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Vertical Flow of a Multiphase Mixture in a Channel

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The flow of a multiphase mixture consisting of a viscous fluid and solid particles between two vertical plates is studied. The theory of interacting continua or mixture theory is used. Constitutive relations for the stress tensor of the granular materials and the interaction force are presented and discussed. The flow of interest is an ideal one where we assume the flow to be steady and fully developed; the mixture is flowing between two long vertical plates. The non-linear boundary value problem is solved numerically, and the results are presented for the dimensionless velocity profiles and the volume fraction as functions of various dimensionless numbers.

Keywords: Multiphase flows; Granular materials; Mixture theory; Continuum mechanics; Interaction force

INTRODUCTION

Study of multiphase flows has become a strong field of research and interest due to its immense number of applications. The classical theories, primarily based on some type of an 'averaging technique' [itself based on continuum mechanics] has been presented in many text books (*cf.* Soo, Ishii, Wallis). These theories, when applied to two-phase flows have also become known as the two-fluid approach. In general, solid particles or bubbles are dispersed in a liquid or a gas. To assume that an assembly of solid particles behaves as a fluid is one of

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the limitations of the two-fluid model. An alternative approach to study the multicomponent systems, similar in many ways to the averaging technique, is the theory of interacting continua which is based on the ideas of diffusion proposed by Fick (*cf.* Truesdell, 1962). This theory has been used successfully in many interesting and diverse fields such as phase transition (*cf.* Gray and Svendsen, 1997), immiscible mixtures (*cf.* Drumheller, 2000), incompressible mixtures (Costa Mattos *et al.*, 1995), mushy zone (*cf.* Hills and Roberts, 1988), debris flows (*cf.* Iverson, 1997), polymer-fluid mixtures (*cf.* Lusitg *et al.*, 1992; Kaloni *et al.*, 1997), avalanches (Hutter, 1983), flow through elastic layers and rubbers (*cf.* Rajagopal and Tao, 1995 for this and many other applications of the theory), particulate mixtures (*cf.* Nunziato *et al.*, 1983; Johnson *et al.*, 1991; Massoudi *et al.*, 1999). The foundation of the theory is given in books by Truesdell (1984); Dobran (1991) and Rajagopal and Tao (1995).

To use the mixture theory to study flows of solid particles entrained in a fluid, we require the particles to be modelled as granular materials. In recent years there has been considerable interest in understanding the behavior of flowing granular materials. The situations encountered include handling of substances such as coal, agricultural products, and other particulate solids, and more complicated processes such as fluidization, combustion, avalanches, *etc.* (*cf.* Gudhe *et al.*, 1993). In earlier studies, Johnson *et al.* (1991a, b) and Massoudi *et al.* (1999) have formulated a two-phase flow theory based on the theory of interacting continua. In this paper we will give a brief review of their formulation. The objective is to study the flow of a mixture of a linearly viscous fluid infused with small solid spherical particles, flowing downward between two vertical plates. First, we present the basic equations of mixture theory and then discuss the constitutive relations for the granular stress tensor and the interaction force. The emphasis in this paper is to give a brief review of the important issues and the various mechanisms for the interactive force.

GOVERNING EQUATIONS

The details of the mixture theory is given in books by Truesdell (1984) and Rajagopal and Tao (1995). At the same time review articles by

Atkin and Craine (1976); Bowen (1976) and Bedford and Drumheller (1983) shed light on specific issues and the important problems in mixture theory. One of the most comprehensive continuum theories developed for polycrystalline mixtures, granular composites, and fluid suspensions is that of Twiss and Eringen (1971, 1972). The theory of interacting continua, *i.e.*, mixture theory, is well established, and therefore we only present the equations of motion for the problem we are investigating. The mixture theory, is in a sense, a homogenization approach in which each component is regarded as a single continuum and at each instant of time, every point in space is considered to be occupied by a particle belonging to each component of the mixture (*cf.* Truesdell, 1984). Balance laws are then written for each component which takes into account interaction with other constituents. Therefore, the way we see or imagine that the mixture is being formed is that the fluid phase in its original configuration has the density ρ_f and volume V_1, and the solid phase in its original configuration consists of 'chunks' of materials, with various shapes and sizes, with voids separating them, in a given volume V_2. The solid phase is then 'homogenized' by introducing the volume fraction field where the solid materials are now distributed, in a sense crushed, over the control volume V_2, and therefore the solid phase acts as a continuum with density ρ_s. We call this configuration the reference configuration for the granular materials. Now the mixture is formed by the motion of this 'new' granular phase and the fluid phase. Figure A shows the details. It is important to notice that before the homogenization process, the volume fraction is either zero or one; zero if we happen to be in the void space, and one if we are on a solid material. After the homogenization process, $0 \leq \nu < 1$.

Since thermal, chemical, and electro-magnetic effects are not considered in the present problem, the balance laws reduce to the conservation of mass and the conservation of linear momentum. And since we are only considering a two-component mixture, we are left only with four governing equations. The conservation of mass and the conservation of linear momentum for the fluid phase and the solid phase are:

$$\frac{\partial \rho_1}{\partial t} + \operatorname{div}(\rho_1 \mathbf{v}_1) = 0 \tag{1}$$

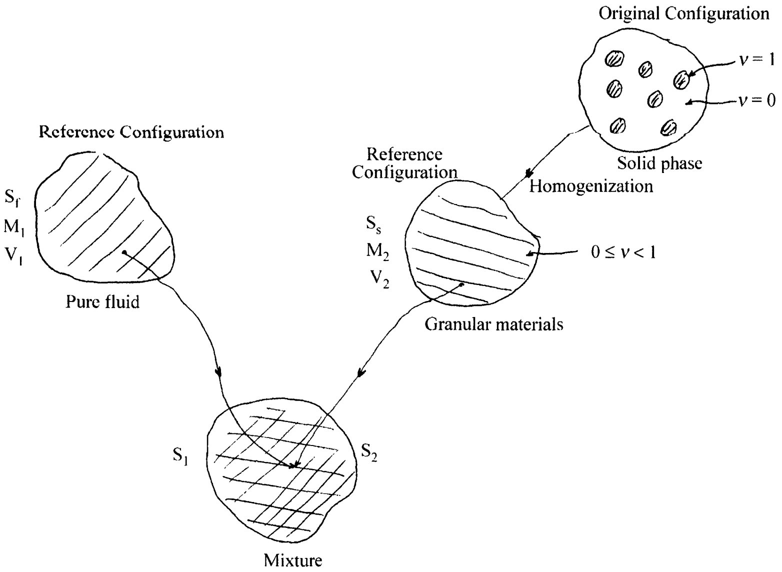

FIGURE A Formation of mixture.

$$\frac{\partial \rho_2}{\partial t} + \text{div}(\rho_2 \mathbf{v}_2) = 0 \tag{2}$$

and,

$$\rho_1 \frac{D_1 \mathbf{v}_1}{Dt} = \text{div}\mathbf{T}_1 + \rho_1 \mathbf{b}_1 + \mathbf{f}_I, \tag{3}$$

$$\rho_2 \frac{D_2 \mathbf{v}_2}{Dt} = \text{div}\mathbf{T}_2 + \rho_2 \mathbf{b}_2 - \mathbf{f}_I \tag{4}$$

where **b** represents the body force, $\mathbf{f}_I$ represents the mechanical interaction (local exchange of momentum) between the two components, ρ_1 and ρ_2 are the bulk densities of the fluid and solid constituents, and $\mathbf{T}_1$ and $\mathbf{T}_2$ denote the partial stress tensors of the fluid and solid respectively. In the above equations, $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocity of the fluid and solid particles, respectively, $(\partial/\partial t)$ denotes the partial derivative with respect to time, (D_1/Dt) and (D_2/Dt) are the material time derivatives following the motion of constituent 1 and 2, *i.e.*, the fluid and solid phases, respectively. The densities ρ_1

and ρ_2 are related to the pure densities of the fluid and solid phases through the volume fractions. For a saturated mixture, the sum of the volume fractions is equal to one, and therefore ρ_1 *and* ρ_2 are related to ρ_f and ρ_s *via*

$$\rho_1 = \rho_f(1-\nu), \tag{5}$$

$$\rho_2 = \rho_s \nu, \tag{6}$$

where ν is the volume fraction of the solid particles and $0 \leq \nu \leq \nu_m < 1$, where ν_m is the maximum packing limit for the particles, and $\nu = 0$ implies that there are no particles.

Equations (5) and (6), in a sense, follow from the definition of mixture density, *i.e.*, $\rho_{\text{mix}} = \rho_1 + \rho_2$, and the 'volume additivity constraint'. This puts a restriction on the motion of the mixture (*cf.* Mills, 1966), where it is assumed that the volume of the mixture is the sum of the volumes of the individual constituents in their reference configurations. Therefore,

$$V_{\text{tot}} = V_1 + V_2 \tag{7}$$

But

$$\begin{aligned} \rho_1 &= \frac{M_1}{V_{\text{tot}}} \\ \rho_2 &= \frac{M_2}{V_{\text{tot}}} \end{aligned} \tag{8}$$

$$\begin{aligned} \rho_f &= \frac{M_1}{V_1} \\ \rho_s &= \frac{M_2}{V_2} \end{aligned} \tag{9}$$

Substituting (9) in (7) we have:

$$V_{\text{tot}} = \frac{M_1}{\rho_f} + \frac{M_2}{\rho_s} \tag{10}$$

Substituting (8) in (10) we get the volume additivity constraint:

$$\frac{\rho_1}{\rho_f} + \frac{\rho_2}{\rho_s} = 1 \tag{11}$$

CONSTITUTIVE RELATIONS

Looking at Eqs. (3) and (4), it is clear that in order to close the system of equations, we need to provide constitutive relations for $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{f}_I$. Since, in our previous studies (*cf.* Johnson *et al.*, 1991a, b; Massoudi *et al.*, 1999) we have given the rationale and the justification for the forms of the constitutive equations for the stress tensors, we will not mention this here. Instead we will focus on an appropriate form for the interactive force $\mathbf{f}_I$. Therefore, we assume that the stress tensors for the fluid phase and the granular materials are given by:

$$\mathbf{T}_f = [-p(\rho_1) + \lambda_f(\rho_1)\text{tr}\mathbf{D}_1]\mathbf{I} + 2\mu_f(\rho_1)\mathbf{D}_1, \tag{12}$$

$$\mathbf{T}_s = [\beta_0(\nu) + \beta_1(\nu)\nabla\nu \cdot \nabla\nu + \beta_2(\nu)\text{tr}\mathbf{D}_2]\mathbf{I} + \beta_4(\nu)\nabla\nu \otimes \nabla\nu + \beta_3(\nu)\mathbf{D}_2, \tag{13}$$

where p is the fluid pressure, λ_f and μ_f are the viscosities of the fluid, $\mathbf{D}_1$ and $\mathbf{D}_2$ are the symmetric part of the velocity gradients associated with the fluid motion and the granular particles, respectively. Rajagopal and Massoudi (1990) give the following interpretation for the various material parameters: $\beta_0(\nu)$ is similar to the pressure in a compressible fluid and is given by an equation of state, $\beta_2(\nu)$ is like the second coefficient of viscosity in a compressible fluid, $\beta_1(\nu)$ and $\beta_4(\nu)$ are the material parameters connected with the distribution of the granular materials and $\beta_3(\nu)$ is the viscosity of the granular materials. Following Rajagopal and Massoudi (1990), we assume that the material parameters corresponding to the solid to be of the form

$$\begin{aligned}
\beta_0(\nu) &= B_0\nu \\
\beta_1(\nu) &= B_1(1 + \nu + \nu^2) \\
\beta_2(\nu) &= B_2(\nu + \nu^2) \\
\beta_3(\nu) &= B_3(\nu + \nu^2) \\
\beta_4(\nu) &= B_4(1 + \nu + \nu^2)
\end{aligned} \tag{14}$$

In their study, Rajagopal and Massoudi (1990) advocate using an orthogonal rheometer to measure some of the material properties, specifically β_1 and β_4. Through a simple analysis they show that if Eq. (13) is to apply at equilibrium (static) case, then $B_0 < 0$. Also, B_3 is related to the shear viscosity of the material and is therefore assumed to be positive. B_2 is related to the second coefficient of viscosity and we have no information about the sign of this parameter; however, since for the flow of interest here tr $\mathbf{D}_2 = 0$, this term does not appear in the equations. With regard to B_1 and B_4 individually, we have no information (about the sign or the relative magnitude), as these terms appear in the equations as a sum, *i.e.* $(B_1 + B_4)$. Johnson *et al.* (1991) assume that $B_1 + B_4 < 0$. Later, Rajagopal *et al.* (1992) prove existence of solutions when $\beta_1 + \beta_4 > 0$, *i.e.*, assuming that β_1 and β_4 are constants. In our study, since we are assuming quadratic variations for β_1 and β_4, we follow the analysis given by Johnson *et al.* (1991) by assuming $B_1 + B_4 < 0$. However, in the absence of any experimental evidence, one should also do a parametric study for cases when $B_1 + B_4 > 0$, and then perform a stability analysis to see if indeed solutions corresponding to positive or negative values are stable. And since we have limited our analysis to isothermal cases, we do not have any thermodynamic restrictions on these material coefficients. Furthermore, since we expect $\mathbf{T}_s \to 0$ as $\nu \to 0$, *i.e.*, when there are no particles, there should be no stress, then we can see that the constant terms in β_2 and β_3 have been set to zero. Finally, we should mention that the structure given by Eq. (14) is only an assumption and is by no means the most general approximation.

With the above formulation, we have already assumed that the individual grains are incompressible in the sense that ρ_s is constant. And therefore, the material parameters $\beta_0 - \beta_4$ now depend on ν instead of ρ_2. We now turn our attention to the formulation of a general constitutive relation for the interaction force in two-phase flows. In general, these flows consist of a large number of solid particles or gas bubbles suspended in a fluid medium (such as air or water). Most of the momentum exchange relations that have been proposed for these types of two-phase flows are based on generalizations of the force balance on a single particle moving through a fluid (*cf.* Zuber, 1964). First, a discussion and an overview of the dynamics of a single particle in a fluid is presented. Then, a constitute relation suitable for continuum theories of mixtures is proposed. The details are given in Johnson *et al.* (1990).

Tchen[1] (1947) synthesizes the work of Basset, Boussinesq, Stokes and Oseen on the motion of a sphere settling under the force of gravity in a fluid at rest. The resulting force balance, sometimes known as the Basset-Boussinesq-Oseen (BBO) equation, is given by:

$$\begin{aligned}\frac{4\pi a^3}{3}\rho_s\dot{u} = &-\frac{2\pi a^3}{3}\rho_f\dot{u} - 6\pi\mu_f au \\ &- 6\pi\mu_f a\frac{a}{\sqrt{\pi\nu_f}}\int_{-\infty}^{t}\frac{\dot{u}(t_1)}{\sqrt{t-t_1}}dt_1 - \frac{4\pi a^3}{3}g(\rho_s-\rho_f),\end{aligned} \tag{15}$$

where u is the velocity of the particle, ρ_f and ρ_s are density of the fluid and particle, respectively, a is the particle radius, g is the acceleration of gravity, μ_f and ν_f are viscosity and kinematic viscosity of the fluid, respectively. The terms on the right hand side of Eq. (15) reflect the presence of virtual mass, Stokes drag, Basset history effects, and buoyancy. Tchen (1947) modified Eq. (15) to describe unsteady Stokes motion of a solid spherical particle in a fluid with a uniform flow field. His modifications include replacing the particle velocity by its relative velocity, and the addition of a term accounting for pressure gradients in the fluid. The resulting expression is:

$$\begin{aligned}\frac{4\pi a^3}{3}\rho_s\dot{u} = &\frac{4\pi a^3}{3}\rho_f\dot{v} - \frac{2\pi a^3}{3}\rho_f(\dot{u}-\dot{v}) - 6\pi\mu_f a(u-v) \\ &- 6\pi\mu_f a\frac{a}{\sqrt{\pi\nu_f}}\int_{-\infty}^{t}\frac{\dot{u}(t_1)-\dot{v}(t_1)}{\sqrt{t-t_1}}dt_1 - \frac{4\pi a^3}{3}g(\rho_s-\rho_f),\end{aligned} \tag{16}$$

where v is the velocity of the fluid in the neighborhood of the particle but far enough away to be unaffected by it. We should note that Eq. (16) is a scalar component of a more general vector equation. Corrsin and Lumley (1956), noting that Eq. (16) applies only in the absence of fluid velocity gradients, propose a more general equation, which takes velocity gradients into account in its expression for the pressure gradient. The source of disagreement over the form of the single particle force balance concerns the pressure gradient term. Tchen (1947) originally proposed the addition of a term accounting for the pressure gradient based on an intuitive argument. Corrsin and Lumley (1956) argued that, for a nonuniform flow field, the full Navier–Stokes equations should be used to determine the pressure gradient.

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[1] Tchen's theory is studied extensively by Gouesbet *et al.* (1984).

Buevich (1966) criticizes both previous studies (Tchen, 1947; Corrsin and Lumley, 1956) in pointing out that adding a term to the BBO equation is not necessary. Soo (1975, 1976) argues that the pressure gradient force is exactly balanced by fluid inertia forces and should not appear in the force balance in any form. Maxey and Riley (1983), based on an analysis similar to that of Buevich, propose the following equation for the force on a sphere in a nonuniform flow:

$$\begin{aligned} m_s\frac{du_i}{dt} = {} & (m_s-m_f)g_i+m_f\frac{Dv_i}{Dt}\Big|_{\mathbf{Y}(t)} \\ & -\frac{1}{2}m_f\frac{d}{dt}\left\{u_i(t)-v_i[\mathbf{Y}(t),t]-\frac{1}{10}a^2\nabla^2 v_i|\mathbf{Y}(t)\right\} \\ & -6\pi a\mu_f\left\{u_i(t)-v_i[\mathbf{Y}(t),t]-\frac{1}{6}a^2\nabla^2 v_i|\mathbf{Y}(t)\right\} \\ & +6\pi a^2\mu_f\int_0^t d\tau\frac{\left((d/d\tau)\left\{u_i(\tau)-v_i[\mathbf{Y}(\tau),\tau]-(1/6)a^2\nabla^2 v_i|\mathbf{Y}(\tau)\right\}\right)}{\sqrt{\pi\nu_f(t-\tau)}} \end{aligned} \tag{17}$$

Note that the inclusion of velocity gradients in their analysis results in modifications to the virtual mass, Stokes drag, and Basset history terms. These velocity gradients correspond to the physical effect known as Faxen forces.

Though Eq. (12) appears to be complete for a single particle in Stokes flow, there are, in general, other forces that must be considered (even for a purely mechanical system). In flows with high relative velocity between phases, or large velocity gradients in the fluid phase, lift may become an important effect (McLaughlin, 1989). Also, spin of the particle is not taken into account in the above equations.

After this brief review, the question remains: What should be an appropriate form for the interactive force $\mathbf{f}_I$ where now we have an assembly of particles? The general tendency has been to generalize Eq. (17) by appropriately generalizing the coefficients appearing in that equation. As Table I indicates there has been a great deal of disagreements on the appropriate form of the pressure term, and in fact, even on the form of the constitutive relations in general. In the generalization approach, taken by many authors, for example Anderson and Jackson (1967), terms proposed for the relative

TABLE I Force due to pressure gradient

Author	*Pressure term*
Tchen, 1947	$V\rho_f((\partial v/\partial t) + v(\partial v/\partial x))$
Corrsin and Lumley, 1956	$V\rho_f((\partial \mathbf{v}/\partial t) + (\mathrm{grad}\mathbf{v})\mathbf{v} - \nu\nabla^2\mathbf{v})$
Buevich, 1966	$V\rho_f((\partial \mathbf{v}/\partial t) + (\mathrm{grad}\mathbf{v})\mathbf{u})$
Soo, 1975, 1976	0
Maxey and Riley, 1983	$V\rho_f((\partial \mathbf{v}/\partial t) + (\mathrm{grad}\mathbf{v})\mathbf{v})\|_{\mathbf{x}=\mathbf{y}(t)}$

acceleration term (virtual mass force) were not frame invariant (*cf.* Truesdell and Noll, 1992). These were pointed out and corrected by other authors (*cf.* Drew, Homsy). A detailed review of these issues are given in the report by Johnson *et al.* (1990). We will only present the form for $\mathbf{f}_I$ based on their findings, recalling that the chemical, thermal, and electro-magnetic effects have not been included.

A summary of the available results for each interaction force is given below:

- *Drag* acts parallel to the direction of flow and should always be included in any model of two-component flow. This is the most studied interactive force, and many experimental correlations for different flow regimes are available in the literature.
- *Diffusion* acts in the direction of density gradients. Little is known about the form of the coefficients, but based on Müller's (1968) analysis, we speculate that a term of this form should be included for all situations.
- *'Slip-Shear' Lift* acts perpendicular to the direction of flow. Its importance increases with increasing volume fraction, relative velocity, fluid density, fluid viscosity, and fluid velocity gradients. It cannot be neglected in most situations. This is usually referred to as Saffman's lift force.
- *'Spin' Lift* acts perpendicular to the direction of flow. Though typically a much smaller effect than 'slip-shear' lift, it may become significant at some points in the system (especially for larger particle sizes, because $\mathbf{W}_1 - \mathbf{W}_2$ becomes larger). Its magnitude increases with increasing volume fraction, fluid density, and relative velocity. This may become important where the vorticity is large.
- *Virtual Mass* effects are present only if there is relative acceleration between the mixture components. The virtual mass force acts in

the direction of flow and becomes larger with increasing volume fraction and particle size. Its relative magnitude, in comparison to the other forces accelerating the particles, depends upon the relative densities of the components. It is especially important for large particles that have smaller densities than the fluid, for example large air bubbles in a liquid medium.

- *Basset Force* is present only in unsteady (or accelerating) flows. It acts in the direction of flow and increases with increasing particle size, fluid density, and fluid viscosity. It has been suggested (but not verified) that it also increases with volume fraction. This force basically includes the history effects.

Therefore, a general (and yet simple enough for computational purpose) constitutive relation for the mechanical interaction between the mixture components, $\mathbf{f}_I$, is written as (Johnson *et al.*, 1990):

$$\begin{aligned}\mathbf{f}_I &= A_1 \operatorname{grad} \nu + A_2 F(\nu)(\mathbf{v}_2 - \mathbf{v}_1) + A_3 \nu (2 \operatorname{tr} \mathbf{D}_1^2)^{-1/4} \mathbf{D}_1 (\mathbf{v}_2 - \mathbf{v}_1) \\ &\quad + A_4 \nu (\mathbf{W}_2 - \mathbf{W}_1)(\mathbf{v}_2 - \mathbf{v}_1) + A_5 \mathbf{a}_{\mathrm{vm}} \qquad (18)\end{aligned}$$

where $\mathbf{a}_{\mathrm{vm}}$ is a properly frame invariant measure of the relative acceleration between the mixture components and $F(\nu)$ represents the dependence of the drag coefficient on the volume fraction. The terms in Eq. (18) reflect the presence of density gradients,[2] drag, "slip-shear" lift, "spin" lift, and virtual mass, respectively. Müller's (1968) work indicates that a term of the form A_1 grad ν must be included in the interactions in order to get well-posed problems. The term multiplying A_3 is a generalization of Saffman's (1965, 1968) single-particle result proposed in this form by McTigue *et al.* (1986). In the above equation, $\mathbf{W}_2$ and $\mathbf{W}_1$ denote the spin tensor (the skew-symmetric part of the velocity gradients) associated with the fluid and solid motions. As we explained earlier, within the framework of mixture theory developed here, we cannot directly account for particle size, or different particle sizes, or particle shape, or surface roughness, or... However, since the

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[2] The actual form of this interaction should include the terms α_1 grad $\rho_1 + \alpha_2$ grad ρ_2 where α_1 and α_2 are constants. If we assume that the system is a saturated mixture with incompressible components, this expression simplifies to A_1 grad ν, where $A_1 = \alpha_2 - \alpha_1$. Since no information concerning the coefficients α_1 and α_2 is available and a term of the same form arises from the granular solid stress tensor, this term will be neglected in the present work.

coefficients in Eq. (18) are determined either by performing simple experiments on an assembly of known size and known shape particles, or by extrapolating and extending the results of a single particle, as a result material or geometrical properties can enter in the constitutive relation, through these coefficients. Therefore, we have, for example (*cf.* Johnson *et al.*, 1990):

$$\begin{aligned} A_2 &= \frac{9}{2}\frac{\mu_f}{a^2} \\ A_3 &= \frac{3(6.46)}{4\pi}\frac{\rho_f^{1/2}\mu_f^{1/2}}{a} \\ A_4 &= \frac{3}{4}\rho_f \\ A_5 &= 2\frac{\pi}{3}a^3\nu\frac{1+2\nu}{1-\nu} \end{aligned} \tag{19}$$

where a is particle radius. If the particles are non-spherical, such as fibers, then the directionality may become an important element, and in that case we need to use (or develop) the constitutive relations which take the microstructure and directionality, *i.e.*, anisotropic nature of the material, into account.

For the present study, we assume $A_1 = A_4 = A_5 = 0$, and therefore Eq. (18) becomes:

$$\mathbf{f}_I = A_2 F(\nu)(\mathbf{v}_2 - \mathbf{v}_1) + A_3\nu(2\operatorname{tr}\mathbf{D}_1^2)^{-1/4}\mathbf{D}_1(\mathbf{v}_2 - \mathbf{v}_1), \tag{20}$$

In the previous studies (*cf.* Johnson *et al.*, 1991; Massoudi *et al.*, 1999) we have used the following relation for $F(\nu)$:

$$F(\nu) = \nu(1 + 6.55\nu) \tag{21}$$

This correlation was proposed by Drew (1976) and is based on the work of Batchelor (1972) for sedimentation of particles. In the present study (following Johnson *et al.*, 1990), we assume $F(\nu)$ to be of the form given by (*cf.* Apazidis, 1985)

$$F(\nu) = \frac{\nu(4 + 3\nu + 3\sqrt{8\nu - 3\nu^2})}{(2 - 3\nu)^2} \tag{22}$$

Notice that the above equation diverges as $\nu \to (2/3)$.

A mixture stress tensor can be defined as (*cf.* Green and Naghdi, 1969):

$$\mathbf{T}_m = \mathbf{T}_1 + \mathbf{T}_2, \tag{23}$$

where,

$$\mathbf{T}_1 = (1-\nu)\mathbf{T}_f, \quad \text{and} \quad \mathbf{T}_2 = \nu\mathbf{T}_s \tag{24}$$

so that the mixture stress tensor reduces to that of a pure fluid as $\nu \to 0$ and to that of a granular solid as $\nu \to 1$. Obviously this is an idealization. At a certain ν_m, called the maximum packing fraction, the particles are in such a close proximity to each other that unless they are crushed or deformed, ν can no longer change. Therefore, in the numerical solution to all the practical problems of interest, we have to set certain limits or restrictions on ν so that ν can never be greater than ν_m. One of the advantages of having formulated the mixture theory as presented here is that in the limit where there are no particles, the momentum equations reduce to the appropriate form for a pure fluid, in this case the Navier–Stokes Equation, and in the limit where $\nu \to \nu_m$, *i.e.*, a dense flow, the equations reduce to that of a flowing granular materials. In this case, the effect or the presence of the fluid in the interstices is ignored.

FLOW BETWEEN TWO VERTICAL PLATES

We now consider the flow of fluid–solid mixture down a vertical channel due to the action of gravity (*cf.* Fig. 1). The plates are assumed to be infinitely long, and therefore the edge effects are ignored. The flow is assumed to be steady, one dimensional, fully developed flow. Let X denote the direction of motion and let the vertical plates be located at $Y=-1$ and $Y=1$. The flow field is assumed to be of the form:

$$\begin{aligned} &\nu = \nu(y) \\ &\mathbf{v}_1 = V(y)\mathbf{i} \quad \text{(Fluid)} \\ &\mathbf{v}_2 = u(y)\mathbf{i} \quad \text{(Solid)} \end{aligned} \tag{25}$$

With the above flow field the equations for conservation of mass are automatically satisfied. The balance of linear momentum in the

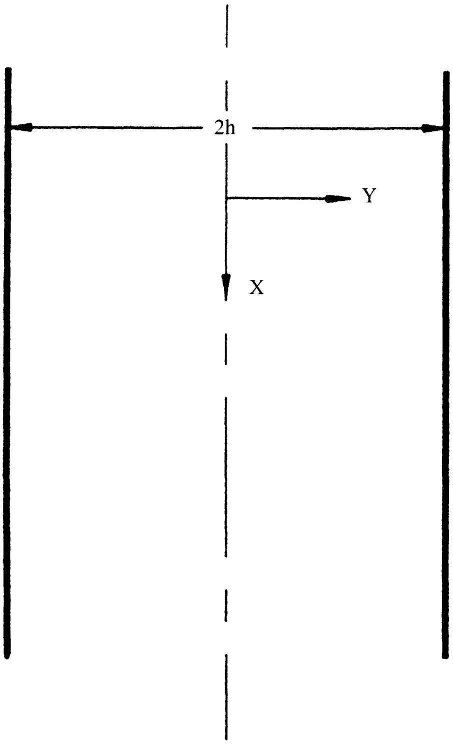

FIGURE 1 Flow between two infinite plates.

non-dimensional form reduces to

$$(1-\nu)\frac{d^3\overline{V}}{d\overline{y}^3}-2\frac{d\nu}{d\overline{y}}\frac{d^2\overline{V}}{d\overline{y}^2}-\frac{d^2\nu}{d\overline{y}^2}\frac{d\overline{V}}{d\overline{y}}-R_1\frac{d\nu}{d\overline{y}}$$

$$+D_1\left[F(\nu)\left(\frac{d\overline{u}}{d\overline{y}}-\frac{d\overline{V}}{d\overline{y}}\right)+\frac{dF}{d\nu}(\overline{u}-\overline{V})\frac{d\nu}{d\overline{y}}\right]=0 \qquad (26a)$$

$$(\nu+\nu^2)\frac{d^2\overline{u}}{d\overline{y}^2}+(1+2\nu)\frac{d\nu}{d\overline{y}}\frac{d\overline{u}}{d\overline{y}}+R_2\nu-D_2F(\nu)(\overline{u}-\overline{V})=0 \qquad (26b)$$

$$2(1+\nu+\nu^2)\frac{d\nu}{d\overline{y}}\frac{d^2\nu}{d\overline{y}^2}+(1+2\nu)\left(\frac{d\nu}{d\overline{y}}\right)^3$$

$$+B\frac{d\nu}{d\overline{y}}+L\nu\left|\frac{d\overline{V}}{d\overline{y}}\right|^{-1/2}\frac{d\overline{V}}{d\overline{y}}(\overline{u}-\overline{V})=0 \qquad (26c)$$

where,

$$\bar{y}=\frac{y}{h},\ \bar{u}=\frac{u}{u_0},\ \overline{V}=\frac{V}{u_0},\ \overline{\rho_1}=\frac{\rho_f}{\rho_0},\ \overline{\rho_2}=\frac{\rho_s}{\rho_0} \tag{27}$$

and

$$\begin{gathered}C_2=\frac{A_2h}{\rho_0u_0};\ Fr=\frac{u_0^2}{hg};\ Re=\frac{\rho_0u_0h}{\mu_f};\ \overline{B_0}=\frac{B_0}{\rho_0u_0^2};\ \overline{B_1}=\frac{B_1}{\rho_1u_0^2h^2}\\ \overline{B_3}=\frac{B_3}{\rho_0u_0h};\ \overline{B_4}=\frac{B_4}{\rho_0u_0^2h^2};\ C_3=\frac{A_3h^{1/2}}{\rho_0u_0^{1/2}}.\end{gathered} \tag{28}$$

where h is some characteristic length, for instance half the space between the plates, u_0 is a characteristic velocity, and ρ_0 is a characteristic density. A look at Eq. (28) reveals that as a result of our formulation we end up with two different kinds of dimensionless numbers. The first is due to material parameters in the constitutive relations for each stress tensor, and the second kind is due to the coefficients appearing in the interactive force. Therefore, we can see that C_2 and C_3 arise because of the interaction mechanisms of drag and lift, respectively, while the other dimensionless numbers are due to material parameters. Also we need to mention that there is no dimensionless number due to pressure (gradient) because the pressure term is eliminated by cross differentiating the two momentum equations for the fluid phase, and therefore we are left with only one momentum equation for the fluid component; however, the order of the equation is raised, as we can see from Eq. (26(a)).

For the sake of simplicity, let us define the following non-dimensional parameters.

$$\begin{gathered}D_1=C_2Re;\ D_2=\frac{2C_2}{\overline{B_3}};\ R_1=\frac{Re}{Fr};\ R_2=\frac{2\overline{\rho_2}}{Fr\overline{B_3}};\\ B=\frac{\overline{B_0}}{\overline{B_1}+\overline{B_4}};\ L=\frac{-C_3}{\overline{B_1}+\overline{B_4}}\end{gathered} \tag{29}$$

Unlike the list of the dimensionless numbers given in Eq. (28), the ones given in Eq. (29) are not 'real' dimensionless numbers in the sense that they represent combination of various effects, and it is only a tool to reduce the number of parameters in the problem. Equations (26(a)–(c)) are to be solved numerically subject to appropriate boundary conditions. Looking at Eq. (26) we can see that we need three boundary conditions for the fluid velocity V, two boundary conditions for the solid velocity u, and two boundary conditions for the volume fraction ν. Assuming the no-slip boundary condition for both phases on both walls, we have:

$$\begin{aligned} &\overline{u} = 0 \\ &\overline{V} = 0 \quad \text{at } \overline{Y} = -1 \end{aligned} \tag{30}$$

$$\begin{aligned} &\overline{u} = 0 \\ &\overline{V} = 0 \quad \text{at } \overline{Y} = 1 \end{aligned} \tag{31}$$

For the volume fraction, we can see that due to the symmetry in the problem, the value of ν at the two walls can be assumed to be equal, and this gives us one condition, given by Eq. (34). The other condition is obtained by providing an integral condition given by Eq. (32). This is simply a measure of the amount of granular materials fed into the system. We still need one more boundary condition for V. We assume that we know the flow rate of the mixture, Q, given by Eq. (33).

$$N = \int_{-1}^{1} \nu \overline{dy} \tag{32}$$

$$Q = \int_{-1}^{1} [(1-\nu)\overline{V} + \nu\overline{u}]\overline{dy} \tag{33}$$

$$\nu(1) = \nu(-1) \tag{34}$$

We realize that Eqs. (32) and (33) would make the numerical scheme more complicated, but we think both of these conditions are experimentally controllable parameters and therefore more realistic values can be assumed *a priori.*

NUMERICAL RESULTS

Equation (26(a)) represents the momentum equation in the direction of the flow for the fluid phase, recalling that the pressure term has been eliminated *via* cross-differentiation. Equations (26(b)–(c)) are the momentum equations for the solid phase in the direction of the flow, and normal to the direction of the flow, respectively. It is important to note that Eq. (26(c)) basically gives us the volume fraction distribution, and in the absence of lift forces, this equation is not coupled to the other two equations and can be integrated directly.

The system of Eq. (26(a)–(c)) subject to the boundary conditions (30) through (34) are solved numerically using a collocation code COLSYS (*cf.* Ascher *et al.*, 1981). Collocation is implemented by COLSYS using B-spline basis functions. COLSYS also features an adaptive mesh-selection procedure based on error estimates. The mesh points are repositioned to roughly equidistribute the error, which is estimated using mesh halving and checked against user prescribed tolerances. The integral conditions are implemented using a secant shooting method to refine the initial guesses. Here, we carry out a parametric study of the equations and delineate how the two of the non-dimensional parameters affect the solution.

In this paper, we will focus our attention only on two dimensionless numbers, B and R_2. We can see from Eq. (29) that B is the ratio of the so-called 'pressure term' (related to β_0) to the 'distribution term' (related to $\beta_1+\beta_4$). Therefore, an increase in B implies either an increase in the pressure term, *i.e.*, β_0 becomes larger, or a decrease in the parameters responsible for the distribution of the particles, *i.e.*, $\beta_1+\beta_4$. Therefore, a smaller B could mean larger density (or volume fraction) effects. The manner in which the volume fraction profiles change with B are shown in Figure 2. Notice, that as B increases the volume fraction increases towards the center of the channel and decreases towards the walls of the channel. This means that as B increases the particles have a tendency to accumulate (or to gather) in the center of the channel, with fewer particles being at the two plates. Also, it is interesting to note that all the profiles cross each other at two locations, a distance of approximately 0.6 unit from the center of the channel. It is hard to say whether this has any relation to the Segre–Silberberg effect (also known as the tubular pinch).

Next we consider the effects of R_2, on the velocity profiles. This number is a combination of three separate factors: (i) It is a measure of the gravitational effects *via* the Froude number, *Fr* (where larger *Fr*

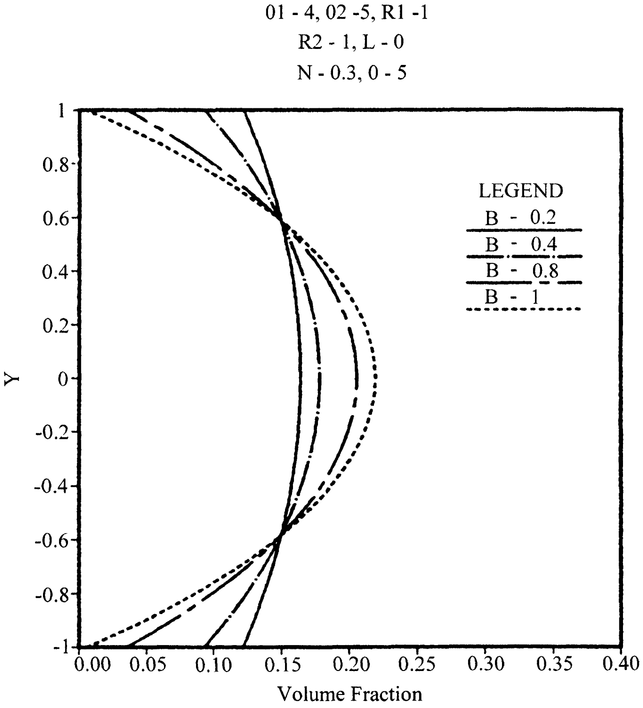

FIGURE 2 Effect of B on the volume fraction profile.

means that gravity is less important, or that the fluid has more inertia), (ii) it is a measure of the heaviness (higher densities) of the particles, (iii) and it is a measure of how 'viscous' the granular material is. Therefore, when R_2 is increased, it could be due to any or a combination of the following factors: the particles are heavier, *i.e.*, ρ_s is larger; B_3 is smaller, *i.e.*, the granules are less viscous (B_3 is the inverse of the solid's Reynolds number); *Fr* number is smaller. Figure 3 shows the effect of R_2 on solid velocity profile. Here, the velocity increases as R_2 is increased; the profiles start to 'flatten out' and they begin to look like turbulent velocity profiles for a viscous fluid. The manner in which R_2 affects the fluid velocity profile is shown in Figure 4. Notice, that the fluid velocity decreases as R_2 is increased. A look at Eq. (26(a)) indicates that R_2 does not directly enter into the picture, and therefore any change in the velocity of the fluid is through a change in the velocity of the solid phase, given by Eq. (26(b)).

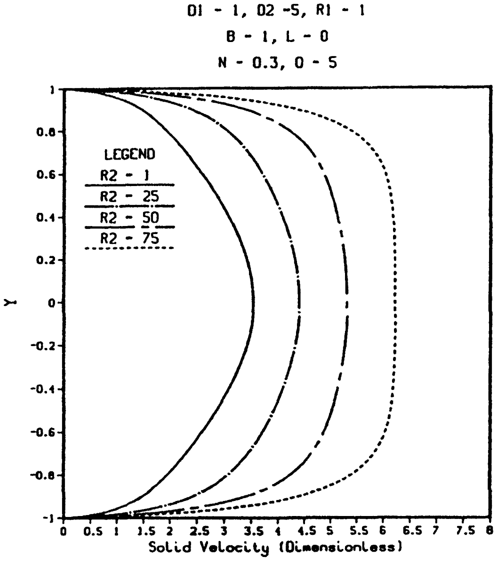

FIGURE 3 Effect of R_2 on the solid velocity profile.

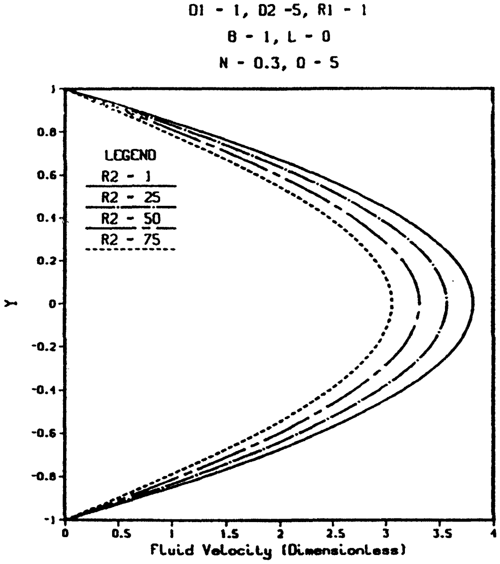

FIGURE 4 Effect of R_2 on the Fluid velocity profile.

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