Journal of Integer Sequences, Vol. 12 (2009),
Article 09.5.4

A Note on a One-Parameter Family of Catalan-Like Numbers

Paul Barry
School of Science
Waterford Institute of Technology
Ireland
pbarry@wit.ie

Abstract

We study a family of sequences of Catalan-like numbers based on the series reversion process. Properties of these sequences are derived, including continued fraction expansions, associated orthogonal polynomials and associated Aigner matrices, which turn out to be Riordan arrays.

1 Introduction

The purpose of this note is to explore some properties of the family of sequences obtained by reverting the expression

$$\frac{x(1+rx)}{1+2rx+r(r+1)x^2}$$

where r is an integer parameter. The analysis involves elements of the theory of Riordan arrays [15], orthogonal polynomials [5, 10], continued fractions [18], the Deleham construction (see A084938) and Hankel transforms [13]. The overall context of this note is that of "Catalan-like" numbers, a notion defined and developed by Martin Aigner [1, 2]. See also [8, 9, 20]. In the sequel, $[x^n]$ denotes the operator that extracts the coefficient of x^n in a power series, Rev denotes the operation of reverting a sequence (thus $\bar{f}(x) = \mathrm{Rev} f(x)$ satisfies $f(\bar{f}(x)) = x$), $[P]$ is the Iverson operator [11] equal to 0 if P is false, and 1 if P is true, and

$$c(x) = \frac{1-\sqrt{1-4x}}{2x} = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}}}$$

is the g.f. of the Catalan numbers A000108. In addition, (g, f) will denote a Riordan array whose k-th column has generating function $g(x)f(x)^k$. In the next section, we shall provide an introduction to the Riordan group. We recall that a number sequence a_n is "Catalan-like" if none of the Hankel determinants $|a_{i+j}|_{i,j=0}^n$ is zero, while a lower-triangular matrix $(a_{n,k})$ is called an *Aigner matrix* if there exist two sequences s_n and t_n such that

$$a_{0,0} = 1, \quad a_{0,k} = 0 \quad (k > 0)$$

and

$$a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1} a_{n-1,k+1} \quad (n, k \geq 1).$$

The (integer) *Hankel transform* of the sequence a_n is the sequence h_n with general term

$$h_n = |a_{i+j}|_{i,j=0}^n.$$

2 The Riordan group

The *Riordan group* [15], [17], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \dots$ and $f(x) = f_1 x + f_2 x^2 + \dots$ where $f_1 \neq 0$ [17]. The associated matrix is the matrix whose i-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair g, f is denoted by (g, f) or $\mathcal{R}(g, f)$. The group law is then given by

$$(g, f) * (h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of (g, f) is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence a_n, is called the *sequence array* of the sequence a_n. Its general term is a_{n-k}. Such arrays are also called *Appell* arrays as they form the elements of the Appell subgroup.

If $\mathbf{M}$ is the matrix (g, f), and $\mathbf{a} = (a_0, a_1, \dots)'$ is an integer sequence with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix (g, f) can thus be considered to act on the ring of integer sequences $\mathbf{Z}^{\mathbf{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbf{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \longrightarrow (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

Example 1. The binomial matrix $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k} m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

The row sums of the matrix (g, f) have generating function

$$(g, f) \cdot \frac{1}{1-x} = \frac{g(x)}{1-f(x)}$$

while the diagonal sums of (g, f) (sums of left-to-right diagonals in the North East direction) have generating function $g(x)/(1 - xf(x))$. These coincide with the row sums of the "generalized" Riordan array (g, xf).i

The bi-variate generating function of the Riordan array (g, f) is given by $\frac{g(x)}{1-yf(x)}$.

3 The sequences $a_n(r)$

The reversion of the function
$$\frac{x(1+rx)}{1+2rx+r(r+1)x}$$
is obtained by solving the equation
$$\frac{u(1+ru)}{1+2ru+r(r+1)u} = x$$
for the unknown u. We obtain
$$u = \frac{\sqrt{1-4rx^2}+2rx-1}{2r(1-(r+1)x)}.$$
We now define
$$\begin{aligned}
a_n(r) &= [x^{n+1}]\mathrm{Rev}\frac{x(1+rx)}{1+2rx+r(r+1)x^2} \\
&= [x^{n+1}]\frac{\sqrt{1-4rx^2}+2rx-1}{2r(1-(r+1)x)} \\
&= [x^n]\frac{\sqrt{1-4rx^2}+2rx-1}{2xr(1-(r+1)x)} \\
&= [x^n]\frac{c(rx^2)}{1-rxc(rx^2)} \\
&= [x^n](c(rx^2), rxc(rx^2)) \cdot \frac{1}{1-x} \\
&= [x^n](c(rx^2), xc(rx^2)) \cdot \frac{1}{1-rx} \\
&= \sum_{k=0}^{n} \frac{k+1}{n+k+2}\binom{n}{\frac{n-k}{2}}(1+(-1)^{n-k})r^{\frac{n-k}{2}}r^k \\
&= \sum_{k=0}^{n} \frac{k+1}{n+k+2}\binom{n}{\frac{n-k}{2}}(1+(-1)^{n-k})r^{\frac{n+k}{2}} \\
&= \sum_{k=0}^{n} [n \le 2k]\binom{n}{k}\frac{2k-n+1}{k+1}r^k.
\end{aligned}$$
The final equalities above follow from the fact that the general term of the Riordan array
$$(c(rx^2), xc(rx^2))$$

is given by

$$\frac{k+1}{n+k+2}\binom{n}{\frac{n-k}{2}}(1+(-1)^{n-k})r^{\frac{n-k}{2}}.$$

A short table of these sequences is given below.

r	A-number	Reversion of
1	A001405	$\frac{x(1+x)}{1+2x+2x^2}$
2	A151281	$\frac{x(1+2x)}{1+4x+6x^2}$
3	A151162	$\frac{x(1+3x)}{1+6x+12x^2}$
4	A151254	$\frac{x(1+4x)}{1+8x+20x^2}$
5	A156195	$\frac{x(1+5x)}{1+10x+30x^2}$
6	A156361	$\frac{x(1+6x)}{1+12x+42x^2}$

For example, $a_n(1)$ is the sequence of central binomial numbers $\binom{n}{\lfloor\frac{n}{2}\rfloor}$, while $a_n(3)$ counts the number of walks within $\mathbf{N}^3$ (the first octant of $\mathbf{Z}^3$) starting at $(0,0,0)$ and consisting of n steps taken from $\{(-1,0,0),(1,0,0),(1,0,1),(1,1,0)\}$ [3].

4 Continued fractions

We have

$$\begin{aligned}
\frac{c(rx^2)}{1-rxc(rx^2)} &= \frac{1}{\frac{1}{c(rx^2)}-rx}\\
&= \frac{1}{1-rx+(\frac{1}{c(rx^2)}-1)}\\
&= \frac{1}{1-rx+\frac{1-c(rx^2)}{c(rx^2)}}\\
&= \frac{1}{1-rx-rx^2c(rx^2)}\\
&= \cfrac{1}{1-rx-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cdots}}}}.
\end{aligned}$$

Thus the generating function of $a_n(r)$ is given by the continued fraction above. An immediate consequence of this is that $a_n(r)$ has Hankel transform $r^{\binom{n+1}{2}}$ [12, 18]. This proves that these numbers are "Catalan-like" ($r \neq 0$). We note that the above implies that the bi-variate generating function of the Riordan array

$$(c(rx^2), xc(rx^2))$$

is given by

$$\cfrac{1}{1-xy-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cdots}}}}$$

while that of the generalized Riordan array

$$(c(rx^2), rxc(rx^2))$$

is given by

$$\cfrac{1}{1-rxy-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cfrac{rx^2}{1-\cdots}}}}.$$

There is another link to continued fractions, via the *Deleham construction.* For the purposes of this note, we define this as follows. Given two sequences r_n and s_n,we use the notation

$$r \quad \Delta \quad s = [r_0, r_1, r_2, \ldots] \quad \Delta \quad [s_0, s_1, s_2, \ldots]$$

to denote the number triangle whose bi-variate generating function is given by

$$\cfrac{1}{1-\cfrac{(r_0x+s_0xy)}{1-\cfrac{(r_1x+s_1xy)}{1-\cfrac{(r_2x+s_2xy)}{1-\cdots}}}}.$$

In this instance, we follow Deleham in A120730 by taking r_n to be the sequence that begins

$$0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1, 0, \ldots$$

and s_n to be the sequence that begins

$$1, 0, 0, -1, 1, 0, 0, -1, 1, 0, 0, -1, 1, 0, \ldots$$

(extending periodically). We thus arrive at the number triangle with bi-variate generating function

$$\cfrac{1}{1-\cfrac{xy}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{xy}{1-\cfrac{xy}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{xy}{1-\cdots}}}}}}}}}.$$

This is the triangle with general term

$$[n \le 2k]\binom{n}{k}\frac{2k-n+1}{k+1}.$$

A consequence of the fact that

$$a_n(r) = \sum_{k=0}^{n}[n \le 2k]\binom{n}{k}\frac{2k-n+1}{k+1}r^k$$

is that the generating function of $a_n(r)$ may also be expressed as

$$\cfrac{1}{1-\cfrac{rx}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{rx}{1-\cfrac{rx}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{rx}{1-\cdots}}}}}}}}}.$$

For example, the generating function of the central binomial numbers $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ can be expressed as

$$\cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1-\cfrac{x}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1-\cdots}}}}}}}}} = \cfrac{1}{1-x-\cfrac{x^2}{1-\cfrac{x^2}{1-\cfrac{x^2}{1-\cdots}}}}.$$

Similarly the generating function of the sequence with g.f. given by $\frac{c(2x^2}{1-2xc(2x^2)}$, or A151281 (the number of walks within $\mathbf{N}^2$ (the first quadrant of $\mathbf{Z}^2$) starting at $(0,0)$ and consisting of n steps taken from $\{(-1,0),(1,0),(1,1)\}$), can be expressed as a continued fraction as follows:

$$\cfrac{1}{1-\cfrac{2x}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1-\cfrac{2x}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1-\cdots}}}}}}}}} = \cfrac{1}{1-2x-\cfrac{2x^2}{1-\cfrac{2x^2}{1-\cfrac{2x^2}{1-\cdots}}}}.$$

5 LDL^t decomposition and orthogonal polynomials

We let $\mathbf{H} = \mathbf{H}(r)$ denote the Hankel matrix of the sequence $a_n(r)$, with general element $a_{i+j}(r)$. The theory of "Catalan-like" numbers ensures us that

$$\mathbf{H} = \mathbf{LDL}^t$$

where $\mathbf{D}$ is a diagonal matrix, and $\mathbf{L} = \mathbf{L}(r)$ is a lower-triangular matrix with 1's on the diagonal. Moreover, $\mathbf{L}^{-1}$ is the coefficient array of a family of orthogonal polynomials [14, 19]. Given that the Hankel transform of $a_n(r)$ is $r^{\binom{n+1}{2}}$, it is clear from the theory that in fact

$$\mathbf{D} = \mathrm{Diag}\{1, r, r^2, r^3, \ldots\}.$$

We obtain

$$\mathbf{L} = \left(\frac{c(rx^2)}{1 - rxc(rx^2)}, xc(rx^2) \right),$$

and

$$\mathbf{L}^{-1} = \left(\frac{1 - rx}{1 + rx^2}, \frac{x}{1 + rx^2} \right).$$

This latter matrix has general term

$$(-r)^{\lfloor \frac{n-k+1}{2} \rfloor} \binom{n - \lfloor \frac{n-k+1}{2} \rfloor}{\lfloor \frac{n-k}{2} \rfloor}.$$

The matrix $\mathbf{L}^{-1}$ is the coefficient array of a set of generalized Chebyshev polynomials of the third kind. To see this, we first let

$$U_n(x; r) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (-r)^k (2x)^{n-2k}.$$

$U_n(x; 1)$ corresponds to the usual Chebyshev polynomials of the second kind. The family of polynomials

$$U_n(x/2; r)$$

has coefficient array

$$\left(\frac{1}{1 + rx^2}, \frac{x}{1 + rx^2} \right).$$

The *generalized Chebyshev polynomials of the third kind* can then be defined to be

$$V_n(x; r) = U_n(x; r) - rU_{n-1}(x; r).$$

(Again, $r = 1$ corresponds to the usual Chebyshev polynomials of the third kind [4]). Then $\mathbf{L}^{-1}$ is the coefficient array of the orthogonal polynomials $V_n(x/2; r)$. We can easily verify that these polynomials satisfy the three-term recurrence

$$V_{n+2}(x/2; r) = xV_{n+1}(x/2; r) - rV_n(x/2; r).$$

Example 2. We take the case r=3. Then we get

$$\mathbf{L}(3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 3 & 1 & 0 & 0 & 0 & 0 & \dots \\ 12 & 3 & 1 & 0 & 0 & 0 & \dots \\ 45 & 15 & 3 & 1 & 0 & 0 & \dots \\ 180 & 54 & 18 & 3 & 1 & 0 & \dots \\ 702 & 234 & 63 & 21 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\mathbf{L}(3)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -3 & 1 & 0 & 0 & 0 & 0 & \dots \\ -3 & -3 & 1 & 0 & 0 & 0 & \dots \\ 9 & -6 & -3 & 1 & 0 & 0 & \dots \\ 9 & 18 & -9 & -3 & 1 & 0 & \dots \\ -27 & 27 & 27 & -12 & -3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$\mathbf{L}(3)$ is an Aigner matrix for the sequences $3, 0, 0, 0, \dots$ and $3, 3, 3, \dots$. Thus for instance

$$702 = 3 \cdot 180 + 3 \cdot 54, \qquad 234 = 180 + 3 \cdot 18.$$

The generalized Chebyshev polynomials of the third kind associated to $a_n(3)$ begin

$$1, x-3, x^2-3x-3, x^3-3x^2-6x-9, \dots$$

6 Moment representation

By means of the Stieltjes transform [10], we can establish that

$$a_n(r) = \frac{1}{2\pi}\int_{-2\sqrt{r}}^{2\sqrt{r}} x^n \frac{\sqrt{4r-x^2}}{r(r+1-x)}dx + \frac{r-1}{r}(r+1)^n.$$

7 Production matrices

It is instructive to examine the production matrices [6, 7] of some of the Riordan arrays involved in this note. The Riordan array

$$(c(rx^2), xc(rx^2))$$

has production matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ r & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & r & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & r & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & r & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & r & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

while the generalized Riordan array

$$(c(rx^2), rxc(rx^2))$$

has production matrix

$$\begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & r & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & r & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & r & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & r & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Riordan array

$$\mathbf{L} = \left(\frac{c(rx^2)}{1 - rxc(rx^2)}, xc(rx^2)\right)$$

has production matrix

$$\begin{pmatrix} r & 1 & 0 & 0 & 0 & 0 & \dots \\ r & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & r & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & r & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & r & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & r & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which confirms that its first column, which is $a_n(r)$, has generating function given by the continued fraction

$$\cfrac{1}{1 - rx - \cfrac{rx^2}{1 - \cfrac{rx^2}{1 - \cdots}}}.$$

$\mathbf{L}$ is thus the Aigner matrix for the sequences s_n given by $r, 0, 0, 0, \ldots$ and t_n given by $r, r, r, \ldots$.

8 $a_n(r)$ as row sums

Following [9], it is possible to exhibit the sequences $a_n(r)$ as the row sums of a given number triangle. In effect, the matrix $\mathbf{LB}^{-1}$, where $\mathbf{B}$ is the Binomial matrix with general term $\binom{n}{k}$ (Pascal's triangle, A007318), has row sums equal to $a_n(r)$. This matrix is the inverse of

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right) \cdot \left(\frac{1-rx}{1+rx^2}, \frac{x}{1+rx^2}\right) = \left(\frac{1-(r+1)x}{1-2x+(r+1)x^2}, \frac{x(1-x)}{1-2x+(r+1)x^2}\right).$$

We may verify this algebraically: the row sums of

$$\mathbf{LB}^{-1} = \left(\frac{c(rx^2)}{1-rxc(rx^2)} \frac{1}{1+xc(rx^2)}, \frac{xc(rx^2)}{1+xc(rx^2)}\right)$$

do have generating function $\frac{c(rx^2)}{1-rxc(rx^2)}$.

References

[1] M. Aigner, Catalan-like numbers and determinants, *J. Combin. Theory Ser. A* **87** (1999), 33–51.

[2] M. Aigner, Catalan and other numbers: a recurrent theme, in: H. Crapo and D. Senato, eds., *Algebraic Combinatorics and Computer Science*, Springer Italia, Milan, 2001, pp. 347–390.

[3] A. Bostan and M. Kauers, Automatic classification of restricted lattice walks, arXiv:0811.2899v1. Available electronically at http://arxiv.org/abs/0811.2899 .

[4] L. Boyadjiev and R. Scherer, On the Chebyshev polynomials, *Kuwait J. Sci. Engrg.* **28** (2001), 227–239.

[5] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[6] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices, *Adv. in Appl. Math* **34** (2005), 101–122.

[7] E. Deutsch, L. Ferrari, and S. Rinaldi, Production matrices and Riordan arrays, `http://arxiv.org/abs/math/0702638v1` .

[8] L. Ferrari, E. Pergola, R. Pinzani and S. Rinaldi, Some applications arising from the interactions between the theory of Catalan-like numbers and the ECO method, available electronically at `http://www.dsi.unifi.it/~ferrari/applecoadm.pdf`

[9] L. Ferrari and R. Pinzani, Catalan-like numbers and succession rules, *Pure Math. Appl.* **16** (2005), 229–250.

[10] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Clarendon Press, Oxford, 2003.

[11] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2e, Addison-Wesley, 1994.

[12] C. Krattenthaler, Advanced determinant calculus, math.CO/9902004. Available electronically at `http://arxiv.org/abs/math/9902004`

[13] J. W. Layman, The Hankel transform and some of its properties, *J. Integer Seq.* **4** (2001), Article 01.1.5.

[14] P. Peart and W-J. Woan, Generating functions via Hankel and Stieltjes matrices, *J. Integer Seq.* **3** (2000), Article 00.2.1.

[15] L. W. Shapiro, S. Getu, W-J. Woan and L.C. Woodson, The Riordan group, *Discr. Appl. Math.* **34** (1991) 229–239.

[16] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at `http://www.research.att.com/~njas/sequences/`.

[17] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* **132** (1994), 267–290.

[18] H. S. Wall, *Analytic Theory of Continued Fractions*, AMS Chelsea Publishing, 2000.

[19] W-J. Woan, Hankel matrices and lattice paths, *J. Integer Seq.* **4** (2001), Article 01.1.2.

[20] Zhizheng Zhang, A characterization of Aigner-Catalan-like numbers, *Indian J Pure Appl. Math.* **32** (2001), 1615–1619.

2000 *Mathematics Subject Classification*: Primary 11B83; Secondary 11Y55, 42C05.
Keywords: Integer sequence, Catalan-like, Riordan arrays, Hankel transform, orthogonal polynomials, Chebyshev polynomials.

(Concerned with sequences A000108,A001405, A007318, A084938, A120730, A151281, A151162, A151254, A156195, and A156361.)

Received March 11 2009; revised version received June 6 2009. Published in *Journal of Integer Sequences*, June 6 2009.

Return to Journal of Integer Sequences home page.