth properties of the others, mostly by pointing out how these quantities arise in abstract analytic number theory. These results extend those of Puri and Ward [17] concerning relations between growth in $\mathcal{F}_T$ and in $\mathcal{O}_T$, and some related asymptotic results are discussed in the paper of Baake and Neumärker [3]. Before listing these, we discuss some of the statements. It is often possible to estimate $\mathcal{F}_T(n)$ (or even to have a closed formula for $\mathcal{F}_T(n)$), and a reasonable combinatorial replacement for "hyperbolicity" is the assumption (12) of a uniform exponential growth rate in $\mathcal{F}_T$, where h plays the role of topological entropy. A similar assumption often used in abstract analytic number theory is (16) (hypotheses of this shape are often called "Axiom A" or "Axiom A$^\sharp$" in number theory). The assumption (16) is weaker than (12): it is pointed out in [10] that there are arithmetic semigroups with $\mathcal{G}_T(n)e^{-hn} - C_3$ converging to zero exponentially fast for which $n\,\mathcal{F}_T(n)e^{-hn}$ does not converge. The hypothesis is weakened further in (17), which is a permissive form of exponential growth rate assumption. The hypothesis (12) fails for many non-hyperbolic systems. If T is a quasihyperbolic toral automorphism or a non-expansive S-integer map with S finite (see [4] or Example 27) then (12) fails since the ratio $\mathcal{F}_T(n+1)/\mathcal{F}_T(n)$ does not converge as $n \to \infty$. In both cases the conclusions of Theorem 20[1] also fail (see Noorani [15] and Waddington [20] for the case of a quasihyperbolic toral automorphism and [6] for the case of S-integer systems with S finite). As pointed out by Lindqvist and Peetre [14], Meissel considered the sum $\sum_p \frac{1}{p(\log p)^a}$ in 1866, and the dynamical analogue of Meissel's theorem is given in Theorem 20[4] below. In Theorem 20, [1] is proved here and [2]–[4] are simply interpretations for orbit–counting of well-known results in number theory.

Theorem 20. *Let T be a map in $\mathfrak{M}$.*
[1] *Assume that there are constants $C_1 > 0, h > 0$ and $h' < h$ with*

$$\mathcal{F}_T(n) = C_1 e^{hn} + \mathrm{O}(e^{h'n}). \tag{12}$$

Then

$$\mathcal{O}_T(n) = \frac{C_1}{n} e^{hn} + \mathrm{O}\left(e^{h'n}/n\right), \tag{13}$$

$$\pi_T(N) = \frac{e^{h(N+1)}}{e^h - 1} + \mathrm{O}\left(e^{hN}/N^{3/2}\right), \tag{14}$$

and

$$\mathcal{M}_T(N) = \sum_{n=1}^{N} \frac{\mathcal{O}_T(n)}{e^{hn}} = C_1 \sum_{n=1}^{N} \frac{1}{n} + C_2 + \mathrm{O}\left(e^{h''N}\right) \tag{15}$$

where $h'' = \max\{h' - h, -h/2\}$, for some constant C_2.

[2] *Assume that there are constants $C_3 > 0, h > 0$ and $h' < h$ with*

$$\mathcal{G}_T(n) = C_3 e^{hn} + \mathrm{O}(e^{h'n}). \tag{16}$$

Then, for any $\alpha > 1$,

$$\pi_T(N) = C_4 \frac{e^{hN}}{N} + \mathrm{O}\left(e^{hN}/N^{\alpha}\right)$$

and (equivalently)

$$\mathcal{F}_T(N) = e^{hN} + \mathrm{O}\left(e^{hN}/N^{\alpha-1}\right).$$

[3] *Assume that there are constants $C_5 > 0$, $h > 0$ with*

$$\mathcal{G}_T(n) = \left(C_5 + r(n)\right) e^{hn}, \tag{17}$$

where $\sum_{n=0}^{\infty} \sup_{k \geqslant n} |r(k)| < \infty$. Then

$$\sum_{n=1}^{N} \frac{n\,\mathcal{O}_T(n)}{e^{hn}} = N + \mathrm{O}(1); \tag{18}$$

$$\sum_{n=1}^{N} \frac{\mathcal{O}_T(n)}{e^{hn}} = \log N + C_6 + \mathrm{O}(1/N); \tag{19}$$

$$\prod_{n=1}^{N} \left(1 - e^{-hn}\right)^{\mathcal{O}_T(n)} = \frac{C_7}{N} + \mathrm{O}(1/N^2);$$

and if in addition $\zeta_T(-e^h) \neq 0$ then, for any $\lambda > 1$,

$$\mathcal{O}_T(n) = \frac{e^{hn}}{n} + \mathrm{O}\left(e^{hn}/n^{\lambda}\right) \tag{20}$$

as $n \to \infty$.

[4] *Assume that there are constants $C_8 > 0$, $h > 0$ with*

$$\frac{\mathcal{G}_T(n)}{e^{hn}} = C_8 + \mathrm{O}\left(1/\log(n)^{2+\varepsilon}\right) \text{ as } n \to \infty. \tag{21}$$

Then

$$\sum_{k=1}^{\infty} \frac{\mathcal{O}_T(k)}{e^{hk} k^a} = \frac{1}{a} + C_9 + \mathrm{O}(a)$$

as $a \to 0$.

Proof. [1] The estimate (13) is easy to see; it is implicit in [17] and [21] for example. By (1), we have

$$\mathcal{F}_T(n) \geqslant n\,\mathcal{O}_T(n) \geqslant \mathcal{F}_T(n) - \sum_{d|n, d<n} \mathcal{F}_T(d),$$

so

$$C_1e^{hn}+\mathrm{O}(e^{h'n})\geqslant n\,\mathcal{O}_T(n)\geqslant C_1e^{hn}-n\left(C_1e^{hn/2}+\mathrm{O}(e^{h'n/2})\right)$$

which gives (13). The proofs of (14) – a dynamical prime number theorem – and (15) – a dynamical Mertens' theorem – use similar arguments to those in [5] where a more delicate non-hyperbolic problem is studied. Turning to (14), notice that (13) implies that

$$\left|\pi_T(N)-\sum_{n=1}^{N}\frac{C_1}{n}e^{hn}\right|=\left|\sum_{n=1}^{N}\mathrm{O}\left(e^{h'n}/n\right)\right|=\mathrm{O}\left(e^{h'N}\right).$$

Now

$$\left|\sum_{n=1}^{N}\frac{C_1}{n}e^{hn}-\sum_{n=N-k(N)}^{N}\frac{C_1}{n}e^{hn}\right|\leqslant\sum_{n=1}^{N-k(N)-1}C_1e^{hn}=\mathrm{O}\left(e^{h(N-k(N))}\right)$$

where $k(N)=\lfloor N^{1/4}\rfloor$. Thus

$$\begin{aligned}\sum_{n=N-k(N)}^{N}\frac{C_1}{n}e^{hn}&=\frac{C_1e^{hN}}{N}\sum_{r=0}^{k(N)}e^{-hr}\left(1-\tfrac{r}{N}\right)^{-1}\\&=\frac{C_1e^{hN}}{N}\left[\frac{e^h}{e^h-1}-\mathrm{O}\left(e^{-hk(N)}\right)+\mathrm{O}\left(\sum_{r=0}^{k(N)}\tfrac{r}{N}\right)\right]\\&=\frac{C_1e^{h(N+1)}}{e^h-1}+\mathrm{O}\left(\frac{e^{hN}}{N^2}\sum_{r=0}^{k(N)}r\right)\\&=\frac{C_1e^{h(N+1)}}{e^h-1}+\mathrm{O}\left(e^{hN}/N^{3/2}\right).\end{aligned}$$

Finally, notice that

$$\frac{\mathcal{F}_T(n)}{ne^{hn}}-\frac{C_1}{n}=\frac{1}{n}\mathrm{O}\left(e^{(h-h')n}\right),\tag{22}$$

so

$$\sum_{n=1}^{N}\frac{\mathcal{O}_T(n)}{e^{hn}}-C_1\sum_{n=1}^{N}\frac{1}{n}=\sum_{n=1}^{N}\frac{1}{n}\left(\frac{\mathcal{F}_T(n)}{e^{hn}}-C_1\right)+\sum\frac{1}{n}\sum_{d|n,d<n}\mu(n/d)\frac{\mathcal{F}_T(d)}{e^{hn}}.\tag{23}$$

The bound (22) shows that the two terms on the right-hand side of (23) converge, giving (15) without error term. To see the error term, notice that

$$\left|\sum_{n=N+1}^{\infty}\frac{1}{n}\left(\frac{\mathcal{F}_T(n)}{e^{hn}}-C_1\right)\right|\leqslant\sum_{n=N+1}^{\infty}\frac{1}{n}\mathrm{O}\left(e^{(h'-h)n}\right)=\mathrm{O}\left(e^{(h'-h)n}\right)$$

and

$$\left|\sum_{n=N+1}^{\infty}\frac{1}{n}\sum_{d|n,d<n}\mu(n/d)\frac{\mathcal{F}_T(d)}{e^{hn}}\right|\leqslant\sum_{n=N+1}^{\infty}\left(\frac{e^{hn/2}}{e^{hn}}+\mathrm{O}\left(e^{-hn/2}\right)\right)=\mathrm{O}\left(e^{-hN/2}\right).$$

[2] These are standard results from Knopfmacher [12, Ch. 8].
[3] The results (18)–(20) are shown in [12] to be consequences of Knopfmacher's Axiom $A^{\#}$ in (17); (20) is due to Indlekofer [9].
[4] This is shown by Wehmeier [22]. □

Standard estimates for the harmonic series allow (15) to be simplified; for example under the hypothesis of Theorem 20[1] we have

$$\mathcal{M}_T(N) = C_1 \log N + \mathrm{O}(1/N).$$

Notice that the statements (15) and (19) are versions of what is usually called a dynamical Mertens' theorem, though they may equally be seen in more elementary terms as consequences of $\frac{\mathcal{O}_T(n)}{e^{hn}}$ being close to $\frac{1}{n}$ and the Euler–Maclaurin summation formula. Theorem 20[1] has the following kind of consequence: If T is a hyperbolic toral automorphism or mixing shift of finite type with entropy h, then

$$\sum_{|\tau|\leqslant n} \frac{1}{e^{h|\tau|}} = \log n + C_{10} + \mathrm{O}(1/n)$$

and

$$\prod_{\tau} \left(1 - e^{-h|\tau|}\right)^{-1} = \frac{C_{11}}{n} + \mathrm{O}(1/n^2)$$

where τ runs over the closed orbits of T. A stronger hypothesis than (21),

$$\zeta_T(z) \sim \frac{C_{12}}{1 - e^h z} \tag{24}$$

as $z \to e^{-h}$ with $0 < z < e^{-h}$, is considered in [11], where it is shown to give

$$\sum_{n=1}^{N} \frac{\mathcal{O}_T(n)}{e^{hn}} = \sum_{n=1}^{N} \frac{1}{n} + C_{13} + \mathrm{o}(1);$$

hence

$$\prod_{n=1}^{N} \left(1 - e^{hn}\right)^{-\mathcal{O}_T(n)} = C_{12} e^{\gamma} N + \mathrm{o}(N)$$

and

$$\sum_{n=1}^{N} \frac{\mathcal{F}_T(n)}{e^{hn}} = N + \mathrm{o}(N). \tag{25}$$

As pointed out in [11], the hypothesis (24) does not permit any smaller error in (25) for the following reason. For any sequence (w_n) of non-negative integers with $\sum_{n=1}^{\infty} \frac{w_n}{n} < \infty$, choose $(a_n)$with $0 \leqslant a_n < n$ and $a_n \equiv 2^n + w_n 2^n \pmod{n}$ and consider a map T with

$$\mathcal{O}_T(n) = 1 + (2^n + w_n 2^n - a_n)/n.$$

This gives property (24), and a calculation shows that the error term in (25) is as big as $\mathrm{O}\left(\sum_{n=1}^{N} w_n\right)$.

For a map $T \in \mathfrak{M}$ with infinitely many orbits, it is clear that $\mathcal{G}_T$ is isomorphic to $\sum_{\mathbb{N}} \mathbb{N}_0$ as a semigroup. The information about how many orbits T has of each length is contained in the weight function ∂, and in each example we compute the size of the level set $\mathcal{G}_T(n)$ for each $n \geqslant 1$. If the sequence $\mathcal{F}_T$ is a linear recurrence sequence, then the relation (11) shows that $\mathcal{G}_T$ is also a linear recurrence sequence. Example 24 shows that $\mathcal{G}_T$ may satisfy a relation of smaller degree, while Example 23 shows that $\mathcal{G}_T$ may be of higher degree.

Example 21. Let $T : X \to X$ be the golden mean shift, so that $\mathcal{F}_T = (1, 3, 4, 7, \dots)$ is the Lucas sequence A000032. By (2), $\mathcal{O}_T$ is A006206. Write τ_i for the unique orbit of length i for $1 \leqslant i \leqslant 4$, and write $\tau_5^{(1)}$, $\tau_5^{(2)}$ for the two orbits of length 5. Then the elements in $\mathcal{G}_T$ with weight 5 are

$$\tau_5^{(1)}, \tau_5^{(2)}, \tau_4 + \tau_1, \tau_3 + \tau_2, \tau_3 + 2\tau_1, \tau_2 + 3\tau_1, 2\tau_2 + \tau_1, 5\tau_1,$$

so $\mathcal{G}_T(5) = 8$. The relation (10) shows that $\mathcal{G}_T(n)$ is the $(n+1)$st Fibonacci number, so $\mathcal{G}_T$ is A000045.

Example 22. Let $T \in \mathfrak{M}$ have $\mathsf{d}_T(s) = \zeta(s)$. Then there is a one-to-one correspondence between elements of $\mathcal{G}_T$ and partitions of natural numbers, so $\mathcal{G}_T$ is the classical partition function A000041.

Example 23. Let $T : X \to X$ be the full shift on a symbols, so that $\zeta_T(s) = \frac{1}{1-as}$; (10) shows that $\mathcal{G}_T(n) = a^n - a^{n-1}$ for all $n \geqslant 1$. Thus $\mathcal{F}_T$ in this case is a linear recurrence of degree 1 while $\mathcal{G}_T$ is a linear recurrence of degree 2.

Example 24. Let $X = \mathbb{Z}[\frac{1}{6}]$ and let $T : X \to X$ be the map dual to $r \mapsto \frac{2}{3}r$ on $\mathbb{Z}[\frac{1}{6}]$. Then $\mathcal{F}_T(n) = 3^n - 2^n$ is A001047 (a linear recurrence of degree 2) by [13] so $\mathcal{G}_T(n) = 3^{n-1}$ by (10), and $\mathcal{G}_T$ is A000244 (a linear recurrence of degree 1). More generally, if $b > a > 0$ are coprime integers, then the dual map to $r \mapsto \frac{a}{b}r$ on $\mathbb{Z}[\frac{a}{b}]$ has

$$\zeta_T(s) = \frac{1 - as}{1 - bs},$$

so $\mathcal{G}_T(n) = (b^n - ab^{n-1})$ for all $n \geqslant 1$.

Example 25. The quadratic map T from Example 8 has a particularly simple monoid: $\mathcal{G}_T$ is the binary partition function A018819 (the number of partitions of n into powers of 2; by Sloane and Sellers [19] this is also the number of "non-squashing" partitions of n).

Example 26. An example similar in growth rate to Example 25 is studied in [6]: the map dual to $x \mapsto 2x$ on the localization $\mathbb{Z}_{(3)}$ at the prime 3 has $\mathcal{F}_T(n) = |2^n - 1|_3^{-1}$, so

$$\mathcal{O}_T(n) = \begin{cases} 1, & \text{if } n = 1 \text{ or } 2 \cdot 3^k, k \geqslant 1; \\ 0, & \text{if not;} \end{cases}$$

and therefore $\mathcal{G}_T(2n+1)$ is the number of partitions of $6n+3$ into powers of 3 for $n \geqslant 0$, and $\mathcal{G}_T(2n) = \mathcal{G}_T(2n+1)$ for $n \geqslant 1$.

Example 27. An example of a map that is not hyperbolic but still has exponentially many periodic orbits is given by the simplest non-trivial S-integer map dual to $x \mapsto 2x$ on $\mathbb{Z}[1/3]$. By [4] this has $\mathcal{F}_T(n) = (2^n - 1)|2^n - 1|_3$, and a calculation shows that $\mathcal{O}_T$ is A060480, and thus $\mathcal{G}_T = (1, 1, 3, 4, 10, 13, 33, 56, 10, \ldots)$.

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